Variation of GIT and Variation of Lagrangian skeletons.

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based on

Peng Zhou. Variation of GIT and variation of Lagrangian skeletons I: flip and flop. *arXiv preprint arXiv:2011.03719*, 2020

Jesse Huang and Peng Zhou. Variation of GIT and variation of Lagrangian skeletons II: Quasi-symmetric case. *arXiv preprint arXiv:2011.06114*, 2020

Outline

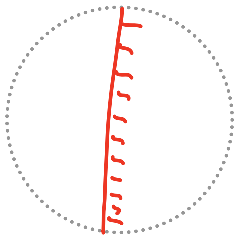
§0. Background on toric mirror symmetry.

§1. Motivations

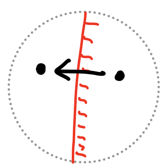
§2. VGIT and Window subcategories

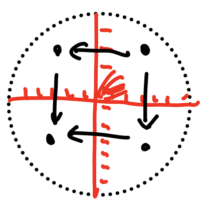
§3. VLag and Window subskeletons.

Cheatsheet for constructible sheaves on $\mathbb{R}^2$

(1) $\Lambda =$  $=$ half conormal to the line $\{x_1=0\}$ + zero section

$$= \left\{ (x_1, x_2, \xi_1, \xi_2) \in T^*\mathbb{R}^2 \;\middle|\; \begin{array}{l} x_1=0, \ \xi_2=0 \\ \xi_1 \geq 0, \ x_2 \text{ free} \end{array} \right\}$$

Sh() $\cong$ Rep$(\bullet \leftarrow \bullet)$

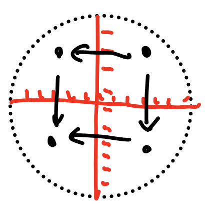
(2) $\Lambda =$  $=$ (zero section) + half conormal to $\{x_1=0\}$

+ half conormal to $\{x_2=0\}$

+ quadrant conormal to $\{x_1=0, x_2=0\}$

$= \{(0,0;\ \xi_1 \geq 0,\ \xi_2 \geq 0)\}$

$$Sh(\Lambda) = \mathrm{Rep}\left(\begin{tikzcd} B \arrow[d] & A \arrow[l] \arrow[d] \arrow[dl, phantom, "2"] \\ C & D \arrow[l] \end{tikzcd} \right)$$

(3) $\Lambda =$  ,

$$sh(\Lambda) = \mathrm{Rep}\left(\begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l] \arrow[d] \arrow[dl, phantom, "\square"] \\ \bullet & \bullet \arrow[l] \end{tikzcd} \right)$$

Cartesian diagram.

Toric Coherent Constructible Correspondence (CCC)

- $N \cong \mathbb{Z}^d$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^d$, $N_{\mathbb{C}^*} \simeq N \otimes \mathbb{C}^*$, $\mathbb{T} = \mathbb{R}/\mathbb{Z} = S^1$

  $M = \mathrm{Hom}(N, \mathbb{Z})$, $M_T = M \otimes \mathbb{T} = (S^1)^d$
- $\Sigma \subset N_{\mathbb{R}}$: smooth toric fan. (cone are simplicial)
- X_Σ : a smooth toric DM stack.
- $\Lambda_\Sigma \subset T^*(S^1)^d = T^* M_T$: FLTZ Lagrangian skeleton.

  $\widetilde{\Lambda}_\Sigma \subset T^*\mathbb{R}^d = T^* M_{\mathbb{R}}$. FLTZ equivariant skeleton.

Theorem (Bondal, Fang-Liu-Treumann-Zaslow, ---, Kuwagaki)

We have equivalence of categories :

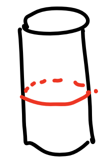
non-eq CCC : $\mathrm{Coh}(X_\Sigma) \simeq Sh^w(M_T, \Lambda_\Sigma)$

equivariant CCC : $\mathrm{Coh}_{(\mathbb{C}^*)^d}(X_\Sigma) \simeq Sh(M_{\mathbb{R}}, \widetilde{\Lambda}_\Sigma)$

$\S 0$

CCC examples

(1) $X_\Sigma = \mathbb{C}^*$, $\Lambda_\Sigma = S^1 \subset T^*S^1$ (zero section)



$\Lambda \to \bullet = \Sigma$

$$\mathrm{Coh}(\mathbb{C}^*) \simeq Sh^w(S^1, \Lambda_\Sigma) \simeq \mathrm{Loc}(S^1)$$

$\forall \lambda \in \mathbb{C}^*$: $\mathcal{O}_{\{\lambda\}} \longleftrightarrow$ rank 1 local system with monodromy λ.

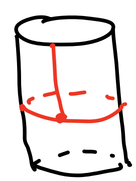
$\mathcal{O}_{\mathbb{C}^*} \longleftrightarrow \pi_* \underline{\mathbb{C}}_{\mathbb{R}}$.

$\uparrow$ an infinite rank locally constant sheaf.

$(\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z})$

$M_T \times N_{\mathbb{R}} \to N_{\mathbb{R}}$

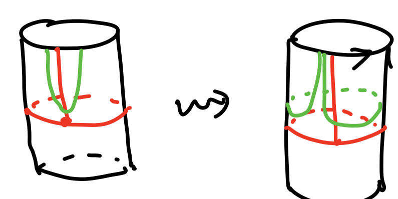
(2). $X_\Sigma = \mathbb{C}$. , $\Lambda_\Sigma =$



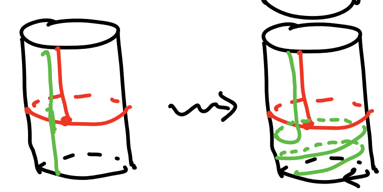
$\longrightarrow \Sigma =$

$$\mathrm{Coh}(\mathbb{C}) \longleftrightarrow Sh^w(S^1, \Lambda_\Sigma)$$

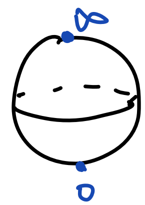
$\mathcal{O}_{\{0\}} \longleftrightarrow \pi_* \mathbb{C}_{[-1,0)}[1]$



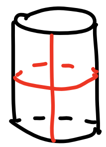
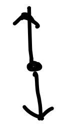
$\mathcal{O}_{\mathbb{C}} \longleftrightarrow \pi_* \mathbb{C}_{(-\infty,0)}[1]$

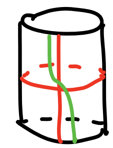


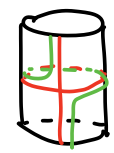
ξ_0

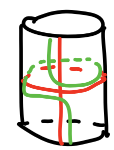
(3) $X = \mathbb{P}^1$,  $\xrightarrow{\mu}$  $= \Delta_X$, moment polytope,

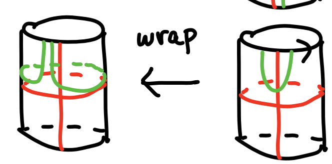
$N_{\mathbb{T}} = S^1$- orbit on $\mathbb{P}^1$.

$\Lambda =$  , $\Sigma =$ 

$\mathcal{O}_{\mathbb{P}^1}(0) \longleftrightarrow \mathbb{C}_{\{0\}}$ 

$\mathcal{O}_{\mathbb{P}^1}(1) \longleftrightarrow \pi_* \mathbb{C}_{(0,1)}[1]$ 

$\mathcal{O}_{\mathbb{P}^1}(-1) \longleftrightarrow \pi_* \mathbb{C}_{[0,1]}$ 

$\mathcal{O}_{\{0\}} \longleftrightarrow \pi_* \mathbb{C}_{[-1,0)}[1]$ 

$\mathcal{O}_{\{\infty\}} \longleftrightarrow \pi_* \mathbb{C}_{(-1,0]}[1]$. 

