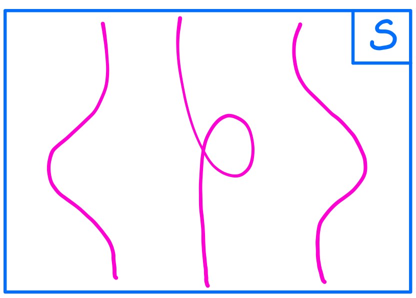
Gromov-Witten Potentials of Local Banana Manifold and Siegel Modular Forms

- Joint with Jim Bryan from ~2019.
- Connects with recent mirror symmetry work of Azam-Cannizzo-Lee-Lin

Outline:

§1. Curve-counting theories for a local banana manifold
§2. Quick survey of Siegel modular forms and Jacobi forms
§3. Main computations
§4. Comments on mirror symmetry

§1. Curve-counting theories for local banana manifold

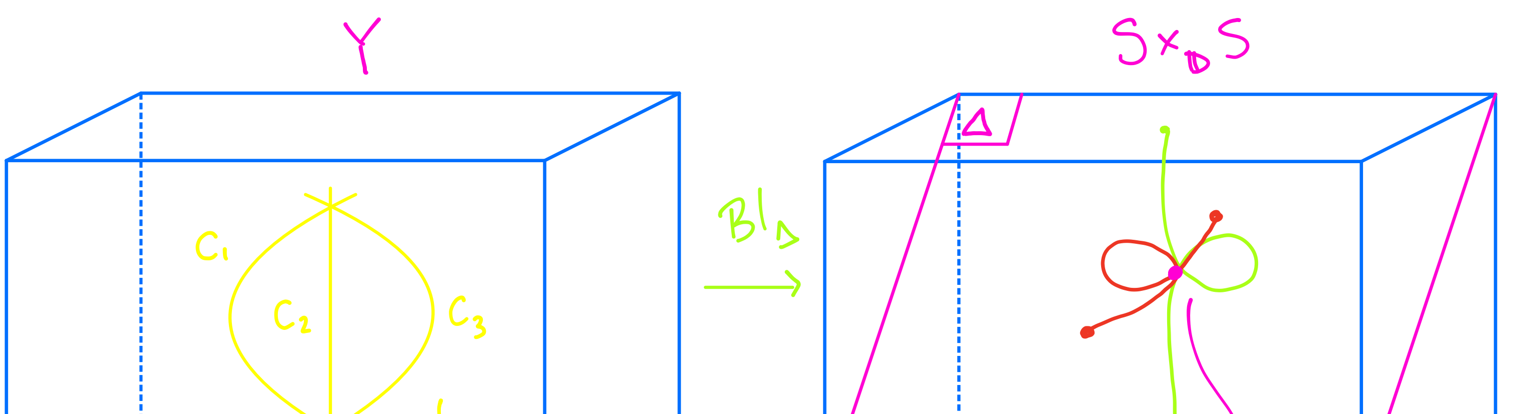


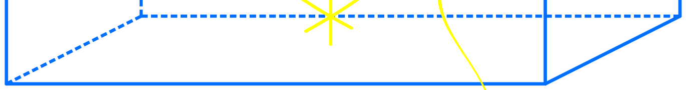
Let $S \to \mathbb{D}$ be an elliptically fibered surface over a disk such that the central fiber is a nodal cubic, and all other fibers are smooth elliptic curves.

For this talk, the local banana manifold

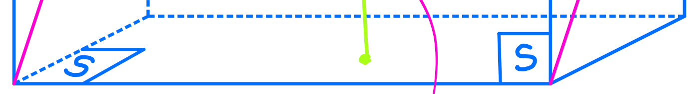
$$Y = \mathrm{Bl}_{\Delta}(S \times_{\mathbb{D}} S)$$

is a small resolution of singularities of $S \times_{\mathbb{D}} S$, blowing up the diagonal Δ.





