HOMOLOGICAL MIRROR SYMMETRY FOR THETA DIVISORS

WORK IN PROGRESS, JOINT WITH

CATHERINE CANNIZZO, HEATHER LEE, CHIU-CHU MELISSA LIU.

Outline of my talk:

§1 Abelian varieties, theta divisors and mirror symmetry.
§2 Complex moduli
§3 Symplectic moduli
§4 Global HMS of theta divisors.

Some motivation:

$\boxed{\text{MS for hypersurfaces in } (\mathbb{C}^*)^n \quad H \subset (\mathbb{C}^*)^n = M \otimes \mathbb{C}^*}$ $\to$ Abouzaid-Auroux-Katzarkov [AAK] (2012)
$\to$ Abouzaid-Auroux [AA] (2021)

$M \cong \mathbb{Z}^n$

H is the zero set of a Laurent poly. $f \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$.

$f \rightsquigarrow \mathrm{trop}(f) = \phi : \mathbb{R}^n \cong M_{\mathbb{R}} \xrightarrow[PL]{} \mathbb{R}.$

f treated as a fn $(\mathbb{C}^*)^n \longrightarrow \mathbb{C}$

$\rightsquigarrow \Delta$ ($\leftarrow$ moment polyhedron.)

of a symplectic toric CY $(n+1)$-fold

$Y = \mathbb{C}^{n+k} /\!/ (\mathbb{C}^*)^k$

$v : Y \longrightarrow \mathbb{C}$ defined by $v([z_1, \dots, z_{n+k}]) = z_1 \cdots z_{n+1+k}$.

Define

$$X := \mathrm{Bl}_{H\times 0} (\mathbb{C}^*)^n \times \mathbb{C} \xrightarrow{\text{blowdown}} (\mathbb{C}^*)^n \times \mathbb{C} \xrightarrow{pr_2} \mathbb{C}$$

W = super potential on X

$$(X, W) \xleftrightarrow{\text{mirror}} (Y, v)$$

MS of hypersurface closely rel. to MS of ↑

[AAK]	A-model on H	$\longleftrightarrow$	B-model on (Y, v)
[AA]	B-model on H	$\longleftrightarrow$	A-model on (Y, v).

matrix factorizations $MF(X, W) \overset{\text{Orlov}}{\cong} \mathrm{Coh}(H) = \mathrm{Perf}(H) \longrightarrow FS(Y, v)$ Fukaya-Seidel cat. of (Y,v).

H is smooth; Derfect complexes.

(Y, v) is the "generalized" SYZ mirror of H.

§1. Principally polarized abelian varieties

- V = abelian variety of $\dim_{\mathbb{C}} n$. (Topologically a cpt $2n$-D torus)

$\boxed{V = (V_\tau, \omega_{V_\tau})}$ polarization; principally polarized abel. var. determined by τ.

$\boxed{\tau \in \mathcal{H}_n := \{\tau = (\tau_{jk}) \in M_n(\mathbb{C}) \mid \tau_{jk} = \tau_{kj},\ \mathrm{Im}(\tau) > 0\}}$ Siegel upper half sp. of genus n. symmetric; pos. def. imaginary part

$$V_\tau := \mathbb{C}^n / (\mathbb{Z}^n + \tau \mathbb{Z}^n) \cong (\mathbb{C}^*)^n / \tau \mathbb{Z}^n.$$

cpx torus; additive action $z_1, \dots, z_n$; $\exp(2\pi i\,\cdot)$; multiplicative action $x_1, \dots, x_n$

$(x_1, \dots, x_n) = (e^{2\pi i z_1}, \dots, e^{2\pi i z_n})$. coordinates on $(\mathbb{C}^*)^n$; coordinates on $\mathbb{C}^n$.

- $\omega_{V_\tau} = \sum_{i=1}^n \hat{a}_i \wedge \hat{b}^i$; $\hat{a}_i, \hat{b}^i$ are harmonic rep. of dual basis of $H^1(V_\tau; \mathbb{Z})$.

closed pos. def. 2-form (symplectic). lattice gens. $\mathbb{Z}^2 + \tau\mathbb{Z}^2 \rightsquigarrow$ loops in the quot. $\rightsquigarrow$ class/basis in $H_1(V_\tau, \mathbb{Z})$.

