3. Constructing Ramanujan graphs

We will describe a procedure that generates a graph given:

- An order $\mathcal{O}$ in $\mathbb{H}$.
- A prime p.

This graph will *usually* be Ramanujan. We will say later exactly what we mean by "usually".

We will keep in mind the example where $\mathcal{O}$ is the order of Example 2.8, i.e. it is generated by

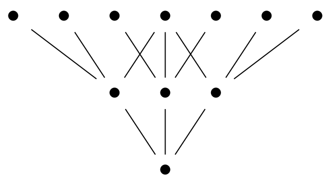
$$1, \quad \frac{i-\sqrt{3}k}{2}, \quad i-\sqrt{3}j, \quad \frac{1+3i+\sqrt{3}j+\sqrt{3}k}{2},$$

and $p = 2$.

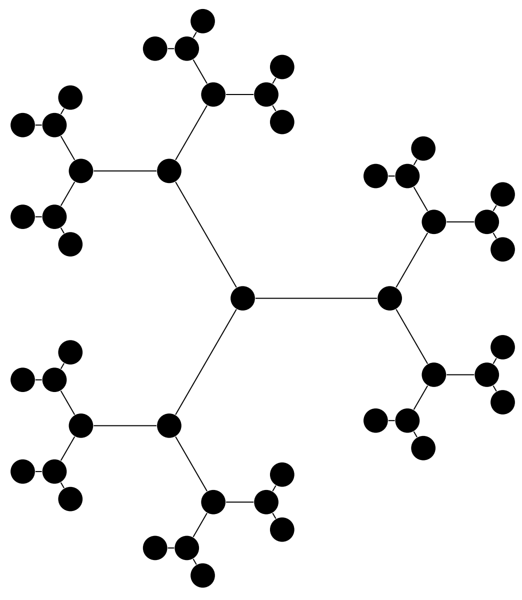
Construct a graph as follows:

- Each left $\mathcal{O}$-ideal is a vertex of the graph.
- An edge is drawn from Λ_1 to Λ_2 if $\Lambda_1 \subset \Lambda_2$ and $[\Lambda_2 \colon \Lambda_1] = p^2 = 2^2 = 4$.

Surprisingly enough, we get the same graph as before:

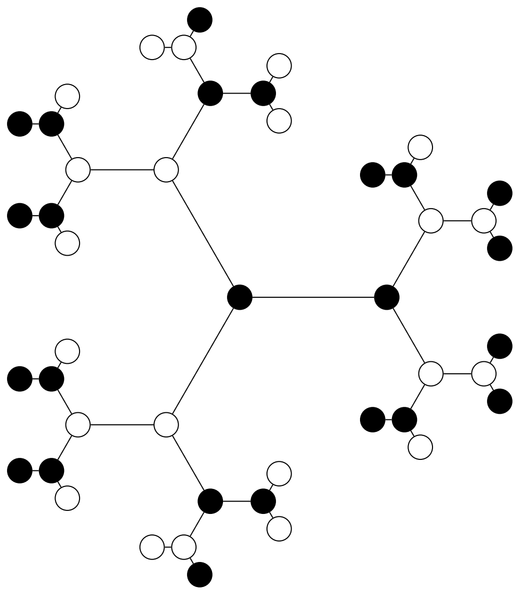


The graph of lattices up to equivalences is again the Bruhat–Tits tree:

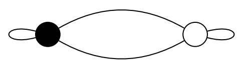


So why consider the four-dimensional lattices if we just get the same tree again? Because we can now consider multiplying a lattice not just by a rational number, but also by a quaternion. If Λ is a left $\mathcal{O}$-ideal and $z \in \mathbb{H}$ is nonzero, then Λz is also a left $\mathcal{O}$-ideal.

If we color the vertices of the Bruhat–Tits tree according to their orbits under the action of right multiplication, it looks like this:



If we identify vertices of the same color, we get the following (multi)graph:



The adjacency matrix is $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Its eigenvectors are $(1,1)$, with eigenvalue 3, and $(1,-1)$, with eigenvalue -1. Since $|-1| < 2\sqrt{3-1} = 2\sqrt{2}$, this graph is Ramanujan.

This example suggests the following general procedure for constructing graphs:

Procedure 3.1.

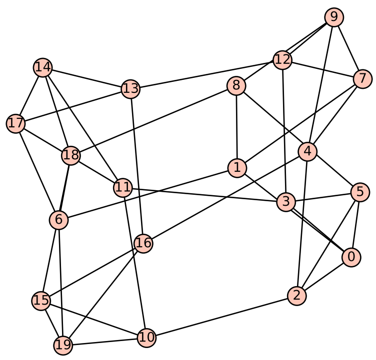
(1) Choose an order $\mathcal{O}$ in $\mathbb{H}$ and a prime p.
(2) Draw a graph whose vertices correspond to left $\mathcal{O}$-ideals, such that the vertices corresponding to Λ_1, Λ_2 are connected by an edge if $\Lambda_2 \subseteq \Lambda_1$ and $[\Lambda_1 : \Lambda_2] = p^2$.
(3) Identify vertices Λ_1, Λ_2 if there exists $z \in \mathbb{H}$ such that $\Lambda_2 = \Lambda_1 z$.

We will denote this graph by $G_p(\mathcal{O})$ and its adjacency matrix by $A_p(\mathcal{O})$. The adjacency matrix is sometimes called a *Brandt matrix.*

The graph from Example 1.11 was constructed by letting $\mathcal{O}$ be

$$\frac{1+i+7j+5k}{2}, \quad i+7j+5k, \quad 25j+5k, \quad 7k\,,$$

and letting $p = 3$.