"Banana configuration" of three rational curves



Conifold singularity $\mathbb{C}[\![x,y,w,z]\!]/(xy-wz)$

So $Y \longrightarrow \mathbb{D}$ is a non-compact Calabi-Yau threefold fibered by Abelian surfaces

- Smooth fibers: $E \times E$, for smooth elliptic curve E.
- The central fiber is a non-normal toric surface:

Normalization of the singular fiber is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at two torus-invariant points.

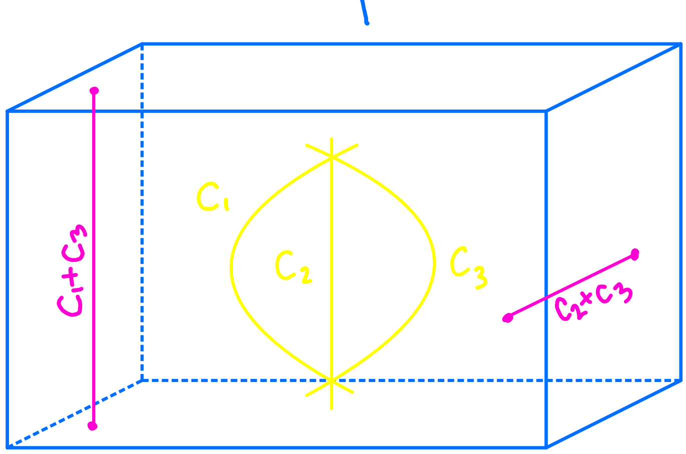
Much of enumerative geometry is concerned with defining curve-counting invariants for Calabi-Yau threefolds. So we want to understand curves in Y:

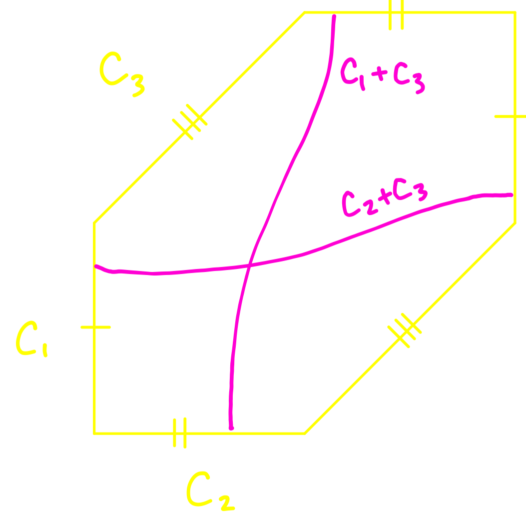
$$\langle C_1, C_2, C_3 \rangle : \text{Lattice of proper curves in } Y$$

For any effective curve class $\beta \in H_2(Y, \mathbb{Z})$, we can write $\beta = d_1 C_1 + d_2 C_2 + d_3 C_3$.

But to see modularity later on, it is convenient to use a different basis:

$$\langle C_1 + C_3,\ C_2 + C_3,\ C_3 \rangle$$





$$\beta_{mn\ell} = m(C_1+C_3) + n(C_2+C_3) + \ell C_3$$

$\hookrightarrow \ell \in \mathbb{Z}$

Introduce formal variables Q, q, y such that the above class corresponds to monomial:

$$Q^m q^n y^\ell$$

Donaldson-Thomas Partition Function:

$$Z_{DT}(Y) = \sum_{m,n,\ell} \sum_{k \in \mathbb{Z}} DT_{\beta_{mn\ell},k}(Y)(-p)^k Q^m q^n y^\ell$$

$$\#_{\text{virtual}} \left\{ \begin{array}{c} \text{1-dimensional subschemes} \\ [\mathcal{O}_Z] \in \mathrm{Hilb}(Y) \end{array} \middle| \begin{array}{c} [\mathrm{Supp}(\mathcal{O}_Z)] = \beta_{mn\ell} \\ \chi(\mathcal{O}_Z) = k \end{array} \right\}$$