In general, polarization is considered to be $\omega_{(\delta_1, \dots, \delta_n)} = \sum_{i=1}^n \delta_i \hat{a}_i \wedge \hat{b}^i$; $\delta_i \mid \delta_{i+1}$.

$\Rightarrow$ "principal" polarization $\Leftrightarrow$ $\omega_{(1, \dots, 1)}$.

- ω_{V_τ} is a positive real $(1,1)$ form: $\frac{i}{2} \sum_{j,k=1}^n \Omega^{jk} dz_j \wedge d\bar{z}_k$.

$\mathrm{Im}(\tau) = \Omega$.

- ω_{V_τ} is a positive real (1,1) form: $\frac{i}{2}\sum_{j,k=1}^{n} \Omega^{jk} dz_j \wedge d\bar{z}_k$. $\Im(\tau) = \Omega$.

- Define a $\boxed{\text{holomorphic line bundle } \mathcal{L}_\tau \text{ over } V_\tau}$ by (deg 1)

$$\mathcal{L}_\tau = ((\mathbb{C}^*)^n \times \mathbb{C}) / \tau\mathbb{Z}^n \longrightarrow V_\tau = (\mathbb{C}^*)^n / \tau\mathbb{Z}^n.$$

ω_{V_τ} = (1,1) real form $\Rightarrow$ ample. $c_1(\mathcal{L}_\tau) = [\omega_{V_\tau}]$.

- The $\boxed{\text{(universal) Riemann-Theta function}}$ is

$$\vartheta : \mathcal{H}_n \times (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$$

for a given $\tau \in \mathcal{H}_n$, $\vartheta_\tau : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$.

$$\vartheta[0,0](\tau, x) = \sum_{n\in\mathbb{Z}^n} e^{\pi i\, n^T \tau n}\, x_1^{n_1}\cdots x_n^{n_n} = \sum_{n\in\mathbb{Z}^n} e^{\pi i\, n^T\tau n + 2\pi i\, n^T z}$$

shift the exponents; $x = \exp(2\pi i\, z)$

$\vartheta[\frac{r_j}{k}, 0](k\tau, x)$ $0 \le r_j \le k$ $\longrightarrow$ basis of hol. sections of $\mathcal{L}_\tau^{\otimes k}$ | satisfy a quasi-periodicity property

- ϑ descends to a section $s_\tau : V_\tau \longrightarrow \mathcal{L}_\tau$

- The zero set of ϑ is called the $\boxed{\text{theta divisor}}$

$$H \equiv \Theta_\tau = Z(\vartheta) = s_\tau^{-1}(0)$$

hypersurface $\subset V_\tau$ — smooth for generic τ.

Mirror of Θ_τ? Following §10.2 [AAK] (suggested by P. Seidel).

- The $\boxed{\text{generalized SYZ mirror to } \Theta_\tau}$ is a $\boxed{\text{Landau-Ginzburg model } (Y, v)}$.

[AAK] A model on Θ_τ ($\underline{\mathrm{Re}\,\tau = 0}$) $\longleftrightarrow$ B-model on (Y, v). * hypersurface in $(\mathbb{C}^*)^n$.

$\tau = \frac{i\log t}{2\pi}\begin{bmatrix} 2 & 1 \\ 1 & 2\end{bmatrix}$ $\longrightarrow$ [Ca 20] + [ACLL] A, Cannizzo, Lee, Lin — B-model on Θ_τ $\longleftrightarrow$ A model on (Y, v)] $n=2$.

t large. any τ "global". Θ_τ $\parallel$ genus 2 curve Σ_2

§2. Complex moduli

We call $(V_\tau, \mathcal{L}_\tau)$ a principally polarized abelian variety (ppav) of dimension 'n'.

$$\mathcal{H}_n = \{\tau = \tau_{jk} \in M_n(\mathbb{C}) \mid \tau_{jk} = \tau_{kj},\ \Im(\tau) > 0\}$$

$\longleftarrow$ sp. of all cpx structures on n-D principally polarized abelian varieties

Siegel upper half space of genus n

$\boxed{\text{moduli of ppav with } \underline{\text{Torelli structure}}}$ — choice of a symplectic basis of $(H_1(V_\tau;\mathbb{Z}), [\omega_{V_\tau}])$: $\{\alpha_i, \beta_i\}_{i=1}^n$.

(since ω_{V_τ} defines a s-form on $\mathbb{Z}^{2n} \cong H_1(V_\tau;\mathbb{Z})$) $\Rightarrow$ $\cong \phi$.

