Variation of GIT and Variation of Lagrangian skeletons.

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based on   Peng Zhou. Variation of GIT and variation of Lagrangian skeletons I: flip
and flop. *arXiv preprint arXiv:2011.03719*, 2020

Jesse Huang and Peng Zhou. Variation of GIT and variation of Lagrangian
skeletons II: Quasi-symmetric case. *arXiv preprint arXiv:2011.06114*, 2020
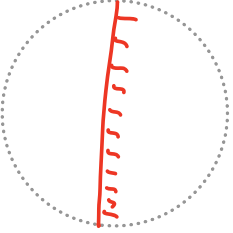
Outline
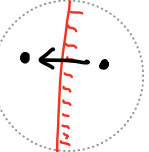
§0. Background on toric mirror symmetry.

§1. Motivations

§2. VGIT and Window subcategories

§3. VLag and Window subskeletons.

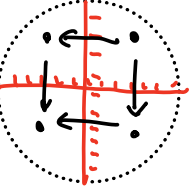# cheatsheet for constructible sheaves on $\mathbb{R}^2$

(1) $\Lambda =$ $= \underset{\text{conormal to the line } \{x_1=0\}}{\overset{\text{half}}{\underline{\qquad\qquad\qquad\qquad}}} +$ zero section

$$\underset{=}{\left\{ (x_1, x_2, \xi_1, \xi_2) \in T^*\mathbb{R}^2 \ \Big| \ \begin{array}{l} x_1=0, \ \xi_2=0 \\ \xi_1 \geqslant 0, \ x_2 \text{ free} \end{array} \right\}}$$

$$\text{sh} \left(\ \bullet \!\leftarrow\! \bullet \ \right) \overset{\sim}{=} \text{Rep} \left(\ \bullet \!\leftarrow\!\!\!\bullet \ \right)$$

(2) $\Lambda =$ $=$ (zero section) $+$ half conormal to $\{x_1=0\}$

$\qquad\qquad\qquad\qquad\qquad\qquad +$ half conormal to $\{x_2=0\}$

$\qquad\qquad\qquad\qquad\qquad\qquad + \underline{\text{quadrant conormal to } \{x_1=0, \ x_2=0\}}$

$\qquad\qquad\qquad\qquad\qquad\qquad = \{(0,0; \ \xi_1 \geqslant 0, \ \xi_2 \geqslant 0)\}$

$$\text{sh}(\Lambda) = \text{Rep} \left(\begin{array}{ccc} B & \leftarrow & A \\ \bullet & & \bullet \\ \downarrow & \boxed{2} & \downarrow \\ \bullet & \leftarrow & \bullet \\ C & & D \end{array} \right)$$

(3) $\Lambda =$ $,$ $\qquad$ $\text{sh}(\Lambda) = \text{Rep} \left(\begin{array}{ccc} \bullet & \leftarrow & \bullet \\ \downarrow & \square & \downarrow \\ \bullet & \leftarrow & \bullet \end{array} \right)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Cartesian diagram.

Toric Coherent Constructible Correspondence (CCC)

- $N \cong \mathbb{Z}^d$, $N_\mathbb{R} = N \underset{\mathbb{Z}}{\otimes} \mathbb{R} \simeq \mathbb{R}^d$, $N_{\mathbb{C}^*} \simeq N \otimes \mathbb{C}^*$, $\mathbb{T} = \mathbb{R}/\mathbb{Z} = S^1$
  $M = \text{Hom}(N, \mathbb{Z})$, $M_\mathbb{T} = M \otimes \mathbb{T} = (S^1)^d$

- $\Sigma \subset N_\mathbb{R}$: smooth toric fan. (cone are simplicial)

- X_Σ : $\qquad$ a smooth toric DM stack.

- $\Lambda_\Sigma \subset T^*(S^1)^d = T^*M_\mathbb{T}$ $\qquad$: FLTZ Lagrangian skeleton.
  $\widetilde{\Lambda}_\Sigma \subset T^*\mathbb{R}^d = T^*M_\mathbb{R}$. $\qquad$ FLTZ equivariant skeleton.

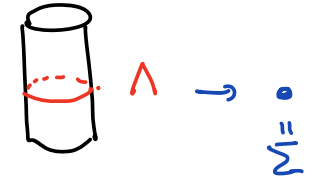## Theorem (Bondal, Fang-Liu-Treumann-Zaslow, ···· , Kuwagaki)

We have equivalence of categories :

non-eq CCC : $\text{Coh}(X_\Sigma) \simeq Sh^\omega(M_\mathbb{T}, \Lambda_\Sigma)$

equivariant CCC : $\text{Coh}_{(\mathbb{C}^*)^d}(X_\Sigma) \simeq Sh(M_\mathbb{R}, \widetilde{\Lambda}_\Sigma)$

CCC examples
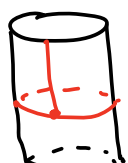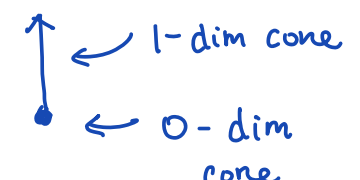
(1) $X_\Sigma = \mathbb{C}^*$, $\Lambda_\Sigma = S^1 \subset T^*S^1$

zero section

$\Lambda \to \bullet$

$\Sigma =$

$$Coh(\mathbb{C}^*) \simeq Sh^w(S^1, \Lambda_\Sigma) \simeq Loc(S^1)$$

$\forall \lambda \in \mathbb{C}^*$; $\mathcal{O}_{\{\lambda\}} \longleftrightarrow$ rank 1 local system with monodromy λ.

$\mathcal{O}_{\mathbb{C}^*} \longleftrightarrow \pi_* \underline{\mathbb{C}}_{\mathbb{R}}$
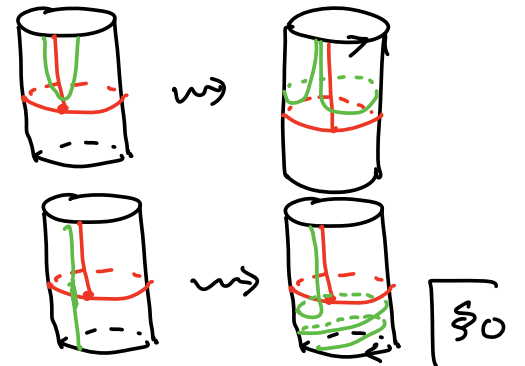
↑ an infinite rank locally constant sheaf.

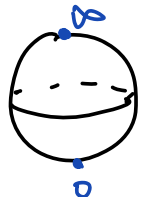$(\pi : \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z})$

$M_T \times N_{\mathbb{R}} \to N_{\mathbb{R}}$

(2). $X_\Sigma = \mathbb{C}$. , $\Lambda_\Sigma =$

$\longrightarrow$ $\Sigma = $

← 1-dim cone

← 0-dim cone

$$Coh(\mathbb{C}) \longleftrightarrow Sh^w(S^1, \Lambda_\Sigma)$$

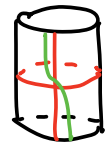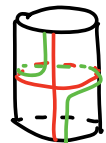$\mathcal{O}_{\{0\}} \longleftrightarrow \pi_* \mathbb{C}_{[-1,0)}[1]$

$\mathcal{O}_{\mathbb{C}} \longleftrightarrow \pi_* \mathbb{C}_{(-\infty,0)}[1]$

§ 0

(3) $X = \mathbb{P}^1$, $\xrightarrow{\mu}$ $= \Delta_X$, moment polytope,

$N_T = S^1$- orbit on $\mathbb{P}^1$.