Theorem: (Bryan) The Donaldson-Thomas partition function of Y is given by the infinite product:

$$Z_{DT}(Y) = \prod_{m,n,\ell,k} \left(1 - p^k Q^m q^n y^\ell\right)^{-c(4nm-\ell^2,k)}$$

where $c(4nm-\ell^2,k)$ are Fourier coefficients of the equivariant elliptic genus of $\mathbb{C}^2$:

$$\mathrm{Ell}_{q,y}(\mathbb{C}^2;p) = \frac{\theta_1(q,yp)\cdot\theta_1(q,yp^{-1})}{\theta_1(q,p)\cdot\theta_1(q,p^{-1})} = \sum_{n,\ell,k} c(4nm-\ell^2,k)\, q^n y^\ell p^k$$

$$\Theta_1(q,y) = -\sum_{k \in \mathbb{Z}+1/2} q^{\frac{1}{2}k^2}(-y)^k$$

Remark: The above theorem looks analogous to the "Igusa cusp form conjecture" for K3 × E. (Oberdieck, Pandharipande, Pixton, Shen)

$$Z_{DT}(K3\times E) = \frac{1}{\chi_{10}(Q,q,y)} = \frac{1}{Qqy}\prod_{m,n,l}(1-Q^m q^n y^l)^{-c(4nm-l^2)}$$

Igusa cusp form

$c(4nm-l^2)$: Fourier coefficients of $Ell_{q,y}(K3)$.

Gromov-Witten Potentials:

$$F_g(Y) = \sum_{m,n,l} GW_{g,\beta_{mnl}}(Y)\, Q^m q^n y^l$$

genus g GW potential

GW invariants: virtual counts of stable maps into Y.

Theorem: (Bryan, P–) For $g \geq 2$, the genus g Gromov-Witten potential $F_g(Y)$ are meromorphic genus two Siegel modular forms of weight $2g-2$. Moreover

$$E_{2g}(q) = 1 - \frac{4g}{B_{2g}}\sum_{n,d=1}^{\infty} n^{2g-1}q^{nd} \longrightarrow E_{2g}(q)\cdot\phi \xrightarrow{\text{"Maass Lift"}} F_g(Y)$$

Eisenstein series (modular weight $2g$, $g \geq 2$)

So each $F_g(Y)$ is built essentially from E_{2g}

§2. Quick Survey of Siegel Modular Forms and Jacobi Forms:

Define the genus two Siegel upper-half space to be:

$$\mathbb{H}_2 := \left\{ \Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \in M_{2\times 2}(\mathbb{C}) \;\middle|\; \mathrm{Im}(\Omega) > 0 \right\}$$

and the integral symplectic group:

$$Sp_4(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{4\times 4}(\mathbb{Z}) \;\middle|\; \begin{array}{l} ab^T = ba^T \\ cd^T = dc^T \\ ad^T - bc^T = \mathbb{1} \end{array} \right\}$$

Notice that $\mathbb{H}_2$ generalizes the ordinary upper half plane, and $Sp_4(\mathbb{Z})$ generalizes $SL_2(\mathbb{Z})$.

We have an action of $Sp_4(\mathbb{Z})$ on $\mathbb{H}_2$ generalizing that of $SL_2(\mathbb{Z})$ on $\mathbb{H}$:

$$\Omega \longmapsto \gamma(\Omega) = (a\Omega + b)(c\Omega + d)^{-1}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_4(\mathbb{Z})$$

(Genus two) Siegel modular forms are functions on $\mathbb{H}_2$ transforming under $Sp_4(\mathbb{Z})$:

More specifically, a <u>Siegel modular form of weight k</u> is a holomorphic function $f: \mathbb{H}_2 \to \mathbb{C}$ such that:

$$f(\gamma(\Omega)) = \det(c\Omega + d)^k f(\Omega)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_4(\mathbb{Z})$.

<u>Geometric Interpretation:</u>

A_2: moduli space of principally polarized Abelian surfaces

$$A_2 \cong \mathbb{H}_2 / Sp_4(\mathbb{Z})$$

(Principally polarized Abelian surface / isom. $\xrightarrow{\sim}$ Period matrix $\Omega \in \mathbb{H}_2$, up to $Sp_4(\mathbb{Z})$.)