- $\boxed{Sp(2n;\mathbb{Z}) \circlearrowright \mathcal{H}_n \quad \begin{bmatrix} A & C \\ D & E\end{bmatrix}\cdot\tau = (A\tau + C)(D\tau+E)^{-1}}$. $\longleftarrow$ identifies equivalent sympl. basis for ω_{V_τ}.

- $\mathcal{H}_n \longrightarrow \mathcal{A}_n = [\mathcal{H}_n / Sp(2n;\mathbb{Z})]$. $\longleftarrow$ $\boxed{\text{moduli of (complex) ppav.}}$

- We write each $\tau = B + i\Omega \in \mathcal{H}_n$

$n\times n$ real symmetric — $S_n(\mathbb{R})$, $P_n(\mathbb{R})$ — $n\times n$ real +ve definite.

- Turn to "tropical" for SYZ mirror symmetry: [CMV] - Chan-Melo-Viviani (2013) (input: Lag. torus fibration on V_τ).

- Turn to "tropical" for SYZ mirror symmetry: [CMV] - Chan-Melo-Viviani (2013) (input: Lag. torus fibration on V_τ).

$\mathrm{Log} = \log|\cdot| : \underline{(\mathbb{C}^*)^n} \longrightarrow \underline{\mathbb{R}^n}$

$x = (x_1, \dots, x_n) \longmapsto \frac{1}{2\pi}(\log|x_1|, \dots, \log|x_n|)$

$T_F \to V_\tau = (\mathbb{C}^*)^n/\Gamma_\tau \to T_\Im = \mathbb{R}^n/\Im\mathbb{Z}^n$ descends to an $\boxed{\text{SYZ fibration on } V_\tau}$, $\Gamma_\tau = \tau\mathbb{Z}^n$.

$\mathcal{H}_n^{trop,p} \subset \mathcal{H}_n^{trop} \subset \overline{\mathcal{H}_n^{trop}}$ — tropical Siegel sp. [illegible]

"tropical" ppav: $(\mathbb{R}^n/\mathbb{Z}^n, \sum \Im_{jk}\, dr_j\, dr_k) \simeq (\mathbb{R}^n/\Im\mathbb{Z}^n)_\xi$ $\quad (x \longmapsto \Im x = \xi)$

$\Im = (\Im_{jk}) \in P_n(\mathbb{R})$, we define $\boxed{P_n(\mathbb{R}) =: \mathcal{H}_n^{trop,p} \text{ — pure tropical Siegel space}}$

- Mirror to the theta divisor is built using "generalized" SYZ mirror symmetry.
- Let $G := \left\{ \begin{bmatrix} A & C \\ D & E \end{bmatrix} \in Sp(2n;\mathbb{Z}) \;\middle|\; D = 0 \right\} \subset Sp(2n;\mathbb{Z})$ be the subgroup preserving fibre tori (in particular, $\Gamma_F = H_1(T_F;\mathbb{Z}) = \bigoplus_{j=1}^{n} \mathbb{Z}\alpha_j \subset H_1(V_\tau;\mathbb{Z})$).

Fact: 1) G is generated by the following subgroups

$\left\{ \begin{bmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{bmatrix} \;\middle|\; A \text{ is } n\times n \text{ invertible} \right\} \cong Gl_n(\mathbb{Z}) \circlearrowright \mathcal{H}_n$, $\tau \longmapsto A\tau A^T$ (preserves $\mathrm{Im}(\tau) = \Im$)

& $\left\{ \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} \;\middle|\; C \text{ is } n\times n \text{ symmetric} \right\} \cong S_n(\mathbb{Z}) \circlearrowright \mathcal{H}_n$, $\tau \longmapsto \tau + C$

also preserves $\mathrm{Im}(\tau) = \Im$, but shifts $\mathrm{Re}(\tau)$ by C.