$\Lambda =$, $\Sigma =$

$\mathcal{O}_{\mathbb{P}^1}(0) \longleftrightarrow \mathbb{C}_{\{0\}}$
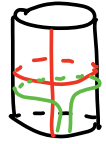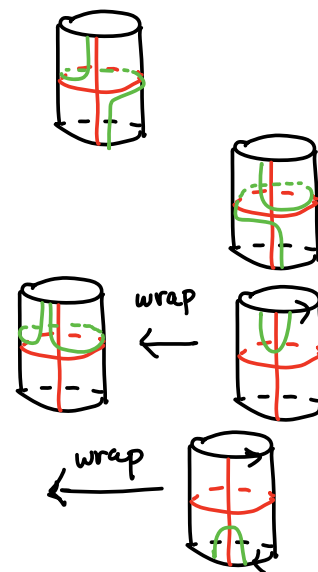
$\mathcal{O}_{\mathbb{P}^1}(1) \longleftrightarrow \pi_* \mathbb{C}_{(0,1)}[1]$

$\mathcal{O}_{\mathbb{P}^1}(-1) \longleftrightarrow \pi_* \mathbb{C}_{[0,1]}$

$\mathcal{O}_{\{0\}} \longleftrightarrow \pi_* \mathbb{C}_{[-1,0)}[1]$

$\mathcal{O}_{\{\infty\}} \longleftrightarrow \pi_* \mathbb{C}_{(-1,0]}[1]$,

wrap

wrap

$\pi: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$

§0

Microlocal Sheaf Theory and Fukaya Category.

- microlocal sheaf theory can be used to compute Fukaya category of a Weinstein mfd (e.g. cotangent bundle).

  Thm (Kontsevich , Nadler, Zaslow, Ganatra-Pardon-Shende).

  (a) Let $(M, \omega = d\lambda)$ be a Weinstein domain, let $\Lambda = skel(M, \lambda)$ be the Liouville skeleton of M. Then

  $$\mu sh^\omega (\Lambda) \simeq Fuk^\omega (M)$$

  (b) If M is a Weinstein domain, $H \subset \partial M$ is a Weinstein hypersurface, $\Lambda = skel(M) \cup \mathbb{R}_{>0} \cdot skel(H)$, then

  (← Liouville flow)

  $$\mu sh^\omega (\Lambda) \simeq Fuk^\omega (M, stop = H)$$

(a) M

Reeb flow on ∂M

$\Lambda = skel(M)$

(b)

H H

Reeb flow on $\partial M \setminus H$.

Λ.

H
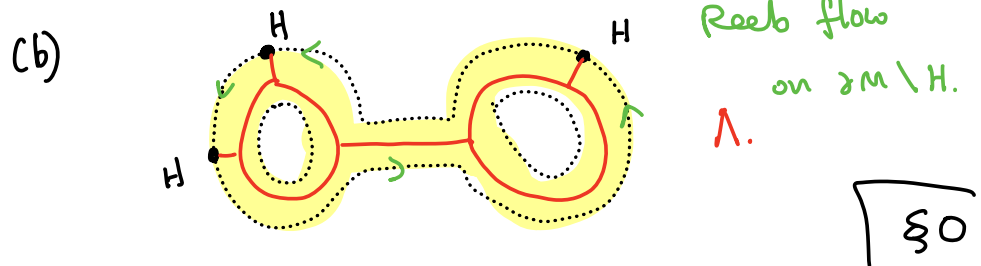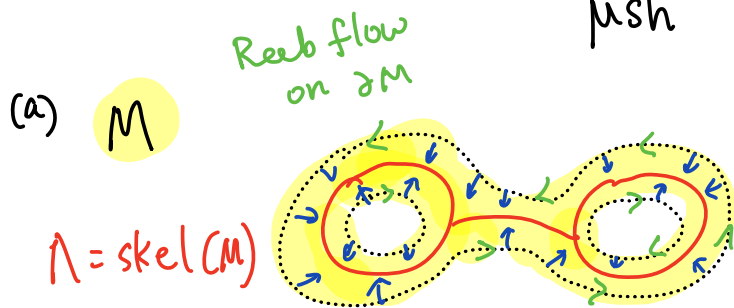
Toric Homological Mirror Symmetry.

- Using previous "microlocal sheaf $\leftrightarrow$ Lagrangian" correspondence. we have

$$\text{Coh}(X_\Sigma) \simeq \text{Sh}^w(M_T, \Lambda_\Sigma) \simeq \text{Fuk}^w(T^*M_T, \text{stop} = \Lambda_\Sigma^\infty)$$

- The traditional toric HMS mirror, uses LG A-model.

$$\text{Coh}(X_\Sigma) \simeq FS\left((\mathbb{C}^*)^d, W_\Sigma\right).$$
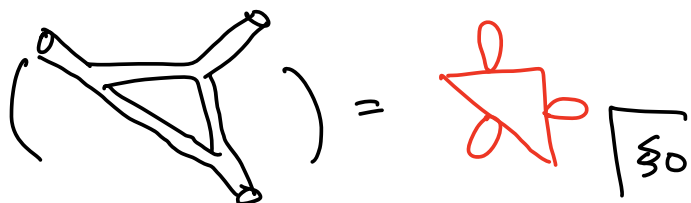
$$\left[\text{e.g.} \quad \text{Coh}(\mathbb{P}^n) \cong FS\left((\mathbb{C}^*)^n, W = z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}\right)\right]$$

- **Thm** (Ruddat-Sibilla-Treuman-Zaslow, Gammage-Shende, Zhou).

Let X_Σ be a smooth toric Fano, $W_\Sigma : (\mathbb{C}^*)^d \to \mathbb{C}$, $\Lambda_\Sigma \subset T^*(S^1)^d$, then

$$\text{"}\Lambda_\Sigma^\infty = \text{skel}(W_\Sigma^{-1}(R))\text{"} \quad \text{for} \quad R \gg 0.$$

- $\underline{\mathbb{P}^2}$. $W = x + y + \frac{1}{xy}$, $\Lambda_\Sigma^\infty = \text{skel}\left(\text{}\right) = \text{}$

$\boxed{\S 0}$

§1 Motivations.

(1.1) • Let $A \subset \mathbb{Z}^d$, $\Delta = \text{Convex Hull}(A) \subset \mathbb{R}^d$,

and let
$$W = \sum_{\alpha \in A} c_\alpha \cdot Z^\alpha$$

be a Laurent polynomial with <u>generic</u> coefficients $c_\alpha \in \mathbb{C}^*$.

Then, $\text{Fuk}^\omega((\mathbb{C}^*)^d, W)$ only depends on Δ.