There exists a bundle $\mathbb{E} \longrightarrow A_2$ called the <u>Hodge bundle</u> with fiber $H^0(X, \Omega_X) \cong \mathbb{C}^2$ over $X \in A_2$
Explicitly, we construct $\mathbb{E}$ as follows:

Universal Family:

$$\begin{tikzcd} \mathcal{X}_2 \arrow[d] & = & \mathbb{H}_2 \times \mathbb{C}^2 / Sp_4(\mathbb{Z}) \ltimes \mathbb{Z}^4 \arrow[d] \\ A_2 \arrow[u, bend right, "s"'] & \cong & \mathbb{H}_2 / Sp_4(\mathbb{Z}) \end{tikzcd}$$

$$\boxed{\mathbb{E} = s^* \Omega^1_{\mathcal{X}/A}}$$

s: zero section.

Remark: I'm ignoring that $\mathcal{X}_2 \longrightarrow A_2$ is really an orbifold family, and $\mathbb{E}$ an orbifold bundle. Need to take an honest family $\mathcal{X}_2(n) \longrightarrow A_2(n)$ for principal congruence subgroup $\Gamma(n) \subset Sp_4(\mathbb{Z})$ and quotient by $Sp_4(\mathbb{Z}/n\mathbb{Z})$.

Proposition: A weight k Siegel modular form f can be viewed as a section of line bundle:

$$f \in H^0(A_2, \det(\mathbb{E})^k)$$

(Roughly: think of transformation law $f(\gamma(\Omega)) = \det(c\Omega + d)^k \cdot f(\Omega)$ as a transition function between charts.)

This is part of the much bigger story of automorphic forms on Shimura varieties X/Γ.

Maass Lifting Jacobi Forms to Siegel Modular Forms:

Standard Change of Variables: $\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \rightsquigarrow Q = e^{2\pi i \sigma}, \; q = e^{2\pi i \tau}, \; y = e^{2\pi i z}$

Theorem: (Eichler–Zagier) A Siegel modular form f of weight k has a "Fourier–Jacobi expansion":

$$f(Q, q, y) = \sum_{m=0}^{\infty} \phi_{k,m}(q, y)\, Q^m$$

where $\phi_{k,m}(q,y)$ are Jacobi forms of weight k and index m. defn.

For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\lambda, \mu \in \mathbb{Z}$:

- $\phi_{k,m}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \exp\left(\frac{2\pi i m c z^2}{c\tau + d}\right) \phi_{k,m}(\tau, z)$ — modular transformation

- $\phi_{k,m}(\tau, z+\lambda\tau+\mu) = \exp\left(-2\pi i m(\lambda^2\tau + 2\lambda z)\right)\phi_{k,m}(\tau,z)$ elliptic transformation

Example: The unique weight -2 index 1 (weak) Jacobi form:

$$\phi_{-2,1}(q,y) = (y^{1/2} - y^{-1/2})^2 \prod_{n=1}^{\infty} \frac{(1-yq^n)^2(1-y^{-1}q^n)^2}{(1-q^n)^4}$$

Given Siegel modular form $f(Q,q,y)$, the Jacobi forms $\{\phi_{k,m}\}_{m=0}^{\infty}$ are in general, unrelated.

But using "Hecke operators" V_m we can build a Siegel modular form from an index 1 Jacobi form:

Theorem: (Eichler-Zagier, Aoki) If $\phi_{k,1}$ is an index 1 Jacobi form, then

$$ML(\phi_{k,1}) := \sum_{m=0}^{\infty} V_m(\phi_{k,1})Q^m$$

is a Siegel modular form of weight k called the Maass lift of $\phi_{k,1}$.

Here $V_m : Jac_{k,1} \longrightarrow Jac_{k,m}$ is the Hecke operator, raising the index of a Jacobi form. For $m>0$,

$$V_m(\phi_{k,1}) = m^{k-1} \sum_{\substack{ad=m \\ a>0}} \sum_{b=0}^{d-1} d^{-k} \phi_{k,1}\left(\frac{a\tau+b}{d}, az\right)$$

Fact: Fourier coefficients only depend on the quantity $4n-\ell^2$.