↳ If $\begin{bmatrix} A & D \\ 0 & (A^T)^{-1} \end{bmatrix} \in G$ and $\tau \in \mathcal{H}_n$ then

$$\boxed{\mathrm{Im}\left(\underbrace{(A\tau + D)}_{C=0,\ E=(A^T)^{-1}} A^T\right) = A\, \mathrm{Im}\,\tau\, A^T = A \Im A^T}$$

2) $Sp(2n;\mathbb{Z})$ is generated by G and $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \circlearrowright \mathcal{H}_n$, $\tau \longmapsto -\tau^{-1}$

- $\mathcal{H}_n \ni \tau = B + i\Im$, $\quad \mathcal{H}_n = S_n(\mathbb{R}) \times \mathcal{H}_n^{trop,p}$

$$\begin{tikzcd}
\mathcal{H}_n = S_n(\mathbb{R}) \times \mathcal{H}_n^{trop,p} \arrow[d] \arrow[drr, "\mathrm{Im}"] & & \tau \arrow[d, mapsto] \\
{[\mathcal{H}_n/S_n(\mathbb{Z})]} = S_n(\mathbb{R})/S_n(\mathbb{Z}) \times \mathcal{H}_n^{trop,p} \arrow[rr] \arrow[d] & \circlearrowleft & \mathrm{Im}(\tau) = \Im \in \mathcal{H}_n^{trop,p} \arrow[d] \\
\mathcal{A}_n^F := [\mathcal{H}_n/G] \arrow[rr, "\text{sends an SYZ fibred ppav to its base}"] \arrow[d] & \circlearrowleft & \mathcal{A}_n^{trop,p} = [\mathcal{H}_n^{trop,p}/Gl_n(\mathbb{Z})] \\
\mathcal{A}_n = [\mathcal{H}_n/Sp(2n;\mathbb{Z})] & & \mathcal{A}_n^{trop}
\end{tikzcd}$$

covering maps

$\mathcal{A}_n^F$: moduli of principally polarized and SYZ fibred abelian variety

$\mathcal{A}_n^{trop,p}$: (sub)moduli of pure tropical ppav $\quad \cap$

$\mathcal{A}_n^{trop}$ — moduli of tropical ppav

$\boxed{\text{Universal SYZ fibration}}$

$$\begin{tikzcd}
\vartheta_{\mathcal{A}_n^F} \arrow[r, "\pi^{SYZ}"] \arrow[dr] & T_{\mathcal{A}_n^F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & T_{\mathcal{A}_n^{trop,p}} \arrow[d] \\
& \mathcal{A}_n^F \arrow[r] & \mathcal{A}_n^{trop,p}
\end{tikzcd}$$

$T_{\mathcal{A}_n^{trop,p}}$: universal family of pure tropical ppav's.

$\mathcal{A}_n^F = [\mathcal{H}_n/G]$, $\quad \mathcal{A}_n^{trop,p} = [\mathcal{H}_n^{trop,p}/Gl_n(\mathbb{Z})]$

For a given τ:

$$\begin{tikzcd}
V_\tau = (\mathbb{C}^*)^n/\Gamma_\tau \arrow[r, "\pi^{SYZ}"] \arrow[dr] & T_\Im = \mathbb{R}^n/\Im\mathbb{Z}^n \arrow[d] \\
& \tau = \beta + i\Im
\end{tikzcd}$$

universal ... SYZ fi...

$$A_n^F \longrightarrow A_n^{trop,p} \qquad \longrightarrow \tau = \beta + i\mathcal{T}$$

universal family of SYZ fibered ppav. $[\mathcal{H}_n/G]$ $\qquad [\mathcal{H}_n^{trop,p}/Gl_n(\mathbb{Z})]$

§3. Kähler moduli:

Theorem (A, Cannizzo, Lee, Liu):

(i) $\mathcal{H}_n^{trop,p} = P_n(\mathbb{R})$ is the Kähler space of (generalized) SYZ mirror of Θ_τ.

(ii) $[\mathcal{H}_n/S_n(\mathbb{Z})] = S_n(\mathbb{R})/S_n(\mathbb{Z}) \times \mathcal{H}_n^{trop,p}$ is the complexified Kähler space of SYZ mirror of Θ_τ.

(iii) $A_n^{trop,p}$ is the Kähler moduli of SYZ mirror of Θ_τ.

(iv) A_n^F is the complexified Kähler moduli of SYZ mirror of Θ_τ.

(v) For complex abel surfaces $(n=2)$, $\dim(\mathcal{H}_2^{trop,p}) = 3$.

Kähler cones $\longleftrightarrow$ 3-cones in Voronoi decomposition of the definite binari forms. (1908)

Q: How to build the SYZ mirror Y of Θ_τ?