$\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$

Microlocal Sheaf Theory and Fukaya Category.

- microlocal sheaf theory can be used to compute Fukaya category of a Weinstein mfd (e.g. cotangent bundle).

<u>Thm</u> (Kontsevich, Nadler, Zaslow, Ganatra-Pardon-Shende)

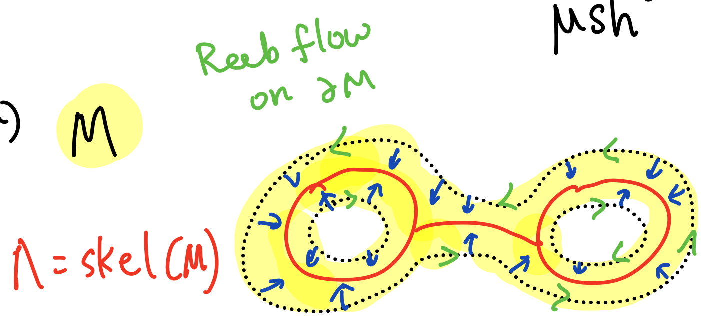
(a) Let $(M, \omega = d\lambda)$ be a Weinstein domain, let $\Lambda = \mathrm{skel}(M,\lambda)$ be the Liouville skeleton of M. Then

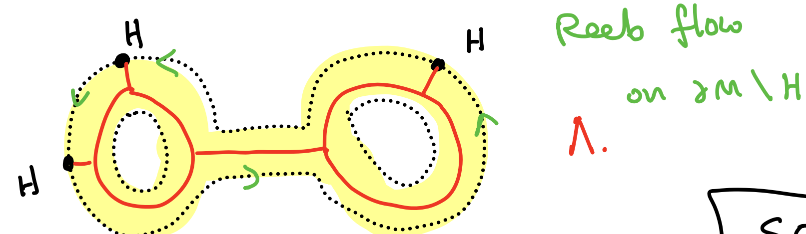
$$\mu sh^w(\Lambda) \simeq \mathrm{Fuk}^w(M)$$

$L_1, L_2 \subset M$
$\mathrm{Hom}(L_1,L_2) := \mathrm{Hom}(R^\infty L_1, L_2)$

(b) If M is a Weinstein domain, $\mathcal{H} \subset \partial M$ is a Weinstein hypersurface, $\Lambda = \mathrm{skel}(M) \cup \underset{\text{Liouville flow}}{\mathbb{R}_{>0}} \cdot \mathrm{skel}(\mathcal{H})$, then

$$\mu sh^w(\Lambda) \simeq \mathrm{Fuk}^w(M, \text{stop} = \mathcal{H})$$

(a)


(b)


§0

Toric Homological Mirror Symmetry.

- Using previous "microlocal sheaf $\leftrightarrow$ Lagrangian" correspondence.

  we have

$$\mathrm{Coh}(X_\Sigma) \overset{\text{CCC}}{\simeq} Sh^w(M_T, \Lambda_\Sigma) \simeq \mathrm{Fuk}^w(T^*M_T,\ \text{stop} = \Lambda^\infty_\Sigma)$$

- The traditional toric HMS mirror, uses LG A-model.

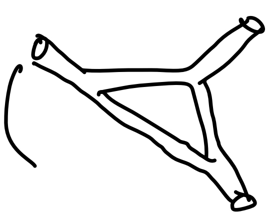
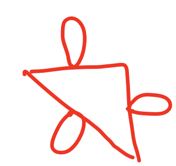
$$\mathrm{Coh}(X_\Sigma) \simeq FS((\mathbb{C}^*)^d,\ W_\Sigma).$$

e.g. $\mathrm{Coh}(\mathbb{P}^n) \cong FS\left((\mathbb{C}^*)^n,\ W = z_1 + \cdots + z_n + \frac{1}{z_1 \cdots z_n}\right)$

- **Thm** (Ruddat-Sibilia-Treuman-Zaslow, Gammage-Shende, Zhou).

  Let X_Σ be a smooth toric Fano, $W_\Sigma : (\mathbb{C}^*)^d \to \mathbb{C}$, $\Lambda_\Sigma \subset T^*(S^1)^d$, then

  "$\Lambda^\infty_\Sigma = \mathrm{skel}(W_\Sigma^{-1}(R))$" for $R \gg 0$.

- $\mathbb{P}^2$. $W = x + y + \frac{1}{xy}$, $\Lambda^\infty_\Sigma = \mathrm{skel}\Big($  $\Big) =$  $\Big|_{\xi \geq 0}$

§1 Motivations.

(1.1) • Let $A \subset \mathbb{Z}^d$, $\Delta = \text{Convex Hull}(A) \subset \mathbb{R}^d$,

and let
$$W = \sum_{\alpha \in A} c_\alpha \cdot z^\alpha$$
be a Laurant polynomial with generic coefficients $c_\alpha \in \mathbb{C}^*$.

Then, $\mathrm{Fuk}^\omega((\mathbb{C}^*)^d, W)$ only depends on Δ.

• One can choose "tropicalization" for W,
$$W = \sum_{\alpha \in A} e^{i\theta_\alpha} \cdot R^{h_\alpha} \cdot z^\alpha$$

$\theta_\alpha \in \mathbb{R}/2\pi\mathbb{Z}$
$h_\alpha \in \mathbb{R}$
$R \to \infty$

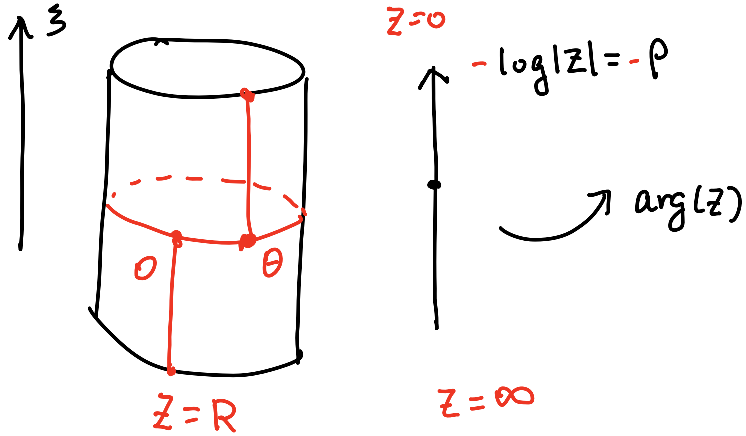
Then different choices results in different
$$\Lambda_W^\infty = \mathrm{skel}(W^{-1}(+\infty)), \text{ however}$$
$$\mathrm{Sh}^\omega((S^1)^d, \Lambda_W) \qquad \Lambda_W = (S^1)^d \cup \mathbb{R}_{>0} \cdot \Lambda_W^\infty$$
should be invariant. How?

