Ramanujan Graphs, Quaternions, and Number Theory homework - Day 2

Some of these exercises are rather time-consuming/difficult. You don't need to do all of them; just choose the ones that look the most interesting to you.

Lattices

1. In class, we saw that two-dimensional lattices can have twofold, fourfold, or sixfold rotational symmetry. Show that a two-dimensional lattice cannot have n-fold rotational symmetry for $n = 5$ or $n > 6$. (Hint: use proof by contradiction. Choose the point P in the lattice that is closest to the origin. Then use the rotation and translation symmetry of the lattice to construct a point that is closer to the origin.)

2. (a) Find a lattice in n dimensions that has n-fold rotational symmetry.

   (b) Find a lattice in $n-1$ dimensions that has n-fold rotational symmetry. (Hint: does the rotation that you found in part (a) fix a vector? Then it also fixes the subspace perpendicular to that vector.)

3. The previous exercise shows that a four-dimensional lattice can have five-fold rotational symmetry. Read the following article, which explains how four-dimensional lattices with fivefold rotational symmetry can be used to construct Penrose tilings: `http://www.ams.org/publicoutreach/feature-column/fcarc-ribbons`.

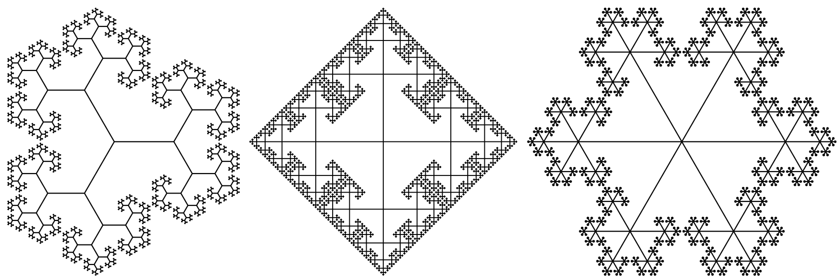
4. Some more questions about symmetries of higher-dimensional lattices:

   (a) Find a lattice in 4 dimensions that has 8-fold rotational symmetry. (Hint: consider your solution to part (a) for $n = 8$. If R is the rotation, look at the eigenspaces of R^4.)

   (b) Find a lattice in 8 dimensions that has 15-fold rotational symmetry. (Hint: consider your solution to part (a) for $n = 15$. If R is the rotation, look at the subspaces fixed by R^3 and R^5 and their orthogonal complements.)

   (c) Let n be a positive integer, and let $\varphi(n)$ be the number of integers between 0 and $n-1$ inclusive that are relatively prime to n. Show that there is a lattice in $\varphi(n)$ dimensions that has n-fold rotational symmetry.

5. Let p be a prime number. In class, we considered a graph whose vertices are two-dimensional lattices up to scaling, such that two vertices are connected by an edge if there are representative lattices Λ_1, Λ_2 such that $\Lambda_2 \subset \Lambda_1$ and $[\Lambda_1 : \Lambda_2] = p$.



We clamimed that this graph is a tree. This exercise will prove the claim.

(a) Show that each vertex has $p+1$ neighbors. This amounts to showing that for any lattice Λ_1, there are $p+1$ lattices $\Lambda_2 \subset \Lambda_1$ satisfying $[\Lambda_1 : \Lambda_2] = p$. (Hint: show that such lattices are in bijection with one-dimensional subspaces of the two-dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space $\Lambda_1/p\Lambda_1$. How many nonzero elements does this vector space have, and how many generators does each one-dimensional subspace have?)

(b) Show that the graph has no loops by the following argument. Let Λ_1 and Λ_2 be lattices in the plane such that for some n, $p^n\Lambda_1 \subseteq \Lambda_2$ and $p^n\Lambda_2 \subset \Lambda_1$. Let A be a matrix sending a basis for Λ_1 to a basis for Λ_2.

Define a function ord_p on the rational numbers by

$$\mathrm{ord}_p\, p^k \frac{m}{n} = k\,,$$

where k is any integer and m and n are any integers not divisible by p, and $\mathrm{ord}_p\, 0 = \infty$. In other words, $\mathrm{ord}_p\, x$ is the number of powers of p dividing x.

Define

$$d(\Lambda_1, \Lambda_2) := \mathrm{ord}_p(\det A) - 2\min(\mathrm{ord}_p(A_{ij}))\,.$$

Show that $d(\Lambda_1, \Lambda_2) \geq 0$, with equality if and only if Λ_1 is a scalar multiple of Λ_2. Show that if $d(\Lambda_1, \Lambda_2) > 0$, then among the $p+1$ lattices Λ_3 satisfying $\Lambda_3 \subset \Lambda_2$ and $[\Lambda_2 : \Lambda_3] = p$, one of them satisfies

$$d(\Lambda_1, \Lambda_3) = d(\Lambda_1, \Lambda_2) - 1\,,$$

and the other p satisfy

$$d(\Lambda_1, \Lambda_3) = d(\Lambda_1, \Lambda_2) + 1\,.$$

Therefore, if we are at the vertex corresponding to Λ_2, there is a unique edge that will take us closer to Λ_1. So there is a unique way to get from the vertex corresponding to Λ_2 to the vertex corresponding to Λ_1 without backtracking.

6. Consider the graph described in Exercise 5, except that we let $[\Lambda_1 : \Lambda_2] = n$, where n is not necessarily prime.

(a) How is the graph for $n = 4$ related to the graph for $n = 2$?

(b) How is the graph for $n = 6$ related to the graphs for $n = 2$ and $n = 3$?

Quaternions

7. Verify that

$$\{a + bi + cj + dk | a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + 1/2\}$$

is an order in $\mathbb{H}$.

8. Verify that the lattice in $\mathbb{H}$ generated by

$$1, \quad \frac{i - \sqrt{3}k}{2}, \quad i - \sqrt{3}j, \quad \frac{1 + 3i + \sqrt{3}j + \sqrt{3}k}{2}$$

is an order.

9. Describe geometrically the following transformations:

(a) $z \mapsto iz$

(b) $z \mapsto zi$

(c) $z \mapsto (i + j)z$