• One can choose "tropicalization" for W,

$$W = \sum_{\alpha \in A} e^{i\theta_\alpha} \cdot R^{h_\alpha} \cdot Z^\alpha$$

$\theta_\alpha \in \mathbb{R}/2\pi\mathbb{Z}$
$h_\alpha \in \mathbb{R}$
$R \to \infty$

Then different choices results in different

$\Lambda_W^\infty = \text{skel}(W^{-1}(+\infty))$, however

$$\text{Sh}^\omega((S^1)^d, \Lambda_W)$$

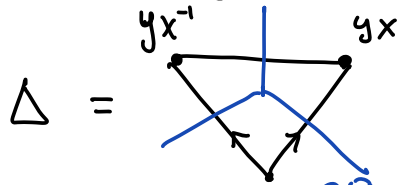$\Lambda_W = (S^1)^d \cup \mathbb{R}_{>0} \cdot \Lambda_W^\infty$

should be invariant. <u>How?</u>

Ex: (a). $\qquad W = Z + \dfrac{e^{i\theta}}{Z}$

$$W^{-1}(R) = \left\{ Z \approx R \right\} \cup \left\{ \dfrac{e^{i\theta}}{Z} \approx R \right\} = \left\{ \right.$$

$\qquad\qquad\qquad\quad \theta_Z = 0 \qquad\qquad\qquad \theta_Z = \theta$

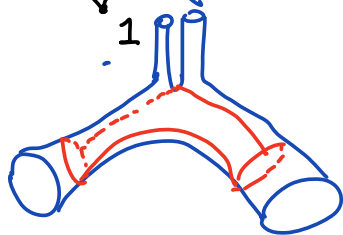$\rightsquigarrow \qquad \Lambda_W =$

$\begin{cases} T^*S^1: & \omega = d\theta \wedge d\mathfrak{z} \\[2mm] \mathbb{C}^*: & \omega = \dfrac{i}{2} \dfrac{dZ \wedge d\bar{Z}}{|Z|^2} = d\rho \wedge d\theta \\ & \qquad\qquad Z = e^{\rho + i\theta} \end{cases}$

$\therefore \quad \mathfrak{z} = -\rho$

$Z = 0$

$-\log|Z| = -\rho$

$\text{arg}(Z)$

$Z = R \qquad Z = \infty$

(b) $\qquad W = y(x^{-1} + x)$

$yx^{-1} \qquad yx$

$\Delta =$

$\Lambda_W =$

$W^{-1}(R) = \quad = \Lambda_W^\infty$

$\cong$

$W = y(x^{-1} + R + x)$

$\Delta =$

$W^{-1}(R) = \quad = \Lambda_W^\infty$

$\Lambda_W =$

(1.2). In the comparison of $FS((\mathbb{C}^*)^n, W)$ and $Sh(T^n, \Lambda)$, one natural question is, where does Lagrangian thimbles go?

Ex: mirror for $\mathbb{P}^1$:

$$W: \mathbb{C}^* \to \mathbb{C}, \qquad W(z) = z + \frac{1}{z}.$$
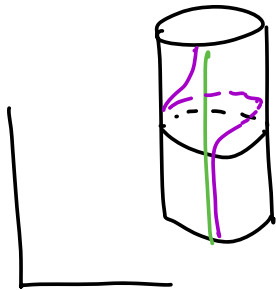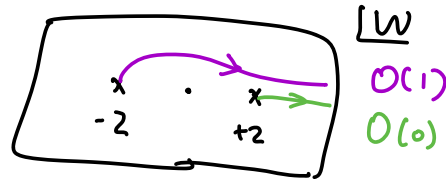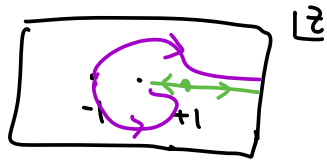
$$L(i) = \pi_* \mathbb{C}_{(0,1)}[1]$$
$$L(0) = \mathbb{C}_{\{0\}}$$

In general, the critical values of W has no pattern, and choice of vanishing path is very ad hoc (straight line paths is complicated)

Semi-orthogonal decomposition (Ballard- Favero- Diemer - Katzarkov- Kerr)
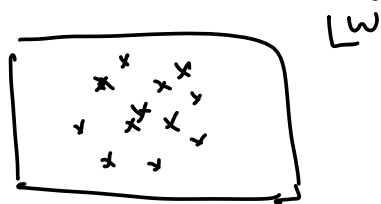
- If we tropicalize W (in a generic way),
then $\mathrm{Crit}(W)$ comes in circle clusters.

$$\mathrm{Fuk}\left((\mathbb{C}^*)^d, W\right) = \left\langle A^{(n)}, \cdots, A^{(2)}, A^{(1)} \right\rangle$$
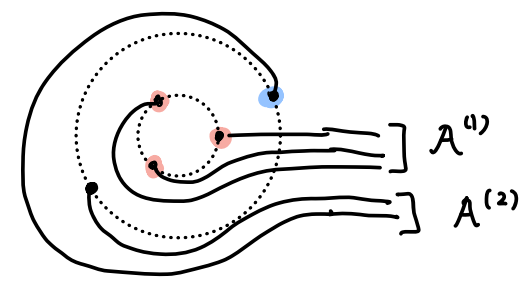
- On the mirror side, we have birational transformation.

$$X = X^{(n)} \to X^{(n-1)} \to \cdots \to X^1$$

$$\mathrm{Coh}(X^{(k)}) = \left\langle B^{(k)}, \mathrm{Coh}(X^{(k-1)}) \right\rangle \qquad \forall k = 1, \cdots, n.$$

$B^{(k)}$ is the coherent sheaves supported on the exceptional
loci in $X^{(k)} \to X^{(k-1)}$.

$\underline{\mathrm{Conj}}$: $\mathrm{Coh}(X^{(k)}) \simeq \left\langle A^{(k)}, \cdots, A^{(1)} \right\rangle$

$\left(\begin{array}{l} \mathrm{Thm}: [\mathrm{Kerr}] \quad \text{associated graded level} \\ \qquad A^{(i)} \cong B^{(i)} \end{array} \right)$

- One trouble that this is still a conj is that Fukaya cat is hard.

(1.2 continue) The bridge between A and B sides is $Sh(T^d, \Lambda)$.

- Let $\Lambda^{(k)} \subset T^* M_T$, such that $Coh(X^{(k)}) \simeq Sh(M_T, \Lambda^{(k)})$.

Then, if we can "measure the difference" between $\Lambda^{(k-1)}$ and $\Lambda^{(k)}$, we can see where the thimbles in SOD component $A^{(k)}$ go.

$(|t| \to 0)$

- Ex:
  - $W = x + y + \frac{1}{xy} + t \cdot xy$

  - $Crit(W) =$
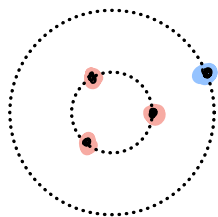
- Triangulation of Newton Polytope of W. Δ_W

- Tropical hypersurfaces: $t^{-k} = x + y + \frac{1}{xy} + t \cdot xy$, $(k \to \infty)$

$X = Bl_0 \mathbb{P}^2$.

- $\phi \to X^{(1)} \longrightarrow X^{(2)}$

  $\phi \to \mathbb{P}^2 \longrightarrow Bl_0 \mathbb{P}^2$

- $\phi \to \Delta_{X^{(1)}} \longrightarrow \Delta_{X^{(2)}}$ moment polytope.

$\phi \to \Lambda^{(1),\infty} \longrightarrow \Lambda^{(2),\infty}$

Goal:

We want to find a family of Lagrangian skeletons $\{\Lambda_t\}_{t \in [0,1]}$, interpolating Λ_0 and Λ_1.
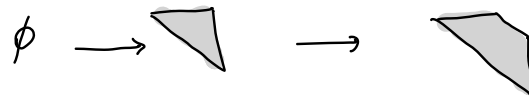
- If $sh(\Lambda_0) \simeq Sh(\Lambda_1)$, we want Λ_t variation to be "non-characteristic", i.e. $Sh(\Lambda_t)$ remain constant

- If $Sh(\Lambda_1) \simeq \langle sh(\Lambda_0), \Upsilon \rangle$ SOD, we want to have critical moments.