Ex: (a). $W = z + \dfrac{e^{i\theta}}{z}$ (c.f. Hanlon's thesis)

$$W^{-1}(\mathbb{R}) = \{ z \approx \mathbb{R} \} \cup \{ \tfrac{e^{i\theta}}{z} \approx \mathbb{R} \} = \{$$

$\theta_z = 0$ $\quad$ $\theta_z = \theta$

$\rightsquigarrow$ $\Lambda_W =$



$$\begin{cases} T^*S^1 : & \omega = d\theta \wedge d\xi \\ \mathbb{C}^* : & \omega = \frac{i}{2}\frac{dz \wedge d\bar z}{|z|^2} = d\rho \wedge d\theta \end{cases}$$

$z = e^{\rho + i\theta}$

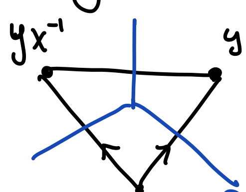
$\therefore\ \xi = -\rho$

$T^*\mathbb{P}^1.$

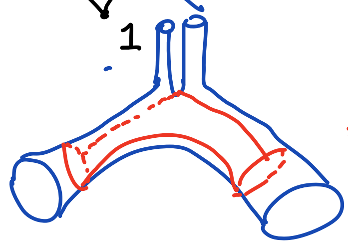
(b) $W = y(x^{-1} + x)$ $\quad$ $\cong$ $\quad$ $W = y(x^{-1} + \underset{\sim}{R} + x)$

$\mathbb{C}^2/\mathbb{Z}_2$

$\Delta =$

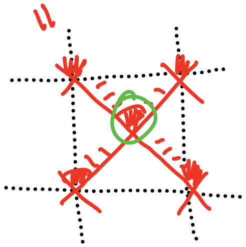


$W^{-1}(R) =$

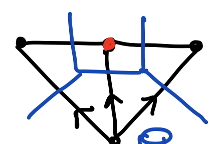


$= \Lambda_W^\infty$

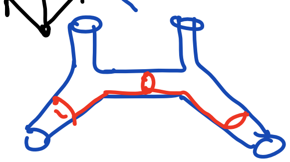
Λ_W $\shortparallel$



$\Delta =$

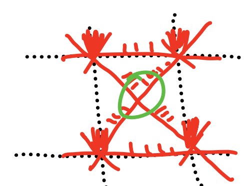


$W^{-1}(R) =$



$= \Lambda_W^\infty$

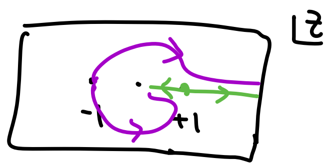
$\Lambda_W =$

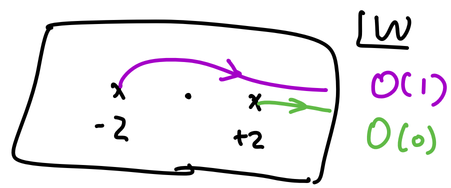


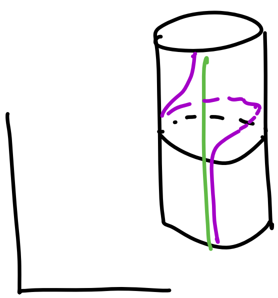
(1.2). In the comparison of $FS((\mathbb{C}^*)^n, W)$ and $Sh(T^n, \Lambda)$, one natural question is, where does Lagrangian thimbles go?

<u>Ex</u>: mirror for $\mathbb{P}^1$:

$W : \mathbb{C}^* \to \mathbb{C}$, $\quad W(z) = z + \frac{1}{z}$.



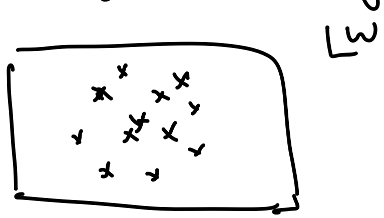




$L(1) = \pi_* \mathbb{C}_{(0,1)}[1]$

$L(0) = \mathbb{C}_{\{0\}}$

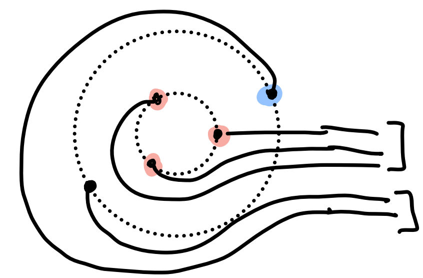
In general, the critical values of W has no pattern, and choice of vanishing path is very ad hoc (Straight line paths is complicated)



(1.2 cont'd) Semi-orthogonal decomposition (Ballard-Favero-Diemer-Katzarkov-Kerr)

- If we tropicalize W (in a generic way), then $\mathrm{Crit}(W)$ comes in circle clusters.

$$\mathrm{Fuk}((\mathbb{C}^*)^d, W) = \langle A^{(n)}, \cdots, A^{(2)}, A^{(1)} \rangle$$



- On the mirror side, we have birational transformation.

$$X = X^{(n)} \to X^{(n-1)} \to \cdots \to X^{1}$$

$$\mathrm{Coh}(X^{(k)}) = \langle B^{(k)}, \mathrm{Coh}(X^{(k-1)}) \rangle \quad \forall k = 1, \cdots, n.$$

$B^{(k)}$ is supported on the exceptional loci.

loci in $X^{(k)} \to X^{(k-1)}$.

<u>Conj</u>: $\mathrm{Coh}(X^{(k)}) \simeq \langle A^{(k)}, \cdots, A^{(1)} \rangle$

(Thm: [Kerr] associated graded level $A^{(i)} \cong B^{(i)}$)

- One trouble that this is still a conj is that Fukaya cat is hard.

(1.2 continue) The bridge between A and B sides is $Sh(T^d, \Lambda)$.

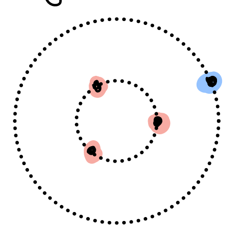
- Let $\Lambda^{(k)} \subset T^*M_T$, such that $Coh(X^{(k)}) \simeq Sh(M_T, \Lambda^{(k)})$.

  Then, if we can "measure the difference" between $\Lambda^{(k-1)}$ and $\Lambda^{(k)}$, we can see where the thimbles in SOD component $A^{(k)}$ go.

$(|t| \to 0)$

- Ex : • $W = x + y + \frac{1}{xy} + t \cdot xy$, $X = Bl_0 \mathbb{P}^2$.

  • $Crit(W) =$

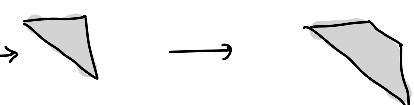


$\cdot\ \phi \to X^{(1)} \longrightarrow X^{(2)}$

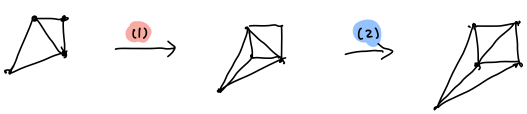
$\phi \to \mathbb{P}^2 \longrightarrow Bl_0 \mathbb{P}^2$

$\phi \to \Delta_{X^{(1)}} \longrightarrow \Delta_{X^{(2)}}$ moment polytope.

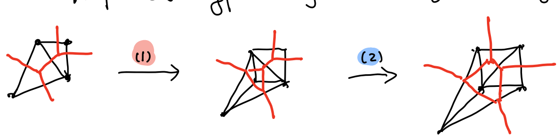
$\phi \longrightarrow$



- Triangulation of Newton Polytope of W, Δ_W

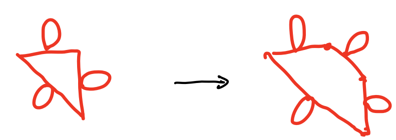


- Tropical hypersurfaces : $t^{-k} = x + y + \frac{1}{xy} + t \cdot xy$, $(k \to \infty)$



$\phi \to \Lambda^{(1),\infty} \longrightarrow \Lambda^{(2),\infty}$

$\phi \to$

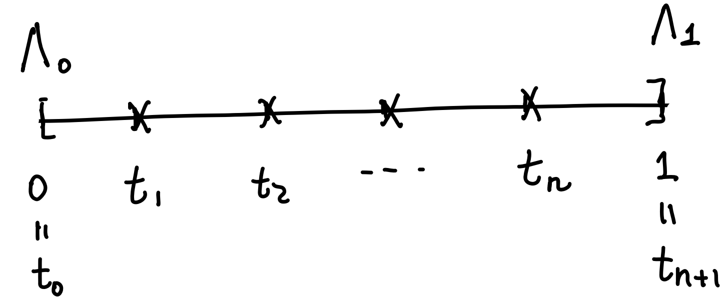


Goal:

We want to find a family of Lagrangian skeletons $\{\Lambda_t\}_{t\in[0,1]}$, interpolating Λ_0 and Λ_1.