It turns out that Procedure 3.1 *usually*, but not always, gives us a $p+1$-regular Ramanujan graph. Let us again consider the case where $\mathcal{O}$ is generated by the vectors

$$1, \quad \frac{i-\sqrt{3}k}{2}, \quad i-\sqrt{3}j, \quad \frac{1+3i+\sqrt{3}j+\sqrt{3}k}{2}\,.$$

Here are the matrices $A_p(\mathcal{O})$ for varying p:

p	2	3	5	7	11	13	17
$A_p(\mathcal{O})$	$\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}6 & 5\\ 5 & 6\end{pmatrix}$	$\begin{pmatrix}4 & 4\\ 4 & 4\end{pmatrix}$	$\begin{pmatrix}4 & 8\\ 8 & 4\end{pmatrix}$	$\begin{pmatrix}6 & 8\\ 8 & 6\end{pmatrix}$	$\begin{pmatrix}10 & 8\\ 8 & 10\end{pmatrix}$

All of these matrices have $(1,1)$ as an eigenvector. The eigenvalue is $p+1$ for all primes p *except* 3 and 5. Also note that when $p=3$, the graph is not Ramanujan as $(1,-1)$ is an eigenvector with eigenvalue -1, whereas the Ramanujan bound is $2\sqrt{1-1}=0$.

So what is different about the primes 3 and 5? To answer that question, we will need to introduce some definitions.

Definition 3.2. The *conjugate* of a quaternion is defined by

$$(a+bi+cj+dk)^* := a-bi-cj-dk\,.$$

The *reduced trace* of a quaternion is defined by $\operatorname{tr} z := z+z^*$, i.e.

$$\operatorname{tr}(a+bi+cj+dk) := 2a\,.$$

The *reduced norm* of a quaternion is defined by $N(z) := zz^*$, i.e.

$$N(a+bi+cj+dk) := a^2+b^2+c^2+d^2\,.$$

Lemma 3.3. *Let $z=a+bi+cj+dk\in\mathbb{H}$. Consider the $\mathbb{R}$-linear map $f_z\colon\mathbb{H}\to\mathbb{H}$ defined by*

$$f_z(z') = zz'\,.$$

The map f_z is represented by the matrix

$$\begin{pmatrix} a & -b & -c & -d\\ b & a & -d & c\\ c & d & a & -b\\ d & -c & b & a\end{pmatrix}.$$

The trace of this matrix is $4a = 2\operatorname{tr} z$, and the determinant of this matrix is $(a^2+b^2+c^2+d^2)^2 = N(z)^2$.

Proof. Left as an exercise to the reader. □

Lemma 3.4. *Let $\mathcal{O} \subseteq \mathbb{H}$ be an order. For any $z \in \mathcal{O}$, $N(z) \in \mathbb{Z}$, $\operatorname{tr}(z) \in \mathbb{Z}$, and $z^* \in \mathcal{O}$.*

Proof. Since left multiplication by z preserves the lattice $\mathcal{O}$, we can choose a basis of $\mathbb{H}$ in which the matrix representing z has integer entries. By Lemma 3.3, $N(z)^2$ must be an integer, and $2 \operatorname{tr} z$ must be an integer. Likewise, for any integer m, $m + z \in \mathcal{O}$, so $N(m+z)^2$ must be an integer. We have

$$N(m+z) = (m+z)(m+z)^* = m^2 + mz^* + mz + zz^* = m^2 + m \operatorname{tr} z + N(z)\,.$$

We know that $m^2 + m \operatorname{tr} z$ is a rational number. In order for $N(m+z)^2$ to be an integer, either $N(z)$ must be an integer or $m^2 + m \operatorname{tr} z$ must be zero. The latter cannot hold for all m, so $N(z)$ must be an integer.

Plugging $m = 1$ into the above formula, we find that $1 + \operatorname{tr} z + N(z)$ must also be an integer. So $\operatorname{tr} z$ is an integer.

Since all integers are in $\mathcal{O}$, $z^* = \operatorname{tr} z - z \in \mathcal{O}$. □

Definition 3.5. Let Λ be a lattice in $\mathbb{H}$, generated by z_1, z_2, z_3, z_4. The *discriminant* of Λ, denoted $\Delta(\Lambda)$, is the determinant of the 4×4 matrix with entries $\operatorname{tr}(z_i^* z_j)$.

Example 3.6. Let $\Lambda = \{a + bi + cj + dk | a, b, c, d \in \mathbb{Z}\}$. Then Λ is generated by 1, i, j, k. We find

$$\Delta(\Lambda) = \det \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = -16\,.$$

Lemma 3.7. *For any order $\mathcal{O} \subset \mathbb{H}$, $\Delta(\mathcal{O}) \in \mathbb{Z}$.*

Proof. This follows from Lemma 3.4. □

Definition 3.8. Let $\mathcal{O} \subseteq \mathbb{H}$ be an order, and let p be a prime number. We say that $\mathcal{O}$ is *unramified at p* if p does not divide the discriminant of $\mathcal{O}$.

Theorem 3.9. *Procedure 3.1 produces a $p+1$-regular Ramanujan graph if $\mathcal{O}$ is unramified at p.*

The key idea in the proof that the graph is $p+1$ regular is that $\mathcal{O}/p\mathcal{O}$ is isomorphic to $M_2(\mathbb{Z}/p\mathbb{Z})$, the space of 2×2 matrices with coefficients in $\mathbb{Z}/p\mathbb{Z}$. An outline of the proof will be given in the homework. The proof that the graph is Ramanujan is much harder.

The proof that the graph is Ramanujan uses a lot of (cool) advanced mathematics. I will explain some of the ideas in the final lecture.