1 General Function Stuff

Problem 1.1:

a) The only thing we need to worry about here is square rooting a negative number. For \( f \), that means that we need \( x \geq 0 \). For \( g \), that means we need \(-2x + 4 \geq 0\), which is equivalent to \( x \leq 2 \).

b) The way I think about this is as follows.

First, the “base point” of the square root for \( g \) is going to be at \((2, 1)\). I know this because the domain of \( g \) ends at \( x = 2 \), and the +1 means that I’ll be shifted up by 1. I can also just evaluate \( g(2) \) and see that I get 1.

Next, because there’s a \(-\) in front of the \( x \), I know I’ll be going to the left (normally the square root function grows as you go to the right, but the minus sign flips that). Because there’s a 2 in front of the \( x \), I know I’ll be growing twice as fast as usual, so \( f \) should look flatter than \( g \) when they’re next to each other.

Finally, I can evaluate \( g(1) = 3 \) and \( g(0) = 5 \), so I know I have to go through the points \((1, 3)\) and \((0, 5)\). I picked these points as guides because they have integer coordinates, but any points will do. Because I just sketched \( f \), I know roughly what a square root function looks like, so from there I’ll just draw a curve which looks roughly right.

Problem 1.2:

a) \((f \circ g)(x) = f(g(x))\). In this case, that’s \( f(cx + d) = a(cx + d) + b = acx + (ad + b) \) (which is a line).

b) Here \( f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2 - 1 = x^4 + 2x^2 \).

Problem 1.3: The inverse function \( f^{-1} \) is, by definition, the function that satisfies \( f^{-1}(f(x)) = x \) for every \( x \). If \( f(f(x)) = x \) for every \( x \), then it follows that \( f \) is its own inverse (since it satisfies the definition). This is a reasonably good characterization of these functions.

However we can do better. If a point \((x, y)\) is on the graph of \( f \), then the point \((y, x)\) is on the graph of \( f^{-1} \). For example, if \( f(1) = 2 \), then \( f^{-1}(2) = 1 \), and that means that \((1, 2)\) if on the graph of \( f \) and \((2, 1)\) is on the graph of \( f^{-1} \). The way to go from \((x, y)\) to \((y, x)\) is to reflect across the line \( y = x \). For example, the graphs of \( x^2 \) and \( \sqrt{x} \) are reflections of one another across the line \( y = x \).

So if \( f = f^{-1} \), then that means that when you reflect \( f \) across the line \( y = x \), you end up with the same graph. In other words, \( f \) is symmetric about the line \( y = x \). I think this is the easiest characterization of these
functions.
Examples include $\frac{1}{x}$, $\sqrt{1 - x^2}$, $1 - x$, etc. With this characterization, it’s easy to sketch examples.

2 Inverse Functions

Problem 2.1:

a) Given a value $y$ which we want the function to take on, we’re trying to find the $x$ value which gives that $y$. In other words, we’re trying to solve $y = \frac{4x - 1}{2x + 3}$ for $x$:

\[
(2x + 3)y = 4x - 1 \\
(2y)x + 3y = 4x - 1 \\
(2y - 4)x = -3y - 1 \\
x = \frac{3y + 1}{4 - 2y}
\]

So the inverse function is $f^{-1}(y) = \frac{3y + 1}{4 - 2y}$.

b) For the same reason as above, we’re trying to solve $y = x^2 - x$ for $x$. We can do this using the quadratic formula: $x^2 - x - y = 0$ happens when

\[
x = \frac{1 \pm \sqrt{1 - 4y}}{2}
\]

So our guess would be that $f^{-1}(y) = \frac{1 + \sqrt{1 - 4y}}{2}$. But there is a problem. What is $f^{-1}(0)$, for example? Our guess is multivalued because of the $\pm$. This is because our original function, $x^2 - x$ admits two solutions to $x^2 - x = y$ (it’s a parabola). As discussed, we can’t talk about inverse functions when this happen. The resolution is to restrict the domain of the function so that $f(x) = y$ has only one solution. Choosing $\pm$ here amounts to saying that $x^2 - x$ is to be considered only for $x \geq \frac{1}{2}$ or $x \leq \frac{1}{2}$.

Problem 2.2 $f^{-1}(3)$ is going to be the $x$ value that gives $f(x) = 3$. By inspection $x = 1$ works, so $f^{-1}(3) = 1$.

$f^{-1}(f(x)) = x$ for every $x$, by definition, so $f^{-1}(f(2)) = 2$. Actually finding the inverse function would be very difficult.

Problem 2.3 $\arcsin(x)$ is an angle $\theta$. Specifically, it’s the angle defining the triangle in the unit circle with a vertical leg of length $x$. Then $\cos(\theta)$ is the length of the horizontal leg of this triangle. Since we know this triangle is inscribed in the unit circle, it satisfies $x^2 + \cos(\theta)^2 = 1$, so $\cos(\theta) = \sqrt{1 - x^2}$ (remember that $\arcsin$ is defined to be what you get after you restrict your domain so that you get a positive $\cos$).
3 Exponentials and Logarithms

Problem 3.1

a) \( x = \exp(\log(x)) \) and \( y = \exp(\log(y)) \), from the definition of \( \log \) as an inverse function of \( x \). Then \( xy = \exp(\log(x)) \exp(\log(y)) \). Using the laws of exponents, we can rewrite the right hand side as \( \exp(\log(x) + \log(y)) \). Finally, take the log of both sides to get \( \log(xy) = \log(x) + \log(y) \).

b) \( x = a^{\log_a(x)} \), by the definition of \( \log_a \) as an inverse function. Take the log of both sides to get \( \log(x) = \log(a^{\log_a(x)}) = \log_a(x) \log(a) \). Divide through by \( \log(a) \) to get the desired result.

Problem 3.2

a) Take \( \log_2 \) of both sides to get \( x = \log_2(10^3) \). This can be rewritten using laws of logarithms as \( x = \frac{3 \log(10)}{\log(2)} \).

b) Exponentiate once to get \( \log(x) = e^1 = e \). Exponentiate again to get \( x = e^e \).

c) Take the log of both sides:

\[
\log(e^{ax}) = \log(Ce^{bx}) \\
ax = \log(C) + bx \\
(a-b)x = \log(C) \\
x = \frac{\log(C)}{a-b}
\]