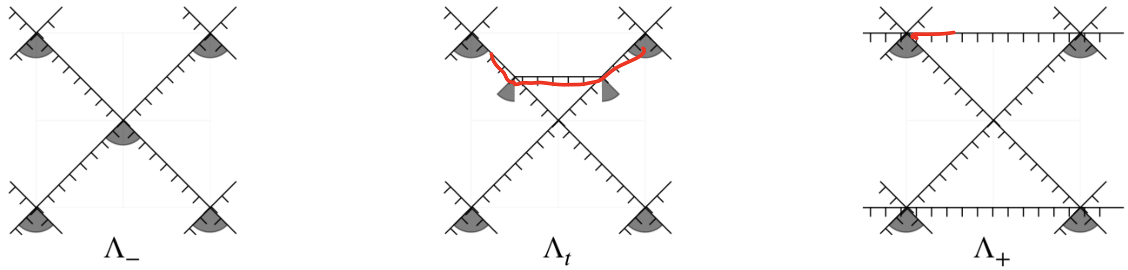
- If $sh(\Lambda_0) \simeq Sh(\Lambda_1)$, we want Λ_t variation to be "non-characteristic", i.e. $Sh(\Lambda_t)$ remain constant

- If $Sh(\Lambda_1) \simeq \langle \overset{\text{thimbles.}}{T}, sh(\Lambda_0) \rangle$ SOD, we want to have critical moments.

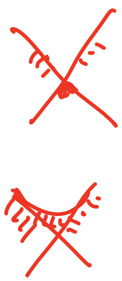


$Sh(\Lambda_t)$ constant over (t_i, t_{i+1})

Ex: (1)

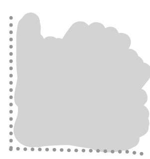


(non-characteristic variation)

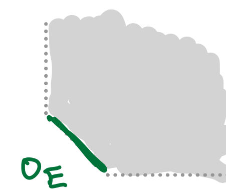


$\mathrm{Coh}([\mathbb{C}^2/\mathbb{Z}_2]) \simeq \mathrm{Coh}(T^*\mathbb{P}^1)$

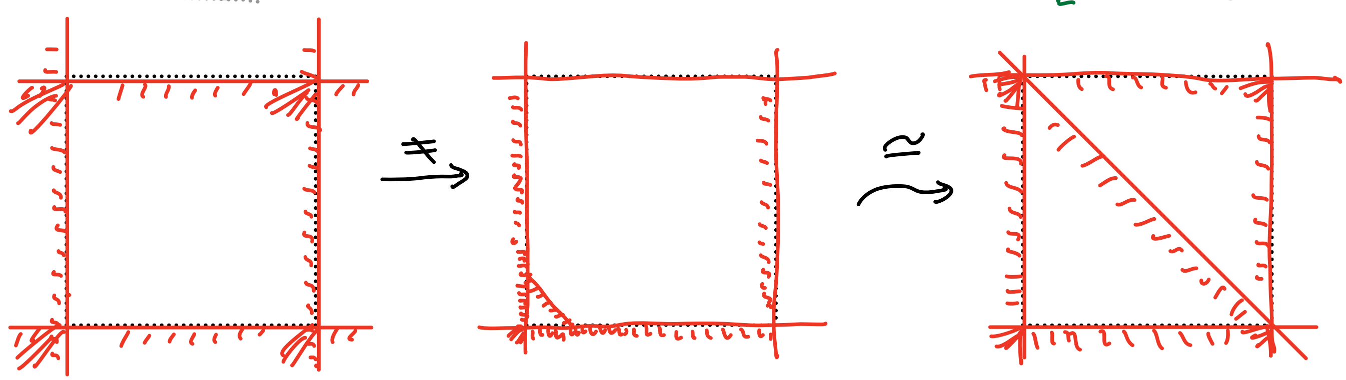
(2) X:

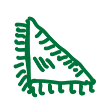


$\mathbb{C}^2 \rightsquigarrow \mathrm{Bl}_0\mathbb{C}^2$



Λ:





$\mathcal{O}_E$

$\mathrm{Coh}(\mathbb{C}^2) \hookrightarrow \mathrm{Coh}(\mathrm{Bl}_0\mathbb{C}^2)$

is $\qquad\qquad$ is

$sh^w(\Lambda_0) \qquad sh^w(\Lambda_1)$

§2 Variation of GIT and Window subcategory.

(Herbst - Hori - Page, E. Segal, Halpern-Leistner, Ballard - Favero - Katzarkov)

§2.1

- Idea:
  - When we study transitions between toric varieties X_- and X_+ (e.g. $X_- = \mathbb{C}^2$, $X_+ = Bl_0 \mathbb{C}^2$), they often come from different "phases" of GIT quotients

$$X_\pm = [\tilde{X} /\!/_{\theta_\pm} \mathbb{C}^*] = [\tilde{X} - \underbrace{\tilde{X}^{us}_{\theta_\pm}}_{\text{unstable loci}} / \mathbb{C}^*] \xhookrightarrow{\iota_\pm} [\tilde{X}/\mathbb{C}^*]$$

-
$$\begin{tikzcd}
 & \mathrm{Coh}([\tilde{X}/\mathbb{C}^*]) \simeq \mathrm{Coh}_{\mathbb{C}^*}(\tilde{X}) \arrow[dl, "\iota_+^*"'] \arrow[dr, "\iota_-^*"] & \\
\mathrm{Coh}(X_+) & & \mathrm{Coh}(X_-)
\end{tikzcd}$$

$$\mathrm{Coh}(X_\pm) = \mathrm{Coh}([\tilde{X}/\mathbb{C}^*]) \Big/ \langle \text{sheaves supported on } \underbrace{\tilde{X}^{us}_\pm}_{\text{unstable loci}} \rangle$$

• **Def**: A window subcategory for a GIT quotient $X_\pm$ is a subcategory $W_\pm \subset \mathrm{Coh}([\tilde{X}/\mathbb{C}^*])$, such that

$$L_i^*|_{W_i} : W_i \xrightarrow{\sim} \mathrm{Coh}(X_i) \qquad i=+,-$$

is an equivalence.

Rmk:

- We can compare $\mathrm{Coh}(X_\pm)$ via comparing $W_\pm$ in $\mathrm{Coh}([\tilde{X}/\mathbb{C}^*])$ now.

- Choices of $W_\pm$ are far from unique.

- Window subcategories exist for general GIT quotients by algebraic group $[\tilde{X}/\!\!/_\theta G] \hookrightarrow [\tilde{X}/G]$. [BFK, HL]

§2.2

Example :

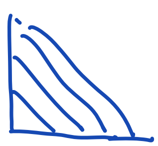
(1) $\mathbb{C}^3/\mathbb{C}^*$, $\mathbb{C}^* \curvearrowright \mathbb{C}^3$ with weight $(1,1,-1)$.