Proposition: If $\phi_{k,1}(q,y) = \sum_{n,\ell} c(4n-\ell^2)q^n y^\ell$ is the Fourier expansion of $\phi_{k,1}$, then:

$$ML(\phi_{k,1}) = -c(0)\frac{B_k}{2k} + \sum_{(m,n,\ell)>0} c(4nm-\ell^2)\sum_{r=1}^{\infty} r^{k-1} Q^{rm} q^{rn} y^{r\ell}$$

where (R. Borcherds)

$$V_0(\phi_{k,1}) := -c(0)\frac{B_k}{2k} + \sum_{(n,\ell)>0} c(-\ell^2)\sum_{r=1}^{\infty} r^{k-1} q^{rn} y^{r\ell}$$

§3. Main Computation:

I want to sketch a proof of the following main theorem:

Theorem: (Bryan, P–) For $g \geq 2$, the genus g Gromov-Witten potential $F_g(Y)$ are meromorphic genus two Siegel modular forms of weight $2g-2$. Moreover

$$F_g(Y) = ML\left(a_{2g} E_{2g}(q) \cdot \phi_{-2,1}(q,y)\right)$$

where $a_{2g} = \frac{(-1)^{g-1}}{(2g-2)!}\frac{B_{2g}}{2g}$ is a constant.

We use the GW/DT Correspondence: Under the change of variables $p = e^{i\lambda}$

$$\log Z'_{DT}(Y) = \sum_{g=0}^{\infty} F'_g(Y)\lambda^{2g-2}$$

reduced DT partition function (divide by power of MacMahon function)

$$F_g(Y) = F_g^{(0)}(Y) + F'_g(Y)$$

degree 0 stable map contributions.

For $g \geq 2$, we have nice formula for degree zero contributions:

$$F_g^{(0)}(Y) = \frac{(-1)^{g-1}}{(2g-2)!}\frac{B_{2g}}{2g}\frac{B_{2g-2}}{2g-2}$$

$$Z'_{DT}(Y) = \prod_{(m,n,l)>0}\prod_{k\in\mathbb{Z}}\left(1-p^kQ^mq^ny^l\right)^{-c(4nm-l^2,k)}$$

$$\Rightarrow \log Z'_{DT}(Y) = \sum_{(m,n,l)>0}\sum_{k\in\mathbb{Z}} -c(4nm-l^2,k)\log\left(1-p^kQ^mq^ny^l\right)$$

$$= \sum_{(m,n,l)>0}\sum_{k\in\mathbb{Z}} c(4nm-l^2,k)\sum_{r=1}^{\infty}\frac{1}{r}p^{rk}Q^{rm}q^{rn}y^{rl}$$

Equivariant Elliptic Genus of $\mathbb{C}^2$:

Under the change of variables $p = e^{i\lambda}$, we have (Zhou):

$$\mathrm{Ell}_{q,y}(\mathbb{C}^2; p) = \sum_{g=0}^{\infty} \lambda^{2g-2} \left(a_{2g} \cdot E_{2g}(q) \cdot \phi_{-2,1}(q,y) \right)$$

Let $C_{2g-2}(4n-\ell)$ be the Fourier coefficients of $\uparrow$ Then we have:

$$\sum_{K \in \mathbb{Z}} C(4nm-\ell^2, K) p^{rK} = \sum_{g=0}^{\infty} C_{2g-2}(4nm-\ell^2)(r\lambda)^{2g-2}$$

$$\log Z'_{DT}(Y) = \sum_{g=0}^{\infty} \lambda^{2g-2} \underbrace{\left(\sum_{(m,n,\ell)>0} C_{2g-2}(4nm-\ell^2) \sum_{r=1}^{\infty} r^{2g-3} Q^{rm} q^{rn} y^{r\ell} \right)}_{F_g'(Y), \text{ by GW/DT Correspondence.}}$$

$$ML\left(a_{2g} E_{2g}(q) \cdot \phi_{-2,1}(q,y)\right) = -C_{2g-2}(0) \cdot \frac{B_{2g-2}}{2(2g-2)} + F_g'(Y)$$