- $\tau = B + i\mathcal{T} \in \mathcal{H}_n$, $B \in S_n(\mathbb{R})$, $\mathcal{T} = P_n(\mathbb{R})$

$$\vartheta(\tau,\cdot): (\mathbb{C}^*)^n \longrightarrow \mathbb{C} \quad \Big| \xrightarrow{\text{tropicalisation}} \phi(\mathcal{T},\cdot): \mathbb{R}^n \longrightarrow \mathbb{R}$$

piece-wise linear

$$\xrightarrow[\text{polyhedron}]{\text{moment}} \Delta_\mathcal{T} = \{(\xi_1,\xi_2,\eta) \in M_\mathbb{R} \times \mathbb{R} \mid \eta \geq \phi(\mathcal{T},\xi)\}$$

Delzante $\Leftrightarrow$ $\mathcal{T}$ is positive definite

$M \otimes \mathbb{R}$

$\Updownarrow$ $L_n(\xi,\eta) \geq 0$

all that lies above the facets $F_n = \{L_n(\xi,\eta)=0\}$

($n=2 \rightsquigarrow$ tiling by hexagons [Ca20], [ACLL])

If $\Delta_\mathcal{T}$ is Delzante we define an infinite fan dual to $\Delta_\mathcal{T}$ using inward normals to each facet $(-n_1,-n_2,1)$

$$\Sigma_\mathcal{T} \subset N_\mathbb{R} \times \mathbb{R} \longrightarrow \tilde{Y} \text{ toric CY } (n+1)\text{-fold (infinite type)}$$

and $\tilde{Y} \xrightarrow{\tilde{v}} \mathbb{C}$ restricts on dense locus $(\mathbb{C}^*)^3$ to $(t_1,t_2,t_3) \longmapsto t_3$.

$\mathbb{Z}^n \underset{\text{holomorphic}}{\circlearrowright} \tilde{Y}$ determined by $\mathbb{Z}^n \circlearrowright \Delta_\mathcal{T}$.

$\mathbb{Z}^n \underset{\text{freely}}{\circlearrowright} \tilde{Y}' = \tilde{v}^{-1}(D)$
$\hookrightarrow$ open unit disc.

$$\boxed{v: \; Y := \tilde{Y}'/\mathbb{Z}^n \longrightarrow D \subseteq \mathbb{C} \quad \Big\| \; Y \text{ is C-Y and } v \text{ is holomorphic.}}$$

$\boxed{\text{Kähler form on on } Y_\tau\text{?}}$ [KL] - Kanazawa-Lau adapt Guillemin's $U(1)^{n+1}$-invariant Kähler form on compact Toric manifolds to toric variety of infinite type.

ω — $U(1)^{n+1}$-invariant and $\mathbb{Z}^n$-invariant Kähler form on $\tilde{Y}'$

$\rightsquigarrow$ ω — Kähler form on Y.

$\Rightarrow$ $v: (Y,\omega) \longrightarrow D$ is a symplectic fibration (since fibers are holomorphic; v is proj to 3rd coord.)

$$\begin{array}{ccccc} \tilde{Y} \supset \tilde{Y}' & \xrightarrow[\text{moment map}]{(\xi,\eta)} & M_\mathbb{R} \times \mathbb{R} & \longrightarrow & M_\mathbb{R} \cong \mathbb{R}^n \\ \downarrow & & & & \downarrow \\ Y = \tilde{Y}'/\mathbb{Z}^n & & \longrightarrow & & T_\mathcal{T} = \mathbb{R}^n/\mathcal{T}\mathbb{Z}^n \end{array}$$

(base of SYZ fib.)

[ACL] S. reduction on some unbounded polyhedra

Now, $B = \mathrm{Re}(\tau)$ determines a $U(1)^{n+1}$-invariant, $\mathbb{Z}^n$-invariant closed 2-form on $\tilde{Y}'$ $\longrightarrow$ B-field B on Y.

angle coordinates

Now, $B = \mathrm{Re}(\tau)$ determines a $U(1)^{n+1}$-invariant, $\mathbb{Z}^n$-invariant closed 2-form on $\tilde{Y}'$ $\longrightarrow$ B-field B on Y.