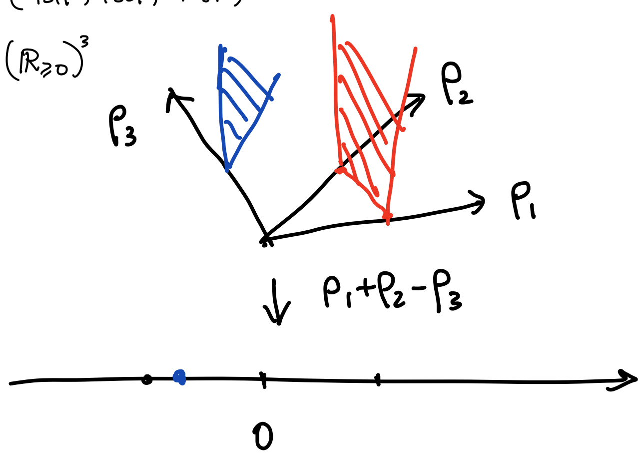
$$X_+ = \Big[\mathbb{C}^3 - \underbrace{\{(0,0,z) \mid z\in\mathbb{C}\}}_{Z_+} \Big/ \mathbb{C}^*\Big] = Bl_0\,\mathbb{C}^2$$
$$= \mathrm{Tot}\,[\mathcal{O}_{\mathbb{P}^1}(-1)]$$

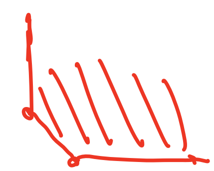
$$X_- = \Big[\mathbb{C}^3 - \underbrace{\{(z_1,z_2,0) \mid z_i\in\mathbb{C}\}}_{Z_-} \Big/ \mathbb{C}^*\Big] \simeq \mathbb{C}^2$$

$$(\mathbb{C}^*)^3 \curvearrowright \mathbb{C}^3 \xrightarrow{\mu} \mathbb{R}^3 = [\mathrm{Lie}(\mathbb{C}^*)^3]^\vee$$
$$(z_1,z_2,z_3) \longmapsto (|z_1|^2, |z_2|^2, |z_3|^2)$$
$$\mu(\mathbb{C}^3) = (\mathbb{R}_{\geq 0})^3$$

$$\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^3 \quad\rightsquigarrow\quad \mathbb{R}^3 \xrightarrow{\mu'} \mathbb{R}$$
$$t \longmapsto (t,t,t^{-1}). \qquad (\rho_1,\rho_2,\rho_3) \longmapsto \rho_1+\rho_2-\rho_3$$

$\Delta_{X_-} =$ 



$\Delta_{X_+} =$ 

- Coherent sheaves on $Z_+ = \{(0,0,z)\}$ is generated by $\mathcal{O}_{Z_+}$.

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}^3}\{-2\} \xrightarrow{(-z_2, z_1)} \mathcal{O}^2_{\mathbb{C}^3}\{-1\} \xrightarrow{(z_1, z_2)} \mathcal{O}_{\mathbb{C}^3}\{0\} \longrightarrow \mathcal{O}_{Z_+}\{0\} \to 0 \qquad \text{(Koszul resolution)}$$

$\{k\}$ means $\mathbb{C}^*$ equivariant degree.

Thus, when restricted to X_+, $\mathcal{O}_{Z_+}$ become 0,

hence we have exact seq.

$$0 \to \mathcal{O}\{-2\} \to \mathcal{O}\{-1\}^{\oplus 2} \longrightarrow \mathcal{O}\{0\} \to 0$$

$\Rightarrow \begin{cases} \bullet\ \mathcal{O}\{k\} \text{ can be expressed using } \mathcal{O}\{k-1\} \text{ and } \mathcal{O}\{k-2\} \\ \bullet\ \mathcal{O}\{k-2\} \text{ can } \text{———} \ \mathcal{O}\{k-1\} \text{ and } \mathcal{O}\{k\}. \end{cases}$

$\Rightarrow$ $\mathrm{Coh}(\mathbb{C}^3 - Z_+ / \mathbb{C}^*)$ can be generated by

$\iota_+^* \langle \mathcal{O}\{k\}, \mathcal{O}\{k+1\} \rangle$

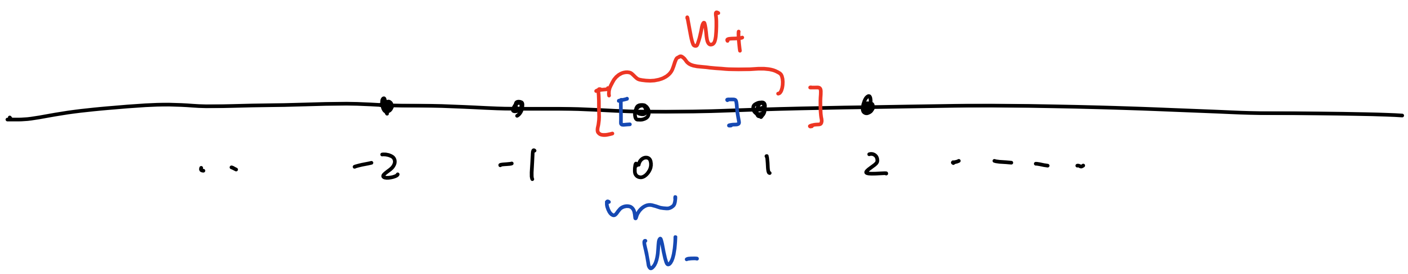
..... $\Rightarrow$ $\forall k \in \mathbb{Z}$, we can choose $W_+ = \langle \mathcal{O}\{k\}, \mathcal{O}\{k+1\} \rangle$

still need to prove $\iota_+^*|_{W_+}$ is fully-faithful

- coherent sheaves on $Z_- = \{(z_1, z_2, 0)\}$ is generated by $\mathcal{O}_{Z_-}$

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}^3}\{+1\} \xrightarrow{z_3} \mathcal{O}_{\mathbb{C}^3}\{0\} \longrightarrow \mathcal{O}_{Z_-}\{0\} \longrightarrow 0$$

same arguments shows $\forall k \in \mathbb{Z}$, we can choose
$$W_- = \langle \mathcal{O}\{k\} \rangle \subset \mathrm{Coh}([\mathbb{C}^3/\mathbb{C}^*]).$$



$$(\underline{1,1},\underline{-1})$$

W_+ : as lattice points in an interval of length $d_+ = 2$

W_- : ______ $d_- = 1$

- SOD : · $\underset{\underset{W_+}{\parallel}}{\mathrm{Coh}(\mathrm{Bl}_0\,\mathbb{C}^2)} = \langle \mathcal{O}_E(-1),\ \underset{\underset{W_-}{\parallel}}{\pi^*\mathrm{Coh}\,\mathbb{C}^2} \rangle$

$$\iota : \quad \mathrm{Coh}(\mathbb{C}^2) \xrightarrow{\sim} W_-^{\{0\}} \hookrightarrow W_+^{\{-1,0\}} \simeq \mathrm{Coh}(\mathrm{Bl}_0\,\mathbb{C}^2).$$

§2.3 Magic windows (HL-Sam, Spenko-van den Bergh)

How about general $[\mathbb{C}^N /\!\!/_\theta (\mathbb{C}^*)^k]$?

- any sm proj toric variety arises from Cox construct (GIT quotient)

- If $(\mathbb{C}^*)^k \circlearrowright \mathbb{C}^N$ preserves $dz_1 \wedge \cdots \wedge dz_N$, then different smooth quotients are all toric CY, and derived equivalent. (in a non-canonical way).

Q: Can we find a universal window

$$W = \langle \text{some line bundles} \rangle \subset \mathrm{Coh}[\mathbb{C}^N /\!\!/ (\mathbb{C}^*)^k]$$

such that,
$$\iota_\theta^* : W \xrightarrow{\sim} \mathrm{Coh}[\mathbb{C}^N /\!\!/_\theta (\mathbb{C}^*)^k]$$

for all GIT param θ in stable chambers ?