Λ_0 Λ_1

$Sh(\Lambda_t)$ constant over (t_i, t_{i+1})

$$0 \quad t_1 \quad t_2 \quad \cdots \quad t_n \quad 1$$

$$\underset{\parallel}{0} \qquad\qquad\qquad\qquad\qquad \underset{\parallel}{1}$$

$$t_0 \qquad\qquad\qquad\qquad\qquad t_{n+1}$$

<u>Ex</u>:   (1)

Λ_- Λ_t Λ_+
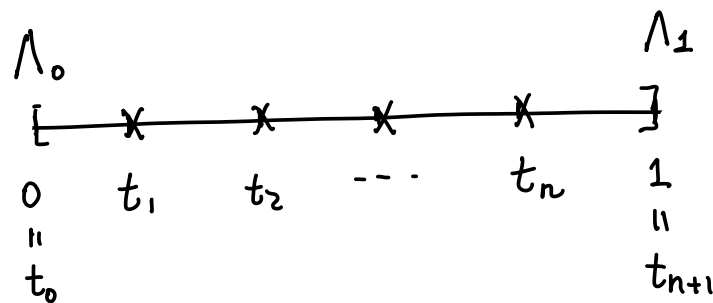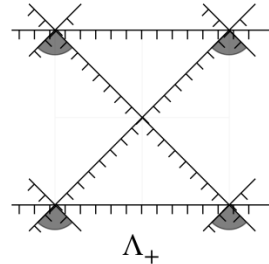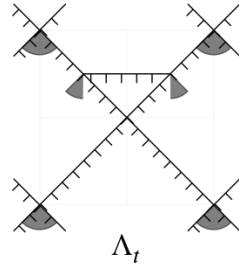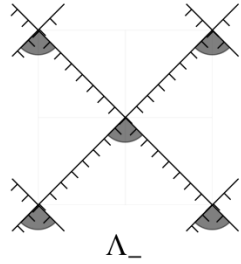
$\begin{pmatrix} \text{non-charateristic} \\ \text{variation} \end{pmatrix}$

$Coh([\mathbb{C}^2/\mathbb{Z}_2])$ $\simeq$ $Coh(T^*\mathbb{P}^1)$

(2) X:

$\mathbb{C}^2$ $Bl_0\,\mathbb{C}^2$

O_E

Λ:

O_E
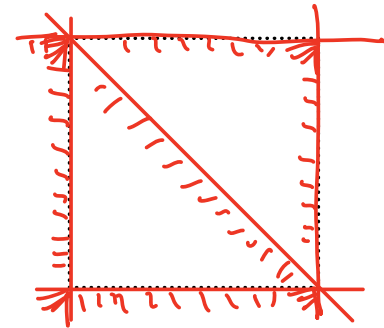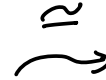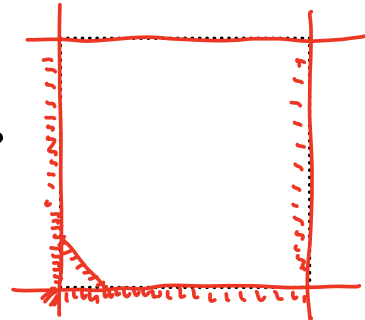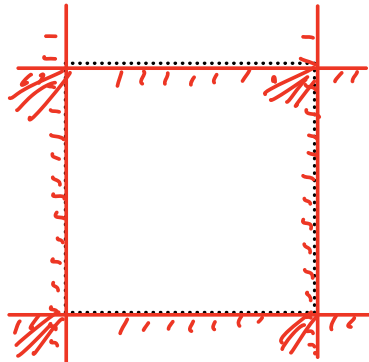
$Coh(\mathbb{C}^2) \hookrightarrow Coh(Bl_0\mathbb{C}^2)$
$\quad\text{is}\qquad\qquad\qquad\text{is}$
$Sh^\omega(\Lambda_0)\qquad\quad Sh^\omega(\Lambda_1)$

§2 — <u>Variation of GIT and Window subcategory.</u>

(Herbst - Hori - Page, E. Segal, Halpern-Leistner, Ballard-Favero-Katzarkov)

§2.1

- <u>Idea:</u>

  - When we study transitions between toric varieties X_- and X_+ (e.g. $X_- = \mathbb{C}^2$, $X_+ = Bl_0\,\mathbb{C}^2$), they often come from different "phases" of GIT quotients

$$X_\pm = [\tilde{X} /\!\!/_{\theta_\pm} \mathbb{C}^*] = [\tilde{X} - \tilde{X}^{us}_{\theta_\pm} / \mathbb{C}^*] \xrightarrow{\;L_\pm\;} [\tilde{X} / \mathbb{C}^*]$$

← unstable loci

  .

$$Coh([\tilde{X}/\mathbb{C}^*]) \simeq Coh_{\mathbb{C}^*}(\tilde{X})$$

$$\swarrow L_+^* \qquad\qquad \searrow L_-^*$$

$$Coh(X_+) \qquad\qquad\qquad Coh(X_-)$$

$$Coh(X_\pm) = Coh([\tilde{X}/\mathbb{C}^*]) \Big/ \Big\langle \text{sheaves supported on} \Big\rangle$$

unstable loci $\tilde{X}^{us}_\pm$

- **Def**: A <u>window subcategory</u> for a GIT quotient $X_\pm$

  is a subcategory $W_\pm \subset \mathrm{Coh}([\tilde{X}/\mathbb{C}^*])$, such that

  $$L_i^*|_{W_i} : W_i \xrightarrow{\sim} \mathrm{Coh}(X_i) \qquad i = +,-$$

  is an equivalence.

<u>Rmk</u>:

- We can compare $\mathrm{Coh}(X_\pm)$ via comparing $W_\pm$ in $\mathrm{Coh}([\tilde{X}/\mathbb{C}^*])$ now.

- Choices of $W_\pm$ are far from unique.

- Window subcategories exist for general GIT quotients by algebraic group $[\tilde{X}/\!\!/_\theta G] \hookrightarrow [\tilde{X}/G]$. $\quad$ [BFK, HL]

Example :

(1) $\mathbb{C}^3 / \mathbb{C}^*$, $\mathbb{C}^* \curvearrowright \mathbb{C}^3$ with weight $(1,1,-1)$.