Complexified Kähler form: $\omega_{\mathbb{C}} = \omega - iB = \omega - i\left[\sum_{j,k=1}^{n} B_{jk}\, dr_j \wedge d\theta_k + d\eta^B \wedge d\theta_\eta\right]$

$\mathrm{Re}(\tau)$; coords that standardise action by lattice $\mathcal{T}\mathbb{Z}^2$ i.e. $\mathcal{T}r = \xi = \log|x|$; angle coordinates ; moment map w.r.t. B (pre-symplectic) s/t $\iota_{\partial/\partial\theta_\eta} B = d\eta^B$

$$\boxed{Y_\tau = \left(Y,\ \omega_{\mathbb{C}} = \omega - iB\right)}$$

Phases and wall-crossing:

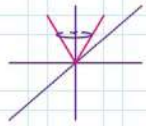
Let $\mathcal{P}_n' \underset{\text{open}}{\subsetneq} \mathcal{P}_n$ then $\mathcal{P}_n \setminus \mathcal{P}_n'$ is a union of hyperplanes $\boxed{\text{walls}}$ and its connected components are called $\boxed{\text{chambers}}$.

$\mathcal{P}_n'$: positive definite symmetric bilinear forms s/t $\Delta_{\mathcal{T}}$'s Delaunte

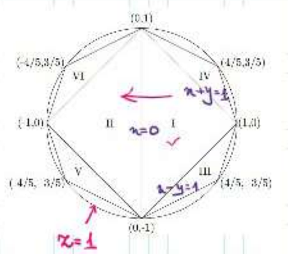
$\mathcal{P}_n$: positive definite symmetric bilinear forms

|| $n=2$ [ACLL]

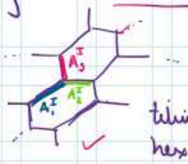
$\left\{\mathcal{T} = \begin{pmatrix} z+y & x \\ x & z+y \end{pmatrix} \,\middle|\, \begin{matrix}\det \mathcal{T} > 0 \\ \mathrm{tr}\, \mathcal{T} > 0\end{matrix}\right\} \iff z > \sqrt{x^2+y^2}$ — a cone.



Cross-section at $z=1$:



$\mathrm{Proj}_{(\xi_1,\xi_2)}(\Delta_{\mathcal{T}}) =$

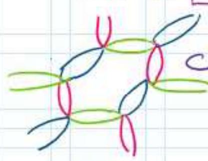


tiling by hexagons.

Phase I:

$\mathcal{T} = \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{pmatrix} = \begin{pmatrix} A_1^I + A_2^I & A_1^I \\ A_1^I & A_1^I + A_3^I \end{pmatrix}$ A_i^I — areas of $\mathbb{P}^1$'s

[CMV]'s tropical period map.

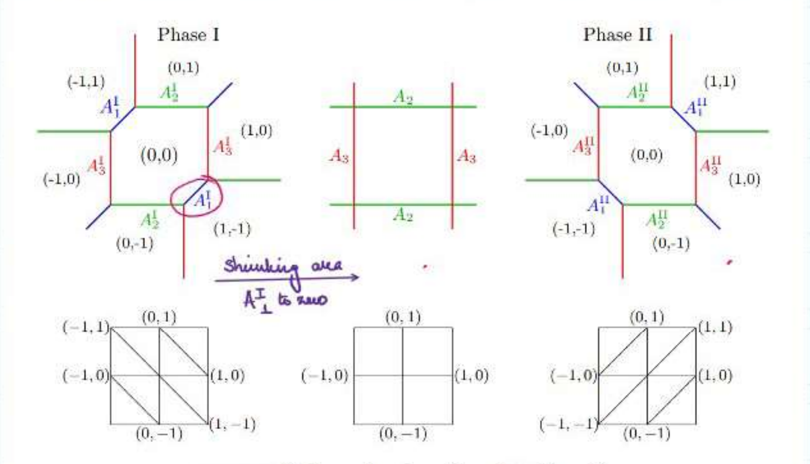


$\subset \tilde{Y}$

$\subset Y$

$\mathbb{P}^1$'s $\cong$ critical locus of v.

Wall crossing from Phase I to Phase II:



Wall-crossing from Phase I to Phase II

§4. Global HMS for theta divisors Θ_τ. (back to $n \geq 2$)

Let $k \in \mathbb{Z}$, $z = a + \tau b$ s/t $[z] \in V_\tau$ for $a, b \in \mathbb{R}^n$.