(in general, I cannot)

• One can achieve this under stronger assumption than toric CY,

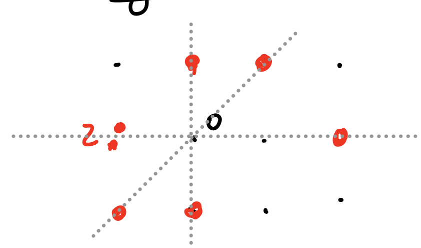
Def: (quasi-symmetric condition)

Let $\beta_1, \cdots, \beta_N \in \mathbb{Z}^k$ denote the collection of weights for $(\mathbb{C}^*)^k \curvearrowright \mathbb{C}^N$. If for any line (passing through 0) $L \subset \mathbb{R}^k$, the sum of weights on L is zero, then we say $(\mathbb{C}^*)^k \curvearrowright \mathbb{C}^N$ is quasi-symmetric.

Ex: • $k=1$, toric CY $\Leftrightarrow$ quasi-symetric.

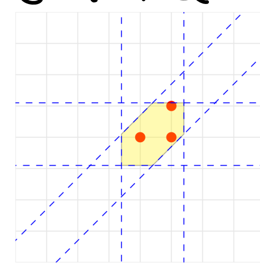
• If $\{\beta_1, \cdots, \beta_N\}$ is invariant under $(-1)\cdot : \mathbb{Z}^k \to \mathbb{Z}^k$, i.e. symmetric under inversion, then it is quasi-symm.

•



- Let $\Delta = \frac{1}{2} \sum_{i=1}^{N} [0, \beta_i]$ Minkowski sum of line segments.

- for any generic $\eta \in \mathbb{R}^k$, we have lattice points.



$$A_\eta = (\eta + \Delta) \cap \mathbb{Z}^k$$

and corresponding windows

$$W_\eta = \Big\langle \bigoplus_{\alpha \in A_\eta} \mathcal{O}\{\alpha\} \Big\rangle \subset \mathrm{Coh}([\mathbb{C}^N /\!/ (\mathbb{C}^*)^k]) = \mathrm{Coh}_{(\mathbb{C}^*)^k}(\mathbb{C}^N)$$

<u>Thm</u> (HL-Sam, SvdB)

For any stable GIT parameter $\theta \in \mathbb{R}^k$, any generic window param η, we have equivalence

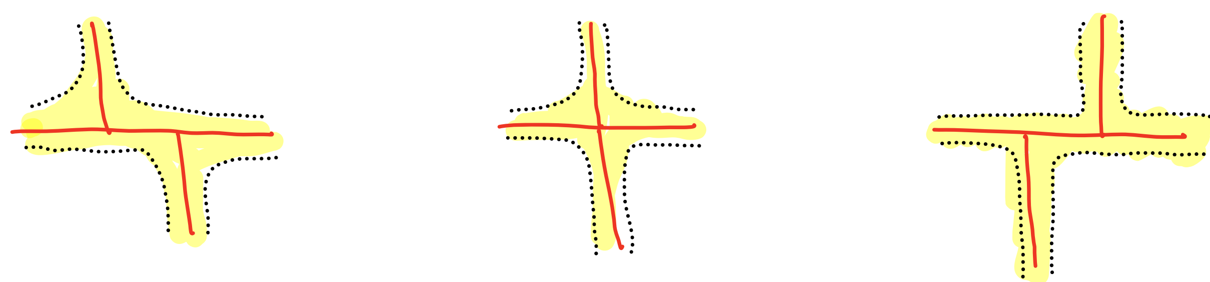
$$\iota_\theta^* : W_\eta \xrightarrow{\sim} \mathrm{Coh}([\mathbb{C}^N /\!/_\theta (\mathbb{C}^*)^k])$$

§3 Variation of Lagrangian (VLag) Skeleton and windows

§3.1 General VLag:

(1) Some variations of Lagrangians induces equivalences of categories. (Nadler)

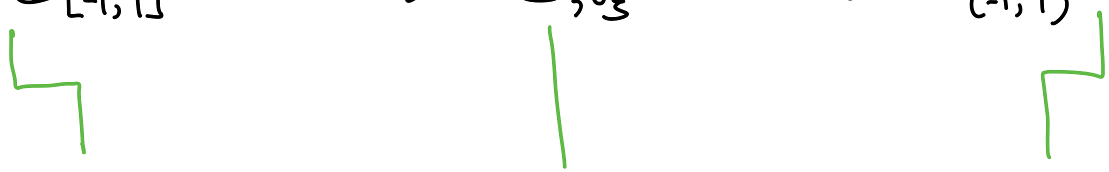
Ex: $\Lambda \subset T^*\mathbb{R}$



- equivalence from invariance of its Weinstein tubular nbhd

- constructible sheaves in $Sh(\mathbb{R}, \Lambda)$ deform along:

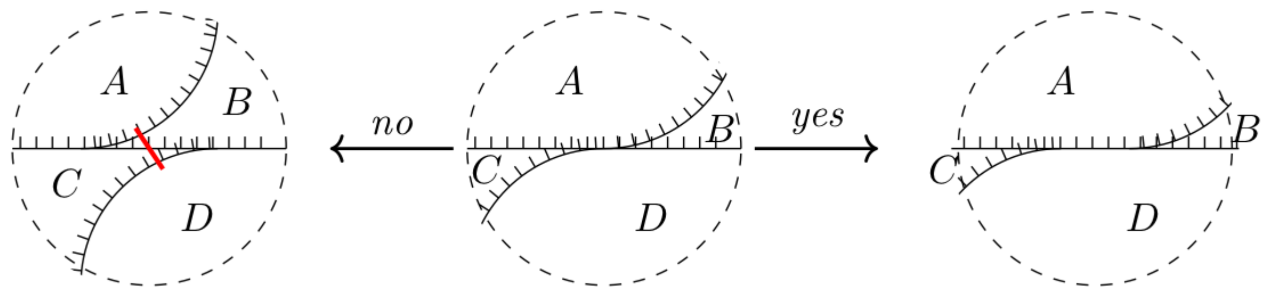
e.g. $\mathbb{C}_{[-1,1]} \rightsquigarrow \mathbb{C}_{\{0\}} \rightsquigarrow \mathbb{C}_{(-1,1)}[1]$



(2) Some VLag are not equivalences :

$\Lambda \subset T^*\mathbb{R}^2$

front projection



the new Reeb chords ending on Λ causes trouble.

$\Lambda_b \subset T^*M$

(3). In general, given a family of skeletons $\{\Lambda_b\}_{b\in B}$.

we can construct a universal skeleton $\Lambda_B \subset T^*(M\times B)$,

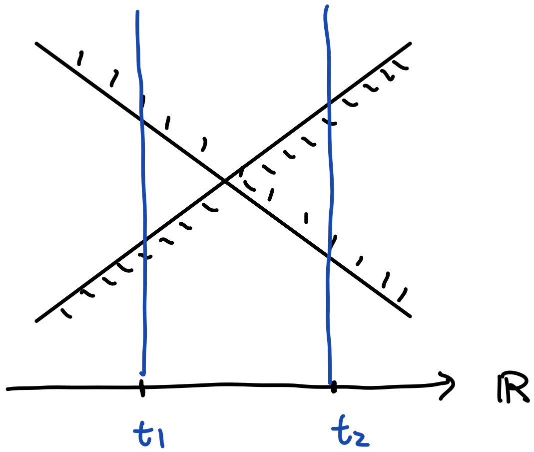
<u>Q:</u> when is restriction

$$\iota_b^* : Sh(M\times B, \Lambda_B) \xrightarrow{\sim} Sh(M, \Lambda_b)$$

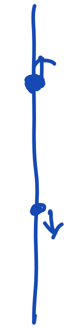
an equivalence of category ?