$C_{2g-2}(0) = -2a_{2g}$ (b/c the q^0 coefficient of $\phi_{-2,1}(q,y)$ is $y^{-1} - 2 + y$)

The key is that $a_{2g} \dfrac{B_{2g-2}}{2g-2}$ is precisely the degree zero contributions to $F_g(Y)$. So:

$$ML\left(a_{2g} E_{2g}(q) \phi_{-2,1}(q,y)\right) = F_g(Y).$$

§4. Mirror Symmetry:

General Comments:

- It is expected that mirror symmetry places serious constraints on the

Gromov-Witten potentials $F_g(X)$ of a Calabi-Yau threefold X.

- A mathematician might think of $F_g(X)$ as just a formal series in curve variables $\vec{Q} = (Q_1, \ldots, Q_r)$.

- In physics, you treat $\vec{Q}$ as coordinates on the complexified Kähler cone, and $F_g(X)$ as an expansion "at large volume".

- But via the mirror map, $F_g(X)$ should live (at least locally) on the moduli space $\mathcal{M}$ of complex structures on the mirror $\hat{X}$:

  - Conjecturally, $F_g(X)$ should be a (local) section of a degree $2g-2$ line bundle (vacuum bundle) on $\mathcal{M}$.
    (See e.g. Dijkgraaf's "Mirror Symmetry and Elliptic Curves")

  - Or, $F_g(X)$ inherits monodromy action present on B-model side.

(Abouzaid-Auroux-Katzarkov) There is a sense in which the "mirror" of a banana configuration is a smooth genus two curve. (Really, two mirror Landau-Ginsburg models)

Roughly: Symplectic volumes of three bananas $\longrightarrow$ Three complex periods, $\Omega \in \mathbb{H}_2$.

$$\begin{tikzcd} & \det(\mathbb{E})^{2g-2} \arrow[d] \\ \mathcal{M}_2 \arrow[r, hook] & A_2 \cong \mathbb{H}_2 / Sp_4(\mathbb{Z}) \end{tikzcd}$$

moduli space of smooth genus two curves.

$$C \longmapsto \mathrm{Jac}(C)$$

So $F_g(Y)$ is a section of a degree $2g-2$ line bundle over $\mathcal{M}_2$ - the complex moduli space of the mirror to a banana configuration.

Global Kähler moduli - known in physics as the stringy Kähler moduli space $\mathcal{M}_{Kah}$ - is not well understood in general.

It is conjecturally related to the Bridgeland stability manifold.

But some recent progress has been made for the local banana manifold:

Theorem: (Azam-Cannizzo-Lee-Liu)

$$\mathcal{M}_{Kah} = \mathbb{H}_2 / G$$

where $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_4(\mathbb{Z}) \;\middle|\; c = 0 \right\} \subset Sp_4(\mathbb{Z})$.

The action of G on $\mathbb{H}_2$ is generated by:

- $\Omega \longmapsto \Omega + b$, b: symmetric 2×2 integral matrix
- $\Omega \longmapsto a \Omega a^T$, $a \in GL_2(\mathbb{Z})$

Fact: $G \subset Sp_4(\mathbb{Z})$ is the subgroup under which F_g transforms trivially.

- All Siegel modular forms are invariant under $\Omega \longmapsto \Omega + b$.
  Necessary to have Fourier expansion: $Q = \exp(2\pi i \sigma)$, $q = \exp(2\pi i \tau)$, ...

- $$F_g(a\Omega a^T) = \underbrace{\det(a)^{2g-2}}_{(\pm 1)^{2g-2}} F_g(\Omega) = F_g(\Omega)$$

So pulling back $F_g(\Omega)$ by the covering map $\mathbb{H}_2/G \longrightarrow \mathbb{H}_2/Sp_4(\mathbb{Z})$ defines a function on $\mathbb{H}_2/G$, i.e. a section of the trivial bundle.