Theorem [ACLL]:

$$\mathcal{D}^b\mathrm{Coh}(V_\tau) \xrightarrow[\text{restriction}]{i^*} \mathcal{D}^b\mathrm{Coh}(\Theta_\tau)$$

$\mathcal{L}_{k,[z]}$ $\qquad$ $i^*\mathcal{L}_{k,[z]}$

$n=1$ [PZ] Polishchuk-Zaslow $\qquad$ $n=2$ [Ca20] [ACLL]

$$\begin{array}{ccccc} \mathcal{L}_{k,[z]} & D^b Coh(V_\tau) & \xrightarrow[\text{restriction}]{i^*} & D^b Coh(\Theta_\tau) & i^*\mathcal{L}_{k,[z]} \\ \big\downarrow & \Big\downarrow \; SYZ & & \Big\downarrow & \big\downarrow \\ \hat{\ell}_{k,[z]} & H^0 Fuk(V_\tau^\vee) & \xrightarrow[\text{parallel transport along U-shaped curves}]{U} & H^0 FS(Y,v) & \hat{L}_{k,[z]} \end{array}$$

(n=1) [PZ] Polishchuk-Zaslow, Fukaya   (n=2) [Ca20] [ACLL]

$V_\tau^\vee = SYZ(V_\tau)$.

$\boxed{D^b Coh(V_\tau)}$

$V_\tau^+ \cong V_\tau \cong Pic^0(V_\tau)$

$\tau = B + i\Omega \rightsquigarrow \mathcal{L}_\tau$

$[x] \longmapsto \mathbb{L}_{[z]} := T_{[z]}^* \mathcal{L}_\tau \otimes \mathcal{L}_\tau^{-1}$

where $T^*_{[z]} : V_\tau \longrightarrow V_\tau$ $v \mapsto v+[z]$
translation by $[z]$.

$\mathbb{L}_{[0]} = \mathcal{O}_{V_\tau}$

→ $\forall$ generic $\tau \in \mathcal{H}_n$, any line bundle on V_τ is of the form

$\mathcal{L}_{k,[z]} := \mathcal{L}_\tau^{\otimes k} \otimes \mathbb{L}_{[z]}$.

$\boxed{H^0 Fuk(V_\tau^\vee)}$

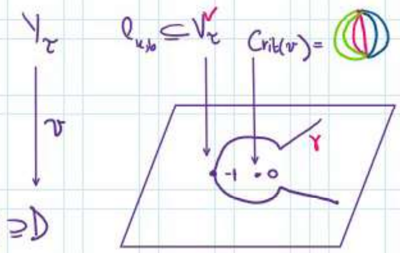
generic fibre $\cong V_\tau^\vee = SYZ(V_\tau)$

$\ell_{k,b} := \{(x_1,x_2,\theta_1,\theta_2) \in \mathbb{R}^4/\mathbb{Z}^4 \mid \theta = \underset{\text{shift}}{b} - \underset{\text{slope}}{k}x\} \subseteq V_\tau^\vee$

$\hat{\ell}_{k,b} := (\ell_{k,b}, \mathcal{E}_a)$ → line bundle on $\ell_{k,b}$ with unitary connection ∇ s/t $F_\nabla = B|_{\ell_{k,b}} = 0$

- inspired from case of elliptic curves [PZ].
- no differential
- there is a product μ^2.

$\boxed{FS(Y,v)}$



Objects are $(L_{k,b}, E_a)$ → trivial line bundle s/t $E_a|_{v_0^{-1}(-\varepsilon)} = \mathcal{E}_a$ with connection having curvature equal to B-field restricted to $L_{k,b}$.

obtained by parallel transporting $\ell_{k,b}$ along γ.

$\boxed{\text{Morphisms on complex side}}$

$Hom_{D^b(V_\tau)}(\mathcal{L}_{k_1,[z_1]}, \mathcal{L}_{k_2,[z_2]}) = H^\bullet(V_\tau, \mathcal{L}_{k_2-k_1,[z_2-z_1]})$

$Hom_{D^b(\Theta_\tau)}(i^*\mathcal{L}_{k_1,[z_1]}, i^*\mathcal{L}_{k_2,[z_2]}) = H^\bullet(\Theta_\tau, i^*\mathcal{L}_{k_2-k_1,[z_2-z_1]})$

computed from the resolution

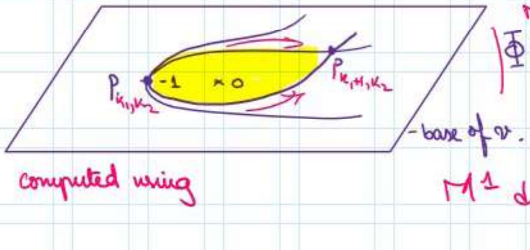
$\mathcal{L}_\tau^{-1} \xrightarrow{\otimes\vartheta} \mathcal{O}_{V_\tau} \xrightarrow{\text{restriction}} \mathcal{O}_{V_\tau}|_{\Theta_\tau} \longrightarrow 0$

ϑ is the defining theta function which is a section of $\mathcal{L}_\tau$.