<u>Ex</u>:

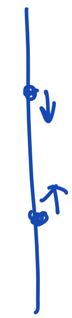
(1) $\Lambda_B =$



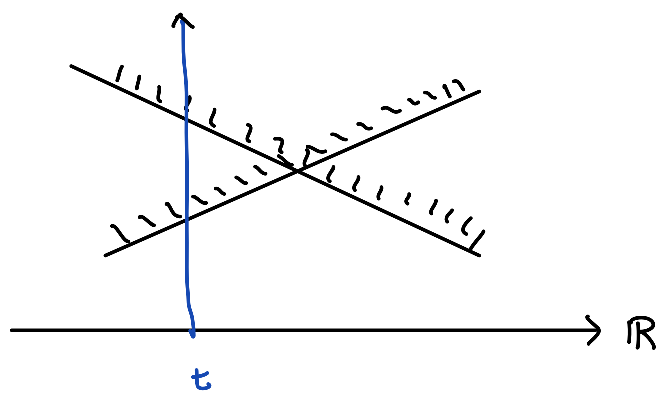
$\Lambda_{t_1} =$



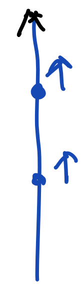
$\simeq \quad \Lambda_{t_2} =$

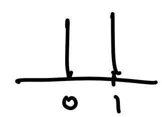


(2) Λ_B



$\Lambda_t =$



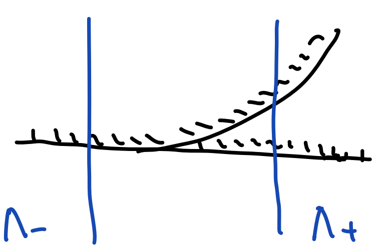


$\subset T^*\mathbb{R}$

$Sh(\mathbb{R}^2, \Lambda_B) \to Sh(\mathbb{R}, \Lambda_t)$ is not an equivalence

$\mathbb{C}_{[0,1)}$ can not be produced.

(3)



$Sh(\Lambda_-) \xleftarrow{\not\simeq} Sh(\Lambda_B) \xrightarrow{\simeq} Sh(\Lambda_+)$

§3.2 Window subskeleton from window subcategory.

idea: Given a window

- $W = \langle \mathcal{L}_1, \cdots, \mathcal{L}_m \rangle \hookrightarrow \mathrm{Coh}\,[\mathbb{C}^N/(\mathbb{C}^*)^k]$, $\mathcal{L}_i = \mathcal{O}\{\alpha_i\}$ equiv. line bundle $\alpha_i \in \mathbb{Z}^k$.

$$\begin{array}{ccc} W = \langle \mathcal{L}_1, \cdots, \mathcal{L}_m \rangle & \hookrightarrow & \mathrm{Coh}\,[\mathbb{C}^N/(\mathbb{C}^*)^k] \\ \downarrow \simeq & & \tau \downarrow \simeq \\ \tau(W) = \langle F_1, \cdots, F_m \rangle & \hookrightarrow & Sh(\mathbb{R}^k \times T^{N-k}, \Lambda_{full}) \end{array}$$

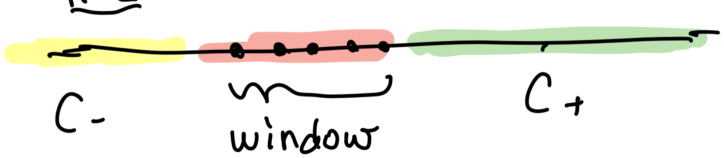
$\tau(W) \hookrightarrow Sh(\Lambda_W)$

- Define $\Lambda_W := \bigcup_{i=1}^{m} SS(F_i) \subset \Lambda_{full}$

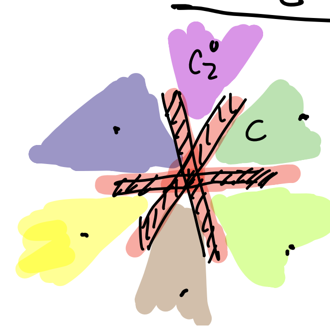
- let $\pi : \underline{\mathbb{R}}^k \times T^{N-k} \longrightarrow \mathbb{R}^k$, for any $b \in \mathbb{R}^k$, let $\Lambda_{W,b} \subset T^*\mathbb{R}^k$ be the restriction of Λ_W.

- For b deep in the GIT chamber $C \subset \mathbb{R}^k$, $\underline{\Lambda_{W,b}} \simeq \Lambda_C^{\subset T^* T^{N-k}}$. Λ_C mirror to $\underline{[\mathbb{C}^N /\!/_C (\mathbb{C}^*)^k]}$

$\underline{k=1}$



$k=2$



Thm A ($\mathbb{C}^* \curvearrowright \mathbb{C}^N$) [Z] Let $\mathbb{C}^* \curvearrowright \mathbb{C}^N$ with weights $(a_1, \cdots, a_N)$, s.t. a_i coprime, nonzero. Let $d_\pm = \sum_{\pm a_i > 0} |a_i|$,

Assume $d_+ \geq d_-$, $\eta = d_+ - d_-$. Then for any $k \in \mathbb{Z}$, define

$$W := \langle \mathcal{O}\{k\}, \cdots, \mathcal{O}\{k+d_+-1\} \rangle \subset \mathrm{Coh}_{\mathbb{C}^*}(\mathbb{C}^N)$$

(1) $\iota : W \xrightarrow{\sim} \mathrm{Sh}(\mathbb{R} \times T^{N-1}, \Lambda_W)$

(2) $\mathrm{Sh}(T^{N-1}, \Lambda_{W,t})$ is locally constant as t vary in $\mathbb{R}$

except at $t \in \{\underbrace{k, k+1, \cdots, k+\eta-1}_{\eta \text{ many.}}\}$

$\forall t \ll 0$ $\mathrm{Sh}(T^{N-1}, \Lambda_{W,t}) \simeq \mathrm{Coh}([\mathbb{C}^N /\!\!/_- \mathbb{C}^*])$

$\forall t \gg 0$ $\mathrm{Sh}(T^{N-1}, \Lambda_{W,t}) \simeq \mathrm{Coh}([\mathbb{C}^N /\!\!/_+ \mathbb{C}^*])$.

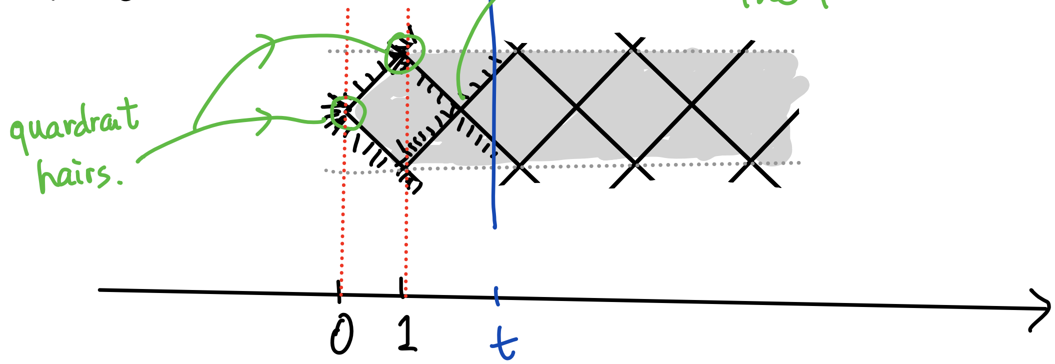
Ex : (1) $\mathbb{C}^* \curvearrowright \mathbb{C}^2$ with weight $(1,1)$.

