An adelic open image theorem for abelian schemes

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The aim of this talk is to extends to arbitrary abelian scheme, Serre's celebrated adelic open image theorem for elliptic curves.

More precisely, let X be a smooth, separated and geometrically connected scheme over a number field k and let $A \to X$ be an abelian scheme. These data give rise to an adelic representation $\rho : \pi_1(X) \to \prod_\ell \mathrm{GL}(T_\ell(A_\eta))$ of the étale fundamental group of X on the Tate module of the generic fiber of $A \to X$. Given a closed point $x \in X$ with residue field $k(x)$, one can regard $\Gamma_{k(x)}$ as a decomposition group inside $\pi_1(X)$ and then the 'local' representation $\rho_x := \rho|_{\Gamma_{k(x)}} : \Gamma_{k(x)} \to \prod_\ell \mathrm{GL}(T_\ell(A_\eta))$ identifies with the representation $\Gamma_{k(x)} \to \prod_\ell \mathrm{GL}(T_\ell(A_x))$. Let G, $\overline{G}$ and G_x denote the image of ρ, $\rho|_{\pi_1(X_{\overline{k}})}$ and ρ_x respectively and, for every prime ℓ, let G_{ℓ^∞}, $\overline{G}_{\ell^\infty}$ and $G_{\ell^\infty,x}$ denote the projection of G, $\overline{G}$ and G_x on the ℓth factor and G_ℓ, $\overline{G}_\ell$ and $G_{\ell,x}$ their reduction modulo ℓ. Given a prime ℓ, it is known that the set of all closed points $x \in X$ such that $G_{\ell,x}$ is not open in G_ℓ is independent of ℓ (by previous work of A. Tamagawa and the speaker, this set is also known to be reasonably sparse in the sense that if X is a curve then its intersection with the set of points of bounded degree is finite). Let us denote it by X^{ex}. Then,

Theorem: *For every closed point $x \notin X^{ex}$ the group G_x is open in G.*

This statement extends as it is to families of 1-motives.

Fix a closed point $x \notin X^{ex}$. The proof decomposes into three steps:

1. **General properties of motivic representations of étale fundamental groups** Up to replacing X with a connected étale cover, reduce to the case where
   - (ℓ-independency) $G = \prod_\ell G_\ell$, $\overline{G} = \prod_\ell \overline{G}_\ell$, $G_x = \prod_\ell G_{\ell,x}$;
   - $\overline{G}_\ell$ is generated by its ℓ-Sylow for $\ell \gg 0$;
   - $\overline{G}_\ell^{ab} = 0$ and $|Z(\overline{G}_\ell)| \leq B$ (independent of ℓ) for every prime ℓ.

2. **Serre-Nori's approximation theory** Up to replacing X with a connected étale cover one has $\overline{G}_\ell \subset G_{\ell,x}$ for $\ell \gg 0$.

3. **Frattini lifting properties of ℓ-adic Lie-groups** The inclusion $\overline{G}_\ell \subset G_{\ell,x}$ lifts to an inclusion $\overline{G}_{\ell^\infty} \subset G_{\ell^\infty,x}$ for $\ell \gg 0$.

I will try and sketch the proofs of the intermediate steps (1) and (2) and state the main group-theoretical result of (3). If time allows, I will also say a few words about the extension to 1-motives.