$$X_+ = \left[\mathbb{C}^3 - \underbrace{\{ (0,0,z) \mid z \in \mathbb{C} \}}_{\color{red}{Z_+}} \middle/ \mathbb{C}^* \right] = Bl_0 \, \mathbb{C}^2$$

$$= Tot \left[\mathcal{O}_{\mathbb{P}^1}(-1) \right]$$

$$X_- = \left[\mathbb{C}^3 - \underbrace{\{ (z_1, z_2, 0) \mid z_i \in \mathbb{C} \}}_{\color{red}{Z_-}} \middle/ \mathbb{C}^* \right] \simeq \mathbb{C}^2$$

$$(\mathbb{C}^*)^3 \curvearrowright \mathbb{C}^3 \xrightarrow{\mu} \mathbb{R}^3 = \left[Lie(\mathbb{C}^*)^3 \right]^\vee$$

$$(z_1, z_2, z_3) \longmapsto (|z_1|^2, |z_2|^2, |z_3|^2)$$
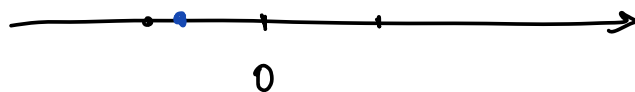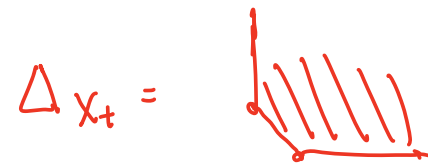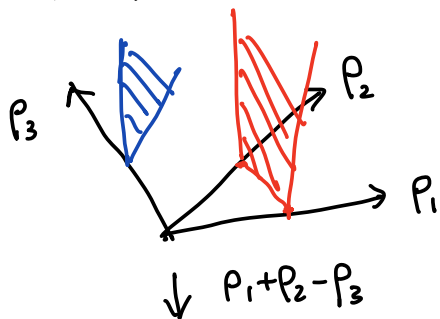
$$\mu(\mathbb{C}^3) = (\mathbb{R}_{\geq 0})^3$$

$$\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^3$$
$$t \longmapsto (t, t, t^{-1}).$$

$$\rightsquigarrow \qquad \mathbb{R}^3 \xrightarrow{\mu'} \mathbb{R}$$
$$(\rho_1, \rho_2, \rho_3) \longmapsto \rho_1 + \rho_2 - \rho_3$$

$$\Delta_{X_-} =$$

$$\Delta_{X_+} =$$

$$\downarrow \ \rho_1 + \rho_2 - \rho_3$$

$$0$$

- coherent sheaves on $Z_+ = \{(0,0,z)\}$ is generated by $\mathcal{O}_{Z_+}$.

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}^3}\{-2\} \xrightarrow{(-z_2, z_1)} \mathcal{O}^2_{\mathbb{C}^3}\{-1\} \xrightarrow{(z_1, z_2)} \mathcal{O}^3_{\mathbb{C}^3}\{0\} \longrightarrow \mathcal{O}_{Z_+}^{\{0\}} \longrightarrow 0 \qquad \left(\begin{matrix} \text{Koszul} \\ \text{resolution} \end{matrix} \right)$$

$\{k\}$ means $\mathbb{C}^*$ equivariant degree.

Thus, when restricted to X_+, $\mathcal{O}_{Z_+}$ become 0, hence we have exact seq.

$$0 \longrightarrow \mathcal{O}\{-2\} \longrightarrow \mathcal{O}\{-1\}^{\oplus 2} \longrightarrow \mathcal{O}\{0\} \longrightarrow 0$$

$\Rightarrow$ $\begin{cases} \bullet \ \mathcal{O}\{k\} \text{ can be expressed using } \mathcal{O}\{k-1\} \text{ and } \mathcal{O}\{k-2\} \\ \bullet \ \mathcal{O}\{k-2\} \text{ can } \underline{\hspace{4cm}} \mathcal{O}\{k-1\} \text{ and } \mathcal{O}\{k\}. \end{cases}$

$\Rightarrow$ $\text{Coh}\left(\mathbb{C}^3 - Z_+ / \mathbb{C}^* \right)$ can be generated by
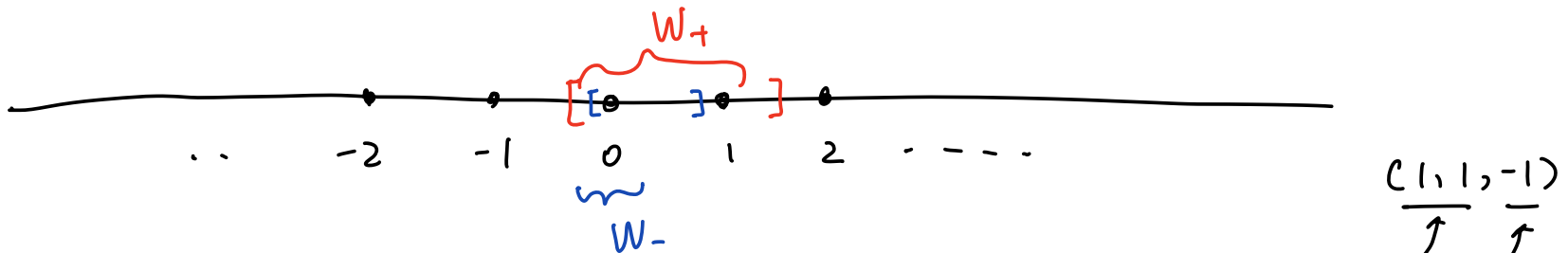
$$\iota_+^* \left\langle \mathcal{O}\{k\}, \mathcal{O}\{k+1\} \right\rangle$$

..... $\Rightarrow$ $\forall k \in \mathbb{Z}$, we can choose $W_+ = \left\langle \mathcal{O}\{k\}, \mathcal{O}\{k+1\} \right\rangle$

still need to prove $\iota_+^* \big|_{W_+}$ is fully-faithful

- coherent sheaves on $Z_- = \{(z_1, z_2, 0)\}$ is generated by O_{Z_-}

$$0 \longrightarrow O_{\mathbb{C}^3}\{+1\} \xrightarrow{z_3} O_{\mathbb{C}^3}\{0\} \longrightarrow O_{Z_-}\{0\} \longrightarrow 0$$

same arguments shows $\forall k \in \mathbb{Z}$, we can choose
$$W_- = \langle O\{k\}\rangle \subset \mathrm{Coh}([\mathbb{C}^3/\mathbb{C}^*]).$$

W_+ : as lattice points in an interval of length $d_+ = 2$

W_- : _____ $d_- = 1$

$(1, 1, -1)$

- SOD : $\cdot$ $\cdot \mathrm{Coh}(\mathrm{Bl}_0 \mathbb{C}^2) = \langle O_E(-1), \pi^* \mathrm{Coh}\,\mathbb{C}^2\rangle$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{W_+} \qquad\qquad\qquad\qquad \overset{\|}{W_-}$

$\iota : \quad \mathrm{Coh}(\mathbb{C}^2) \xrightarrow{\sim} W_- \overset{\{0\}}{\hookrightarrow} W_+ \overset{\{-1,0\}}{\simeq} \mathrm{Coh}(\mathrm{Bl}_0 \mathbb{C}^2).$

§2.3   Magic windows   (HL-Sam ,   Spenko- van den Bergh)

How about   general $[\mathbb{C}^N /\!\!/_\theta (\mathbb{C}^*)^k]$?

- any sm proj toric variety arises from Cox construct (GIT quotient)

- If $(\mathbb{C}^*)^k \curvearrowright \mathbb{C}^N$ preserves $dz_1 \wedge \cdots \wedge dz_N$, then different smooth quotients are all toric CY, and derived equivalent.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (in a non-canonical way).

Q: Can we find a universal window
$$W = \langle \text{ some line} \rangle \subset \text{Coh}[\mathbb{C}^N /\!\!/ (\mathbb{C}^*)^k]$$
$\qquad\qquad\quad$ bundles

such that ,
$$\iota_\theta^* : W \xrightarrow{\sim} \text{Coh}[\mathbb{C}^N /\!\!/_\theta (\mathbb{C}^*)^k]$$

for all GIT param θ in stable chambers ?

