Assignment 1 Solutions

1.1 The human answer is "Yes, f = g." The function in both cases returns what it was given plus its square. The notation "f(x) = x + x^2" is just shorthand for "f(0) = 0, f(1) = 2, f(2) = 6, f(\frac{1}{2}) = \frac{5}{4}" and from this point of view f = g. In general we can say that "f = g if and only if (iff) for every input w, f(w) = g(w)."

The robot answer, however, is that "there is not enough information in the statement of the question to be sure one way or the other." The issue is that the domains of f and g may be different. It's natural to assume that both have domain \( \mathbb{R} \), but it could be that f is only defined for non-negative integers or something. If f and g have different domains they are not equal.

1.2 a) f(1) = 3. Find x = 1 on the x-axis. The curve is at a height of 3 directly above x = 1.

b) f(-1) = \frac{1}{4}, found the same way.

c) x = 0 and x = 3. We'll have f(x) = 1 when the curve is at a height of 1 above the x-axis.

d) x = \frac{-2}{3}, for the same reason.

e) [-2, 4]. The curve is only drawn for [-2, 4]. We couldn't evaluate f(-2.5) or f(3.5), for example.

f) [-1, 3]. The height of the curve is always between -1 and 3 wherever it's drawn. We can't find x such that f(x) = -2, e.g.
1.3. There are many possible answers here. I’ll give the first ones I thought of.

a) \( f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \)

b) ![Graph](image)

c) \( f(x) = x, \text{ Dom } f = \mathbb{Q} \)

or

\( f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \) \text{ (which is essentially the same thing)}

Challenge for bored students: Find \( f(x) \) such that \( \text{Range } f = \mathbb{Q} \) and \( \text{Dom } f = \mathbb{Z} \). (This proves that there are as many rationals as there are integers.)

1.4. There are two things to worry about here:

a) we can’t square-root numbers less than 0
b) we can’t divide by 0.

a) Condition a) means that we need \( 1 - x \geq 0 \), since there’s a \( 1 - x \) under the square-root. This is equivalent to \( x \leq 1 \)

b) Condition b) means that we need \( \sqrt{1 - x} + 1 \neq 0 \) and \( 2x - 3 \neq 0 \). The first of these is automatic, since \( \sqrt{u} \geq 0 \) for all \( u \) (the square-root of something is never negative). The second is only violated when \( x = \frac{3}{2} \). Normally we would remove this \( x \) value from the domain, but we already established that \( x \leq 1 \) from condition a), so we’re fine.

All in all, \( \text{Dom } f = (-\infty, 1] \)
2.1. a) The rock is dropped at $t=0$. Evaluating, $h(0) = 5$, so the rock was dropped from 5m.

b) The rock hits the ground when its height above the ground is 0, so we need to solve $h(t) = 0$. This equation has two solutions: +1 and -1. We disregard $t=-1$ based on the context of the problem, so the unique answer is $t=1$; the rock hits the ground after falling for 1 second.

c) $h(t) = \frac{5}{2}$ means $5 - 5t^2 = \frac{5}{2}$

$$\Rightarrow 1 - t^2 = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} = t^2$$

$$\Rightarrow t = \sqrt{\frac{1}{2}} \text{ (again disregarding the solution with } t < 0)$$

d) We know that the rock hits the ground at $t=1$. That means we want to evaluate $h(1-0.01) = h(0.99)$

$$h(1-0.01) = 5 - 5(1-0.01)^2$$

$$= 5 - 5(1 - 0.02 + 0.0001)$$

$$= 5 - 5(0.98 - 0.0001)$$

$$= 5 - 0.9995$$

$$= 0.0005$$

(or just ask a calculator)
2.2. 
a) 
\[(x-2)(x-3), \text{ by inspection (which is a fine way to do things)}\]

b) 
\[(x-2)(x-3) \text{ is } 0 \text{ iff } x=2 \text{ or } x=3, \text{ so the roots are } 2 \text{ and } 3.\]

c) 
The factor theorem says that \((x-r_1)\) and \((x-r_2)\) are factors of \(p(x)\). Since the degrees and leading coefficients match up, we can conclude that
\[p(x) = (x-r_1)(x-r_2) = (x-\frac{1+\sqrt{13}}{2})(x-\frac{1-\sqrt{13}}{2})\]

d) 
\[r_1 = \frac{1+\sqrt{13}}{2}, \text{ using the quadratic formula}\]

e) 
The factor theorem tells us that \((x-r_1)\) and \((x-r_2)\) are factors. Since \(p(x)\) has degree only 2, and we have two factors, we can guess that \(p(x)\sim (x-r_1)(x-r_2)\). However the leading coefficients don't match this way (\(p(x)\) has "a" as a leading coefficient, while \((x-r_1)(x-r_2)\) has "1" as a leading coefficient). Multiplying a polynomial by a constant doesn't change its roots, so we can just multiply \((x-r_1)(x-r_2)\) so that the leading coefficients match. That means that
\[ax^2+bx+c = a\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right)\]

(f) 
\[r_\pm = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad \text{(quadratic formula)}\]

(g) 
The factor theorem tells us that \((x-r_1)\) and \((x-r_2)\) are factors. Since \(p(x)\) has degree only 2, and we have two factors, we can guess that \(p(x)\sim (x-r_1)(x-r_2)\). However the leading coefficients don't match this way (\(p(x)\) has "a" as a leading coefficient, while \((x-r_1)(x-r_2)\) has "1" as a leading coefficient). Multiplying a polynomial by a constant doesn't change its roots, so we can just multiply \((x-r_1)(x-r_2)\) so that the leading coefficients match. That means that
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(and you can multiply out to check)
I imagine most people just looked this up, but it's entirely possible to figure this out.

Reflecting about the x and y axes give you everything if you can figure out the first quadrant \( \left( \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3} \right) \).

Reflection about the 45° line \( x = y \) shows you that the coordinates for \( \theta = \frac{\pi}{6} \) and \( \theta = \frac{\pi}{3} \) are just swaps of each other.
For $\theta = \frac{\pi}{4}$, we'll have $x = y$. The equation for the unit circle is then $2x^2 = 1$, giving $x = \frac{1}{\sqrt{2}}$ and $y = \frac{1}{\sqrt{2}}$.

For $\theta = \frac{\pi}{2}$, flip the triangle about the x-axis like this:

The triangle is isosceles because we reflected half to make the other half, and the angle at the tip is $2\theta = \frac{\pi}{2}$. Together this means that the triangle is equilateral. One of the sides is a radius of the unit circle, which has length 1. That means that the vertical side has length 1. Since $y$ is exactly half that side, $y = \frac{1}{2}$ and $x = \sqrt{1 - y^2} = \frac{\sqrt{3}}{2}$.

3.2. a) $\sin^2 x + \cos^2 x = 1$, because you're on the unit circle.
   b) $\sin 2x = 2 \sin x \cos x$? It's fine to just memorize.
   c) $\cos 2x = \cos^2 x - \sin^2 x$: these two.