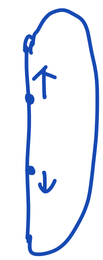
$X_- = \phi$, $\quad X_+ = \mathbb{P}^1$

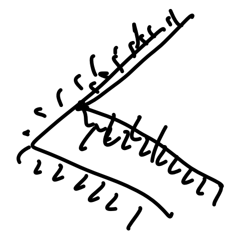
$W = \langle \mathcal{O}\{0\}, \mathcal{O}\{1\} \rangle$

$$w = z + \frac{e^{i\theta}}{z}$$

$\mathbb{R} \times S^1 = \mathbb{R} \times [0,1]/\sim$

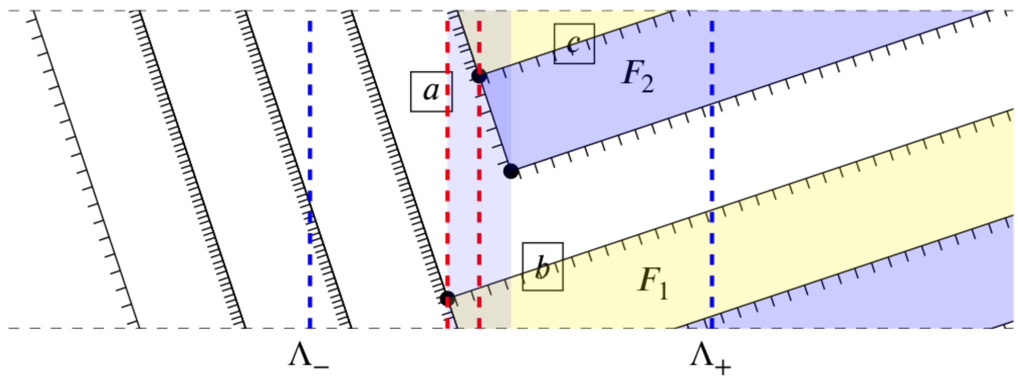


$\Lambda_t =$  $\simeq \Lambda_{\mathbb{P}^1}$

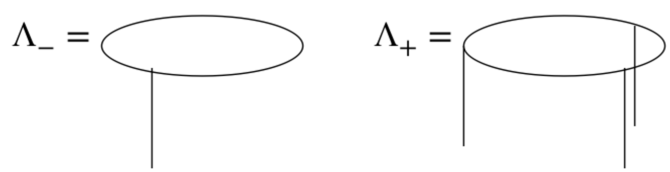


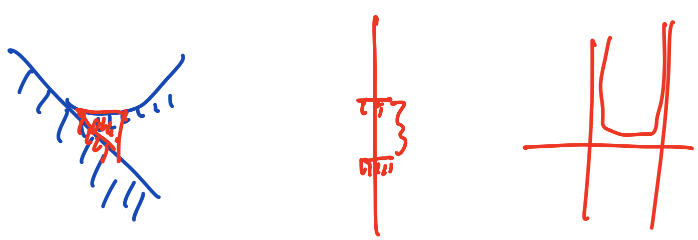
Ex 2 :

Example 1.3. *Consider $\mathbb{C}^*$ acting on $\mathbb{C}^2$ with weight $(3, -1)$. The window skeleton is shown as below, living over $S^1 \times \mathbb{R}$ (drawn as $\mathbb{R} \times [0, 1]$ with top and bottom edge identified).*



The window skeleton is the union of three skeleton $\Lambda(0), \Lambda(1), \Lambda(2)$, whose vertices are marked in black nodes. The window region is marked in shadow. Take a vertical slice on the right of the window region, we get the skeleton Λ_+ for $[\mathbb{C}/\mathbb{Z}_3]$; and the vertical slice on the left of the window region gives skeleton Λ_- for $\mathbb{C}$.





Thm B [Huang-Z]. Suppose $(\mathbb{C}^*)^k \curvearrowright \mathbb{C}^N$ satisfies quasi-symmetric condition, then for any $\delta \in \mathbb{R}^k$, we have B-side window subcat $W_\delta \subset Coh([\mathbb{C}^N/(\mathbb{C}^*)^k])$, and A-side window skeleton $\Lambda_\delta \subset T^*(\mathbb{R}^k \times T^{N-k})$, and

(1) $W_\delta \simeq Sh(\mathbb{R}^k \times T^{N-k}, \Lambda_\delta)$

(2) For any η deep in GIT chamber $C \subset \mathbb{R}^k$,

$$\begin{aligned} Sh(T^{N-k}, \Lambda_{\delta,\eta}) &\simeq Sh(T^{N-k}, \Lambda_C) \\ &\simeq Coh([\mathbb{C}^N /\!\!/_C (\mathbb{C}^*)^k]). \end{aligned}$$

(3). For generic δ, Λ_δ defines a non-characteristic k-parameter variation of skeleton $\{\Lambda_{\delta,\eta}\}_{\eta \in \mathbb{R}^k}$.

(4) More generally, $\pi_*(sh_{\Lambda_S})$ defines a sheaf of categories over $\mathbb{R}^k$, with singular support along some thickened hyperplanes.

Ex:

Example 1.10 ($N = 6, k = 2$). *Consider the example of $(\mathbb{C}^*)^2$ acting on $\mathbb{C}^6$ with weight vectors β_i (as column vectors) given by*

$$(\beta_1, \beta_2, \cdots, \beta_6) = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix}$$

There are 6 GKZ chambers, separated by the 6 rays generated by β_i.

The stratification of the shift parameter space $\mathbb{R}^k_\delta$ (subscript is used to indicate the name of the coordinate) is shown in Figure 1. We consider three sample choices of δ as shown above, with δ_1 being the most non-generic and δ_3 being generic. For each δ_i, we illustrate in Figure 2 the zonotope, window points, and the singular support of C_δ. Note that in the first figure, over the vertices the zonotope, we have Lagrangian cones in the cotangent fiber, marked by the blue arcs, and over other intersections of the blue hairy lines, we don't have anything extra in the cotangent fiber.

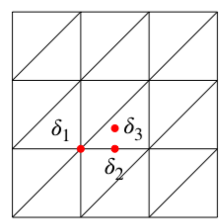
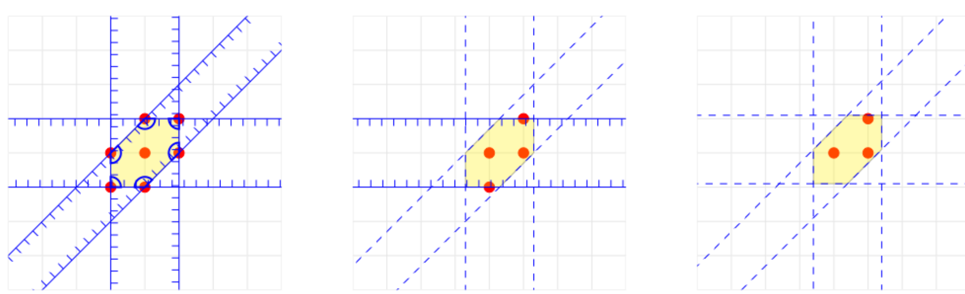


FIGURE 1. Stratification of the shift parameter space $\mathbb{R}^k_\delta$.



(A) $\delta = \delta_1$ (B) $\delta = \delta_2$ (C) $\delta = \delta_3$

$$W = \sum \underline{\underline{c_\alpha}}\, z^\alpha$$

$$\underline{\underline{Z_{disc}}} \subset \{\text{space of } c_\alpha\}$$

$$= \underline{\{\text{space of } W\}}$$