$\qquad\qquad\qquad\qquad$ (in general , I cannot)

- One can achieve this under stronger assumption than toric CY,

Def: (quasi-symmetric condition)

Let $\beta_1, \cdots, \beta_N \in \mathbb{Z}^k$ denote the collection of weights for $(\mathbb{C}^*)^k \curvearrowright \mathbb{C}^N$. If for any line (passing through 0) $L \subset \mathbb{R}^k$, the sum of weights on L is zero, then we say $(\mathbb{C}^*)^k \curvearrowright \mathbb{C}^N$ is quasi-symmetric.

Ex: • $k = 1$, toric CY $\iff$ quasi-symmetric.

• If $\{\beta_1, \cdots, \beta_N\}$ is invariant under $(-1)\cdot : \mathbb{Z}^k \to \mathbb{Z}^k$, i.e. $\underline{symmetric}$ under inversion, then it is quasi-symm.
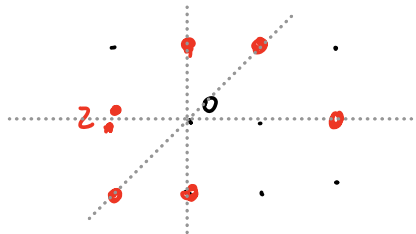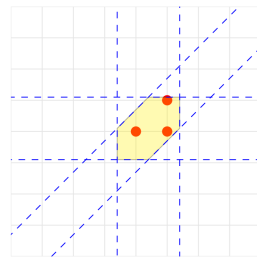
- Let $\Delta = \frac{1}{2} \sum_{i=1}^{N} [0, \beta_i]$ Minkowski sum of line segments.

- for any generic $\eta \in \mathbb{R}^k$, we have lattice points.

$$A_\eta = (\eta + \Delta) \cap \mathbb{Z}^k$$

and corresponding windows

$$W_\eta = \left\langle \bigoplus_{\alpha \in A_\eta} \mathcal{O}\{\alpha\} \right\rangle \subset \text{Coh}\left([\mathbb{C}^N /\!\!/ (\mathbb{C}^*)^k]\right)$$

$$\| $$

$$\text{Coh}_{(\mathbb{C}^*)^k}(\mathbb{C}^N)$$

<u>Thm</u> (HL-Sam, SvdB)

For any stable GIT parameter $\Theta \in \mathbb{R}^k$, any $\overset{\text{generic}}{\text{window}}$ param η, we have equivalence
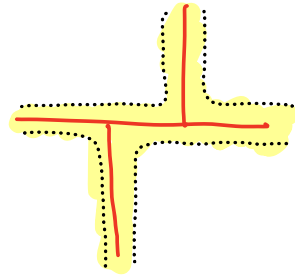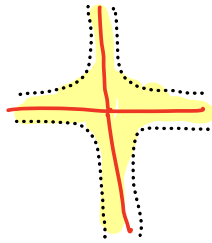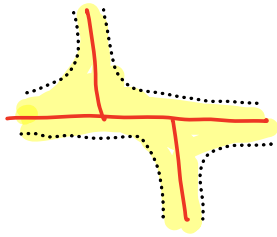
$$\iota_\Theta^* : W_\eta \xrightarrow{\sim} \text{Coh}\left([\mathbb{C}^N /\!\!/_\Theta (\mathbb{C}^*)^k]\right)$$

§3   Variation of (VLag) Lagrangian Skeleton and windows

§3.1 General VLag :

(1) Some variations of Lagrangians induces equivalences of categories.

EX: $\Lambda \subset T^*\mathbb{R}$
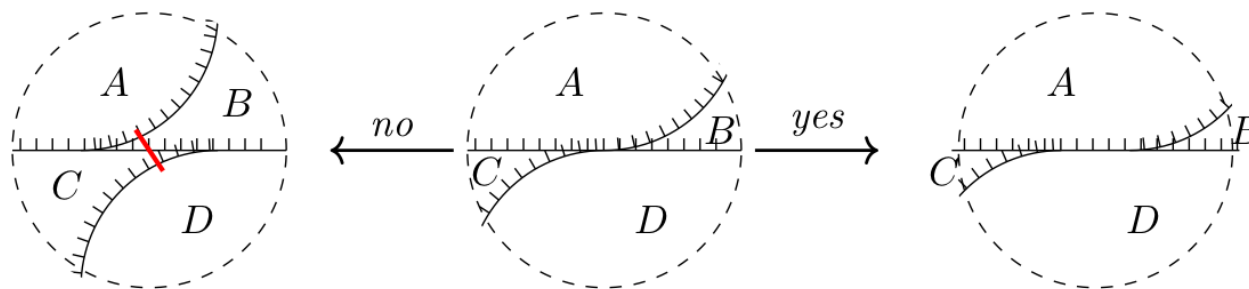

- equivalence from invariance of its Weinstein tubular nbhd

- constructible sheaves in $Sh(\mathbb{R}, \Lambda)$ deform along:

e.g. $\mathbb{C}_{[-1,1]} \rightsquigarrow \mathbb{C}_{\{0\}} \rightsquigarrow \mathbb{C}_{(-1,1)}^{[1]}$

(2) Some VLag are not equivalences :

$\Lambda \subset T^* \mathbb{R}^2$
front projection

$$\xleftarrow{\ no\ } \qquad \xrightarrow{\ yes\ }$$

the new   Reeb chords ending on Λ causes trouble.
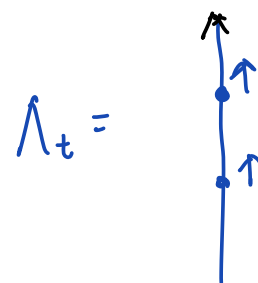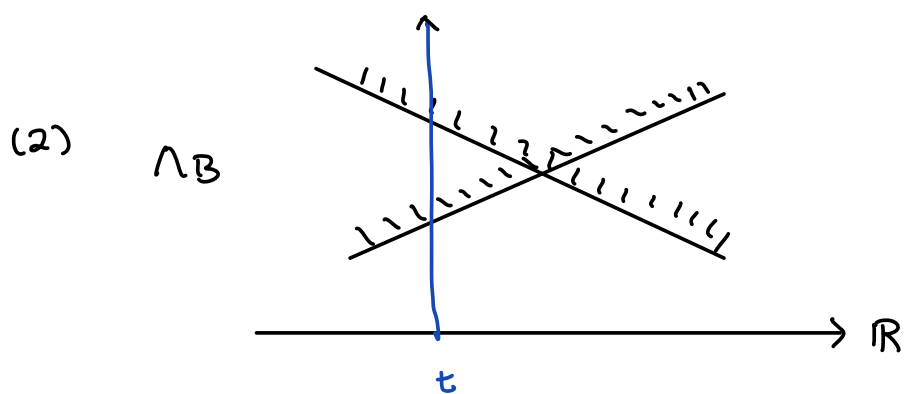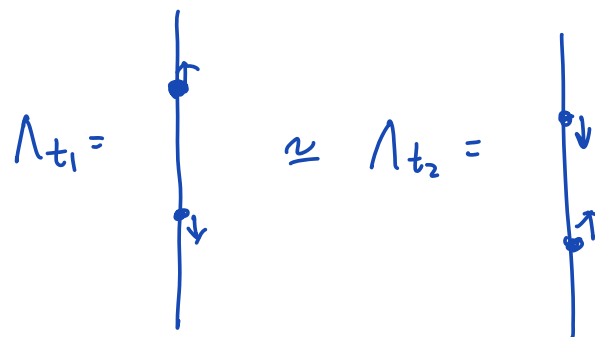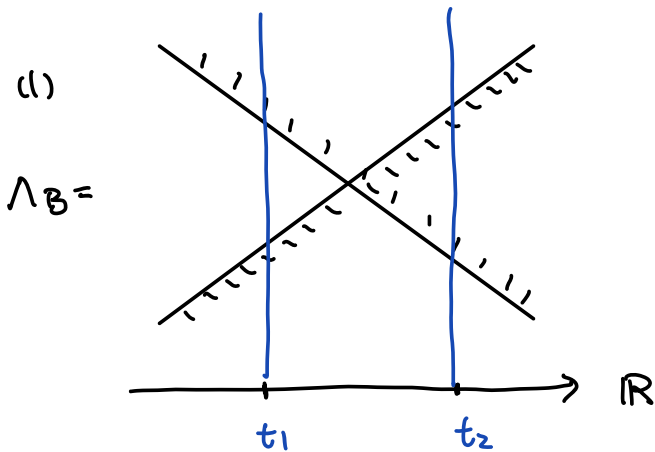
$$\Lambda_b \subset T^* M$$

