Quantum K-theory of flag varieties via non-abelian localization

Xiaohan Yan

University of California, Berkeley

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Outline

Some background

The study of a K-theoretic analogue of the quantum cohomology, namely the quantum K-theory, was initiated at the beginning of this century.

- Givental *On the WDVV equation in quantum K-theory*
- Lee *Quantum K-theory. I. Foundations*
- Givental-Lee *Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups*

quasimap

About a decade later, relations of such invariants to integrable systems and representation theory were explored.

- Okounkov *Lectures on K-theoretic computations in enumerative geometry*
- Aganagic-Okounkov *Elliptic stable envelopes*

Main result (non-rigorous formulation)

The permutation-invariant **big $\mathcal{J}$-function**, which is a generating function of the invariants, plays a crucial role in the theory.

- $X = \mathrm{Flag}(v_1, \cdots, v_n; N)$: the flag variety $(v_1 < \cdots < v_n < N)$,
- V_i: tautological bundles of X $(1 \leqslant i \leqslant n)$, rank $V_i = v_i$
  P_{ij}: K-theoretic Chern roots of V_i $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant v_i)$, $V_i = P_{i1} + \cdots + P_{iv_i}$
- $Q_i (1 \leqslant i \leqslant n)$: Novikov variables of X corresponding to the determinant bundles of V_i.
  $H_2(X;\mathbb{Z})$

Theorem (X.Y.)

The image of the big $\mathcal{J}$-function of X is covered by the orbit of $\widetilde{J}$ with respect to a family of pseudo-finite-difference operators, where

$$\widetilde{J} = (1-q) \sum_{d_{ij} \geqslant 0} \prod_{i,j} Q_{ij}^{d_{ij}} \frac{\prod_{i=1}^{n} \prod_{r \neq s}^{1 \leqslant r,s \leqslant v_i} \prod_{l=1}^{d_{is}-d_{ir}} (1 - y \frac{P_{is}}{P_{ir}} q^l)}{\prod_{i=1}^{n} \prod_{1 \leqslant r \leqslant v_{i+1}}^{1 \leqslant s \leqslant v_i} \prod_{l=1}^{d_{is}-d_{i+1,r}} (1 - \frac{P_{is}}{P_{i+1,r}} q^l)}.$$

Questions

- Permutation-invariant big $\mathcal{J}$-function?
- Pseudo-finite-difference operators??
- Why $\widetilde{J}$???

Reconstruction

- The main theorem can be regarded as a **reconstruction theorem** of the big $\mathcal{J}$-functions of flag varieties, generalizing the result of Givental [3] where the target variety is required to have its K-ring generated by line bundles (e.g. toric varieties and complete flag varieties).

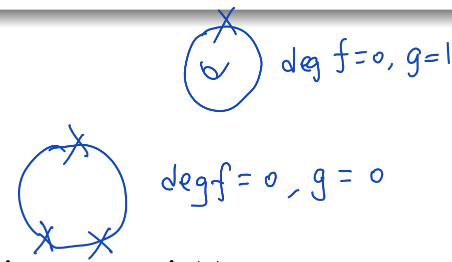
- Reconstruction of a different flavor is provided in Iritani-Milanov-Tonita [7], where the big quantum K-ring is recovered from the small J-function through analysis of q-shift operators.

Permutation-invariant quantum K-theory

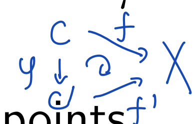
Assume X is a smooth projective variety and $d \in H_2(X;\mathbb{Z})$.

Definition

$\overline{\mathcal{M}}_{g,m}(X,d)$ is the moduli of stable maps $f:(C;p_1,\cdots,p_m)\to X$ of homological degree d and genus g with m marked points.



- **Stability**: C connected, nodal and projective;
  $p_1,\cdots,p_m$ smooth points on C;
  $|\mathrm{Aut}(f,(C;p_1,\cdots,p_m))| < \infty$.
- **Equivalence**: $(f,(C;p_1,\cdots,p_m)) \sim (f',(C';p'_1,\cdots,p'_m)) \Leftrightarrow \exists\varphi$: $(C;p_1,\cdots,p_m) \xrightarrow{\sim} (C';p'_1,\cdots,p'_m)$ with $f'\circ\varphi = f$.



S_m acts naturally on $\overline{\mathcal{M}}_{g,m}(X,d)$ by permuting the marked points.

