Definitions from yesterday's lecture

$$\mathbb{H} := \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\} .$$

- Multiplication of quaternions is defined by the rules

$$i^2 = j^2 = k^2 = -1 ,$$

$$ij = -ji = k, \quad ki = -ik = j, \quad jk = -kj = i .$$

- An *order* in $\mathbb{H}$ is a lattice $\mathcal{O} \subset \mathbb{H}$ such that:
  - $1 \in \mathcal{O}$.
  - For all $z_1, z_2 \in \mathcal{O}$, $z_1 z_2 \in \mathcal{O}$.
- Let Λ be a lattice in $\mathbb{H}$, and let $\mathcal{O}$ be an order in $\mathbb{H}$. We say that Λ is a *left $\mathcal{O}$-ideal* if

$$\{z \in \mathbb{H} | z\Lambda \subseteq \Lambda\} = \mathcal{O} .$$

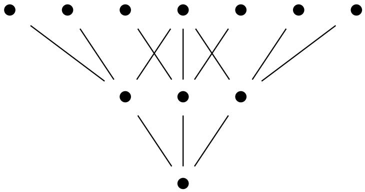
Constructing Ramanujan graphs

- We will describe a procedure that constructs a graph given:
  - An order $\mathcal{O} \subset \mathbb{H}$.
  - A prime p.
- It will turn out that this graph is *usually* Ramanujan.
- We will show how to construct the Ramanujan graph from the first lecture using this procedure.

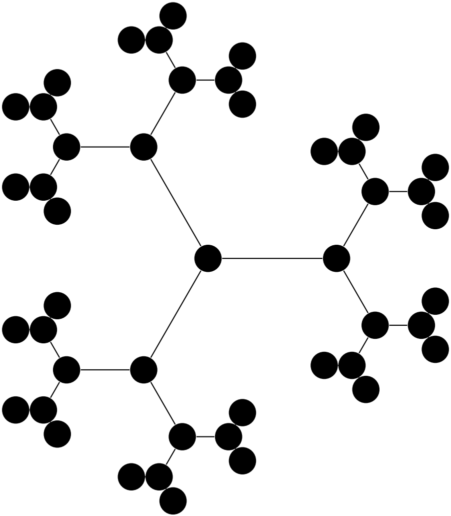
- Let $\mathcal{O} \subset \mathbb{H}$ be the order generated by

$$1, \quad \frac{i - \sqrt{3}k}{2}, \quad i - \sqrt{3}j, \quad \frac{1 + 3i + \sqrt{3}j + \sqrt{3}k}{2},$$

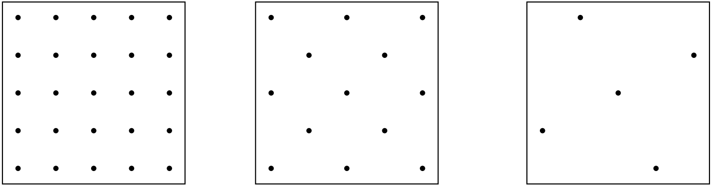
  and let $p = 2$.
- Construct a graph as follows:
  - Each left $\mathcal{O}$-ideal is a vertex of the graph.
  - An edge is drawn between Λ_1 to Λ_2 if $\Lambda_2 \subset \Lambda_1$ and $[\Lambda_1 : \Lambda_2] = p^2 = 2^2 = 4$.
- It turns out that we get the same graph that we had before for 2-dimensional lattices.



If we identify lattices related by scaling, we again get the Bruhat–Tits tree:



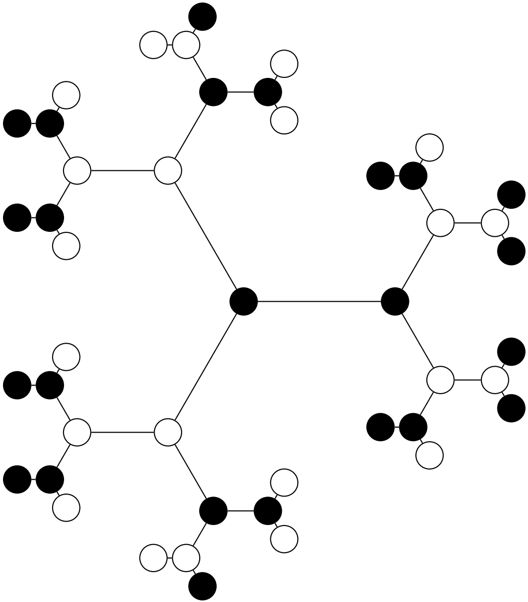
- Why introduce this fancy setup just to get the same tree?
- If Λ is a left $\mathcal{O}$-ideal and $z \in \mathcal{O}$, then Λz is also a left $\mathcal{O}$-ideal.
- The analogous statement is also true for lattices in $\mathbb{C}$: the following lattices are all $\mathbb{Z}[i]$-ideals.



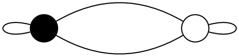
Λ $\quad$ $(1+i)\Lambda = \Lambda(1+i)$ $\quad$ $(2+i)\Lambda = \Lambda(2+i)$

- So we can consider identifying lattices related not just by scaling, but also by right multiplication by elements of $\mathcal{O}$.

- Consider Λ_1, Λ_2 to be equivalent if $\Lambda_1 = \Lambda_2 z$ for some $z \in \mathbb{H}$.
- There are only two equivalence classes in the tree:



- Identifying vertices of the same color yields this graph:



- Adjacency matrix:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

- Eigenvalues:

$$3, \text{ eigenvector } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad -1, \text{ eigenvector } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

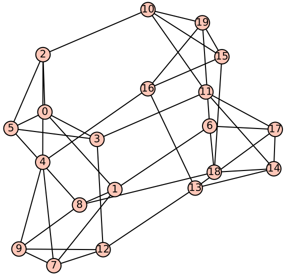
- This graph is Ramanujan since $|-1| < 2\sqrt{3-1} = 2\sqrt{2}$.

- We have a general procedure for constructing a graph given an order $\mathcal{O} \subset \mathbb{H}$ and a prime p:
    - Draw a vertex for each left $\mathcal{O}$-ideal Λ.
    - Draw an edge between vertices Λ_1 and Λ_2 if $\Lambda_2 \subseteq \Lambda_1$ and $[\Lambda_1 : \Lambda_2] = p^2$.
    - Identify vertices Λ_1 and Λ_2 if $\Lambda_1 = \Lambda_2 z$ for some $z \in \mathbb{H}$.
- We call this graph $G_p(\mathcal{O})$ and its adjacency matrix $A_p(\mathcal{O})$.

- The Ramanujan graph from the first lecture is $G_p(\mathcal{O})$, where:
  - $\mathcal{O} \subset \mathbb{H}$ is the order generated by

$$\frac{1+i+7j+5k}{2}, \quad i+7j+5k, \quad 25j+5k, \quad 7k\,,$$

  - $p = 3$.



Some more examples

- Again, we consider the order $\mathcal{O}$ generated by

$$1, \quad \frac{i - \sqrt{3}k}{2}, \quad i - \sqrt{3}j, \quad \frac{1 + 3i + \sqrt{3}j + \sqrt{3}k}{2}.$$

p	2	3	5	7	11	13
$A_p(\mathcal{O})$	$\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}6 & 5\\5 & 6\end{pmatrix}$	$\begin{pmatrix}4 & 4\\4 & 4\end{pmatrix}$	$\begin{pmatrix}4 & 8\\8 & 4\end{pmatrix}$	$\begin{pmatrix}6 & 8\\8 & 6\end{pmatrix}$

- Again, we consider the order $\mathcal{O}$ generated by

$$1, \quad \frac{i-\sqrt{3}k}{2}, \quad i-\sqrt{3}j, \quad \frac{1+3i+\sqrt{3}j+\sqrt{3}k}{2}.$$

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λ_1	3	1	11	8	12	14
λ_2	3	-1	1	0	-4	-2

- We get a $p+1$-regular Ramanujan graph for all primes p *except* 3 and 5.