(3). In general, given a family of skeletons $\{\Lambda_b\}_{b \in B}$.
we can construct a universal skeleton $\Lambda_B \subset T^*(M \times B)$,
Q: when is restriction
$$l_b^* : Sh(M \times B, \Lambda_B) \xrightarrow{\sim} Sh(M, \Lambda_b)$$
an equivalence of category ?

Ex:

(1)

$\Lambda_B =$

$\Lambda_{t_1} = $ $\simeq$ $\Lambda_{t_2} = $

$\mathbb{R}$

t_1 t_2

(2) Λ_B

$\Lambda_t = $

$\mathbb{R}$
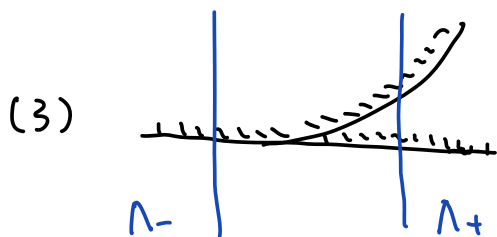
t

$\underset{0 \quad 1}{\underline{\quad\quad}} \subset T^* \mathbb{R}$
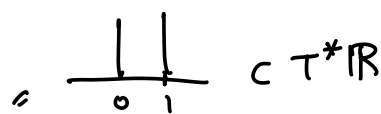
$Sh(\mathbb{R}^2, \Lambda_B) \longrightarrow Sh(\mathbb{R}, \Lambda_t)$ is not an equivalence

$\mathbb{C}_{[0,1)}$ can not be produced.

(3)

Λ_- Λ_+

$Sh(\Lambda_-) \xleftarrow{\neq} Sh(\Lambda_B) \xrightarrow{\sim} Sh(\Lambda_+)$

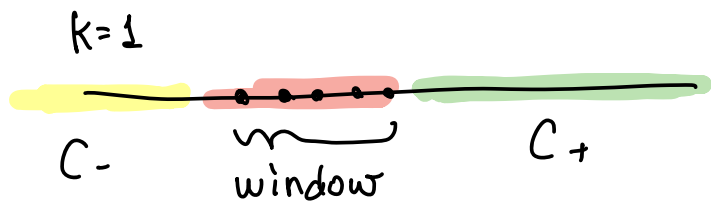§3.2　Window subskeleton from window subcategory.

Given a window

- idea:
  - $W = \langle \mathcal{L}_1, \cdots, \mathcal{L}_m \rangle \hookrightarrow \text{Coh} \, [\mathbb{C}^N / (\mathbb{C}^*)^k]$,　$\mathcal{L}_i = \mathcal{O}\{\alpha_i\}$

    equiv. line bundle

    $\alpha_i \in \mathbb{Z}^k$.

    $\downarrow \simeq$　　　　　　$\tau \downarrow \simeq$

    $\tau(W) = \langle F_1, \cdots, F_m \rangle \hookrightarrow \text{Sh}(\mathbb{R}^k \times T^{N-k}, \Lambda_{full})$

  - Define $\Lambda_W := \bigcup\limits_{i=1}^{m} SS(F_i) \subset \Lambda_{full}$

  - Let $\pi : \mathbb{R}^k \times T^{N-k} \longrightarrow \mathbb{R}^k$, for any $b \in \mathbb{R}^k$, let
    $\Lambda_{W,b} \subset T^* \mathbb{R}^k$ be the restriction of Λ_W.

  - For b deep in the GIT chamber $C \subset \mathbb{R}^k$,
    $\Lambda_{W,b} \simeq \Lambda_C \subset T^* T^{N-k}$, Λ_C mirror to $[\mathbb{C}^N /\!/_C (\mathbb{C}^*)^k]$

$k=1$

C_-　　window　　C_+

$k=2$

C_2

C
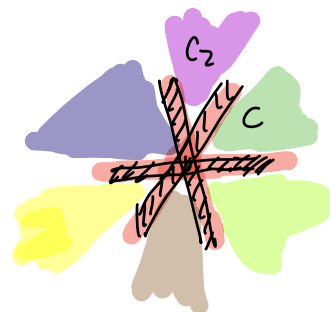
$$\text{Thm A}\left(\mathbb{C}^* \overset{[Z]}{\curvearrowright} \mathbb{C}^N\right) \quad \text{Let} \quad \mathbb{C}^* \curvearrowright \mathbb{C}^N \quad \text{with weights}$$

$$(a_1, \cdots, a_N), \quad \text{s.t. } a_i \text{ coprime, nonzero.} \quad \text{Let } d_\pm = \sum_{\pm a_i > 0} |a_i|$$

$$\text{Assume} \quad d_+ \geq d_-, \overset{\eta = d_+ - d_-}{\text{Then}} \text{ for any } k \in \mathbb{Z}, \text{ define} \qquad ,$$

$$W := \langle \mathcal{O}\{k\}, \cdots, \mathcal{O}\{k + d_+ - 1\} \rangle \subset \text{Coh}_{\mathbb{C}^*}(\mathbb{C}^N)$$

(1) $\tau: W \overset{\sim}{\longrightarrow} \text{Sh}(\mathbb{R} \times T^{N-1}, \Lambda_W)$

(2) $\text{Sh}(T^{N-1}, \Lambda_{W,t})$ is locally constant as t vary in $\mathbb{R}$