$\boxed{\text{Morphisms on symplectic side}}$

$Hom_{H^0 Fuk(V_\tau^\vee)}(\hat{\ell}_{k_1,[z_1]}, \hat{\ell}_{k_2,[z_2]}) = HF^\bullet(\hat{\ell}_{k_1,[z_1]}, \hat{\ell}_{k_2,[z_2]}) \overset{(m_1=0)}{=} CF^\bullet(\hat{\ell}_{k_1,[z_1]}, \hat{\ell}_{k_2,[z_2]})$

$Hom_{H^0 FS(Y,v)}(\hat{L}_{k_1,[z_1]}, \hat{L}_{k_2,[z_2]}) = HF^\bullet(\hat{L}_{k_1,[z_1]}, \hat{L}_{k_2,[z_2]})$



computed using

Monodromy: $\Phi(\ell_{k_1,[z_1]}) = \ell_{k_1+1,[z_1]}$. Hamiltonian isotopi.

μ^1 differential in $FS(Y,v)$.

$\boxed{\text{Global HMS:}}$

$$\begin{array}{ccccc} H^\bullet(V_\tau, \mathcal{L}_{k_2-k_1-1,[z_2-z_1]}) & \xrightarrow{\vartheta} & H^\bullet(V_\tau, \mathcal{L}_{k_2-k_1,[z_2-z_1]}) & \longrightarrow & H^\bullet(\Theta_\tau, i^*\mathcal{L}_{k_2-k_1,[z_2-z_1]}) \\ \big\downarrow \cong_1 & \circlearrowright & \big\downarrow \cong_2 & & \big\downarrow \\ CF^\bullet(\hat{\ell}_{k_1+1,[z_1+B\tau]}, \hat{\ell}_{k_2,[z_2]}) & \xrightarrow{\partial = \mu^1} & CF^\bullet(\hat{\ell}_{k_2,[z_2]}, \hat{\ell}_{k_1,[z_1]}) & \longrightarrow & HF^\bullet(\hat{L}_{k,[z_2]}, \hat{L}_{k,[z_1]}) \end{array}$$

$$CF^\bullet(\hat{\ell}_{k_1+1,[z_1+B\tau]}, \hat{\ell}_{k_2,[z_2]}) \xrightarrow{\partial = m^1} CF^\bullet(\hat{\ell}_{k_2,[z_2]}, \hat{\ell}_{k_1,[z_1]}) \xrightarrow{4} HF^\bullet(\hat{L}_{k_2,[z_2]}, \hat{L}_{k_1,[z_1]})$$

- Fukaya [HMS for abelian var., any dimension] $\Rightarrow$ $\uparrow \cong_1$ and $\uparrow \cong_2$ for V_τ.
- First $\square$ commutes by Leibniz rule: $m^1(m^2(\cdot,\cdot)) = m^2(m^1(\cdot),\cdot) + m_2(\cdot, m_1(\cdot))$ ↳ product.

$$\Rightarrow \quad H^\bullet(\Theta_\tau, i^* \mathcal{L}_{k_2-k_1,[z_2-z_1]}) \xrightarrow{\cong} HF^\bullet(\hat{L}_{k_2,[z_2]}, \hat{L}_{k_1,[z_1]}).$$

$$\boxed{D^b\,Coh(\Theta_\tau) \xhookrightarrow{\text{fully faithful}} H^0 FS(Y_\tau, v) \text{ globally!}}$$

consider all powers of $\mathcal{L}$ and $\otimes\, \mathbb{L}_{a_j+\tau b_j}$

$\forall$ generic choice of τ $\rightsquigarrow$ mirror Y_τ.

[ACLL] + [ACLL 2] generalizes [Ca 20] → general τ; → gen. line bundles