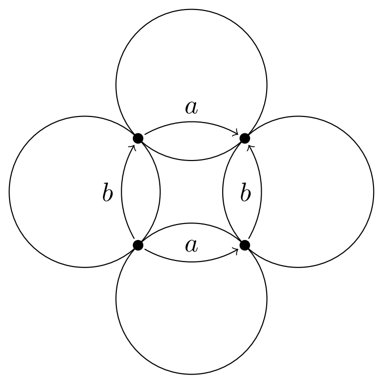
Kleinian Groups and Fractals homework - Day 3

1. In class, we claimed that in order to get a connected limit set, we should take $aba^{-1}b^{-1}$ to be "parabolic", i.e. it should have a single fixed point. Actually, there are a few other ways of getting a connected limit set.

   In class, we assumed that a and b map tangency points of circles to each other in the following way:

   So $aba^{-1}b^{-1}$ fixes the upper right tangent point.

   But there are a few other ways of matching up tangency points. What are they, and which elements of the group fix corners of the tile?

2. In class, we saw a connection between degenerate Schottky groups with $aba^{-1}b^{-1}$ parabolic, and Fuchsian uniformization of a torus with one puncture. In exercise 1, you considered some other degenerate Schottky groups. Explain how these are related to uniformization of a Klein bottle with one puncture, or a projective plane with two punctures. (Hint: A Klein bottle with a puncture and a projective plane with two punctures can both be written as quotients $\mathbb{H}/\Gamma$, where Γ contains both Möbius transformations and "anti-Möbius" transformations of the form $z \mapsto \frac{a\overline{z}+b}{c\overline{z}+d}$. Composing an "anti-Möbius" transformation with the map $z \mapsto 1/\overline{z}$ (which is an inversion about the boundary of $\mathbb{H}$) gives you a Möbius transformation.)

3. Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its *Hermitian conjugate* $A^\dagger$ is defined by

   $$A^\dagger := \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} .$$

   The matrix A is called *Hermitian* if $A = A^\dagger$. Similarly, given a column vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$, its Hermitian conjugate is defined by

   $$v^\dagger := \begin{pmatrix} \overline{x} & \overline{y} \end{pmatrix} .$$

Verify that if A is a 2×2 Hermitian matrix with negative determinant, then

$$\begin{pmatrix} z \\ 1 \end{pmatrix}^\dagger A \begin{pmatrix} z \\ 1 \end{pmatrix} = 0$$

is the equation of a line or a circle. Conversely, verify that every line or circle is defined by an equation of this form. Verify that if the matrix A corresponds to the circle C and B is an arbitrary invertible matrix, then the matrix $(B^{-1})^\dagger A B^{-1}$ corresponds to the circle BC.