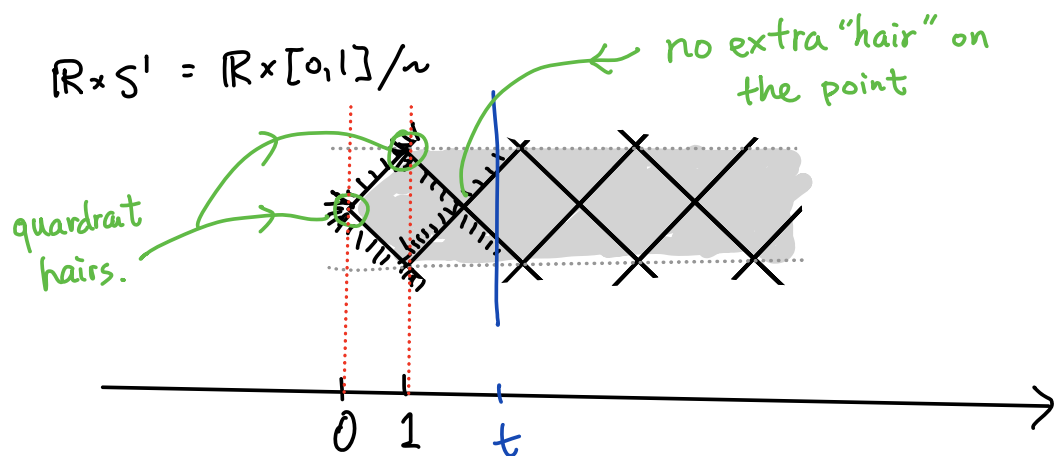$$\text{except at} \quad t \in \underbrace{\{k, k+1, \cdots, k+\eta-1\}}_{\eta \text{ many.}}$$

$\forall t \ll 0$
$$\text{Sh}(T^{N-1}, \Lambda_{W,t}) \simeq \text{Coh}([\mathbb{C}^N /\!\!/_- \mathbb{C}^*])$$

$\forall t \gg 0 \quad \text{Sh}(T^{N-1}, \Lambda_{W,t}) \simeq \text{Coh}([\mathbb{C}^N /\!\!/_+ \mathbb{C}^*]).$

Ex: $^{(1)}$ $\mathbb{C}^* \curvearrowright \mathbb{C}^2$ with weight $(1,1)$.

$X_- = \phi$, $\qquad X_+ = \mathbb{P}^1$

$W = \langle O\{0\}, O\{1\} \rangle$

$\mathbb{R} \times S^1 = \mathbb{R} \times [0,1]/\sim$

no extra "hair" on the point

quardrant hairs.

$\Lambda_t = $ $\simeq \Lambda_{\mathbb{P}^1}$

$0 \quad 1 \quad t$

Ex 2 :

Example 1.3. *Consider* $\mathbb{C}^*$ *acting on* $\mathbb{C}^2$ *with weight* $(3, -1)$. *The window skeleton is shown as below, living over* $S^1 \times \mathbb{R}$ *(drawn as* $\mathbb{R} \times [0, 1]$ *with top and bottom edge identified).*
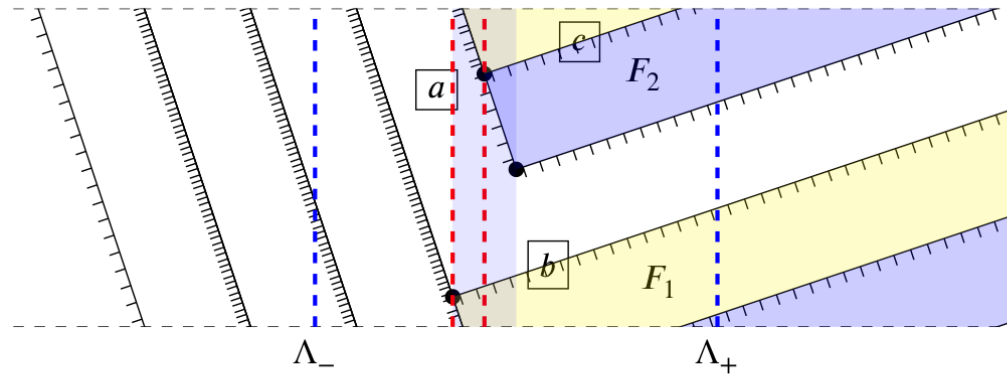
The window skeleton is the union of three skeleton $\Lambda(0), \Lambda(1), \Lambda(2)$, *whose vertices are marked in black nodes. The window region is marked in shadow. Take a vertical slice on the right of the window region, we get the skeleton* Λ_+ *for* $[\mathbb{C}/\mathbb{Z}_3]$; *and the vertical slice on the left of the window region gives skeleton* Λ_- *for* $\mathbb{C}$.

Thm B [Huang-Z]. Suppose $(\mathbb{C}^*)^k \curvearrowright \mathbb{C}^N$ satisfies quasi-symmetric condition, then for any $\delta \in \mathbb{R}^k$, we have B-side window subcat $W_\delta \subset \text{Coh}\left([\mathbb{C}^N/(\mathbb{C}^*)^k]\right)$, and A-side window skeleton $\Lambda_\delta \subset T^*(\mathbb{R}^k \times T^{N-k})$, and

(1) $\qquad W_\delta \simeq Sh(\mathbb{R}^k \times T^{N-k}, \Lambda_\delta)$

(2) $\qquad$ For any η deep in GIT chamber $C \subset \mathbb{R}^k$,

$$Sh(T^{N-k}, \Lambda_{\delta,\eta}) \simeq Sh(T^{N-k}, \Lambda_C)$$
$$\simeq \text{Coh}\left([\mathbb{C}^N /\!/_C (\mathbb{C}^*)^k]\right).$$

(3). For generic δ, Λ_δ defines a non-characteristic k-parameter variation of skeleton $\{\Lambda_{\delta,\eta}\}_{\eta \in \mathbb{R}^k}$.

(4) More generally, $\pi_* (\mathit{sh}_{\Lambda_\delta})$ defines a sheaf of categories over $\mathbb{R}^k$, with singular support along some thickened hyperplanes.

Ex:

Example 1.10 ($N = 6, k = 2$). *Consider the example of $(\mathbb{C}^*)^2$ acting on $\mathbb{C}^6$ with weight vectors β_i (as column vectors) given by*

$$(\beta_1, \beta_2, \cdots, \beta_6) = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix}$$

There are 6 GKZ chambers, separated by the 6 rays generated by β_i.

The stratification of the shift parameter space $\mathbb{R}^k_\delta$ (subscript is used to indicate the name of the coordinate) is shown in Figure 1. We consider three sample choices of δ as shown above, with δ_1 being the most non-generic and δ_3 being generic. For each δ_i, we illustrate in Figure 2 the zonotope, window points, and the singular support of C_δ. Note that in the first figure, over the vertices the zonotope, we have Lagrangian cones in the cotangent fiber, marked by the blue arcs, and over other intersections of the blue hairy lines, we don't have anything extra in the cotangent fiber.
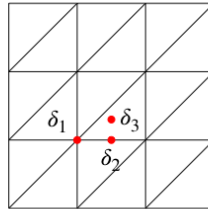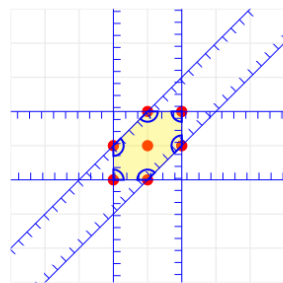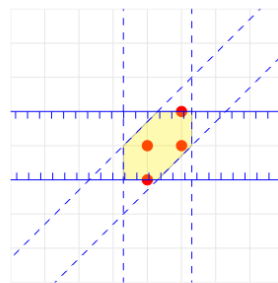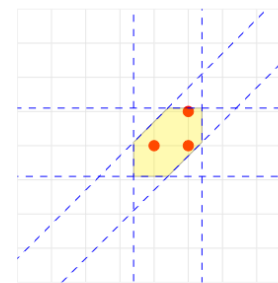

FIGURE 1. Stratification of the shift parameter space $\mathbb{R}^k_\delta$.

(A) $\delta = \delta_1$ (B) $\delta = \delta_2$ (C) $\delta = \delta_3$