Correlators

With the virtual structure sheaf defined by Lee [8], one can define **K-theoretic permutation-invariant correlators** (of genus 0):

Definition

$$\langle aL_1^k, \cdots, aL_m^k \rangle_{0,m,d}^{S_m} = \chi^{S_m}(\overline{\mathcal{M}}_{0,m}(X,d), \mathcal{O}^{\mathrm{virt}} \otimes \bigotimes_{l=1}^{n} \mathrm{ev}_l^*(a) L_l^k),$$

$S_n \times S_{m-n}$

$f \longmapsto f(p_l)$

where $a \in K(X)$, $\mathrm{ev}_l : \overline{\mathcal{M}}_{0,m}(X,d) \to X$ *is the evaluation map at the* l*-th marked point, and* L_l *is the universal cotangent bundle at the* l*-th marked point over the moduli space.*

S_m in the above construction may be replaced by any subgroup.

Big $\mathcal{J}$-function

Let $\{\phi_\alpha\}$ be an additive basis of $K(X)$ and $\{\phi^\alpha\}$ be its dual basis.

The **K-theoretic permutation-invariant big $\mathcal{J}$-function** is defined by

Definition

$\sum_k t_k L_1^k$

$$\mathcal{J}^X(\mathbf{t}; q) = 1 - q + \mathbf{t}(q) + \sum_{m,d,\alpha} Q^d \phi^\alpha \left\langle \frac{\phi_\alpha}{1 - qL_0}, \mathbf{t}(L_1), \cdots, \mathbf{t}(L_n) \right\rangle^{S_m}_{0,m+1,d}$$

where $Q^d = \prod_i Q_i^{d_i}$ with $\{Q_i\}$ are the Novikov variables, and Laurent polynomial $\mathbf{t} = \mathbf{t}(q) = \sum_k \mathbf{t}_k q^k$ is the input (with coefficients $\mathbf{t}_k \in K(X)[[Q_1, \cdots, Q_n]]$).

Loop space formalism

Denote

$$\mathcal{K} = K(X)[[Q_1, \cdots, Q_n]](q^{\pm 1})$$
$$\mathcal{K}_+ = K(X)[[Q_1, \cdots, Q_n]][q, q^{-1}]$$
$$\mathcal{K}_- = \{\mathbf{f} \in \mathcal{K} | \mathbf{f}(0) \neq \infty, \mathbf{f}(\infty) = 0\}$$

Fact

$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ *is a* **Lagrangian polarization** *under the symplectic pairing*

$$\Omega(\mathbf{f}, \mathbf{g}) = \mathrm{Res}_{q \neq 0, \infty} \langle \mathbf{f}(q^{-1}), \mathbf{g}(q) \rangle \frac{dq}{q}$$

where $\langle \cdot, \cdot \rangle$ is the K-theoretic Poincaré pairing.

Loop space formalism

Under this polarization,

$$\mathcal{J}^X : \mathbf{t} \longmapsto 1 - q + \mathbf{t}(q) + \sum_{m,d,\alpha} Q^d \phi^\alpha \langle \frac{\phi_\alpha}{1-qL_0}, \mathbf{t}(L_1), \cdots, \mathbf{t}(L_n) \rangle_{0,m+1,d}^{S_m}$$

is a map from $\mathcal{K}_+$ to $\mathcal{K}$.

Fact

The image $\mathcal{L}^X$ of $\mathcal{J}^X$ is an overruled cone in $\mathcal{K}$.

$\mathcal{J}^X(0) \in \mathcal{L}^X$ is called the **small J-function**.

Pseudo-finite-difference operators

$(V\otimes\cdots\otimes V)^{S_m} = \psi^k(V)$

$(Pq^{Q\partial_Q})\cdot Q^k = P\cdot q^k\cdot Q^k$

Fact ([6][3][5])

Let D be any Laurent polynomial. Then,

Adams operation

- *ruling spaces of $\mathcal{L}^X$ are invariant under operators like $e^{D(Pq^{Q\partial_Q},Q,q)}$;*
- *$\mathcal{L}^X$ is invariant under operators like $e^{\sum_{k>0}\frac{\Psi^k(D(Pq^{kQ\partial_Q},Q,q))}{k(1-q^k)}}$.*

$\psi^k: K(X)\to K(X)$

$L\mapsto L^k$

$Q\mapsto Q^k$

Here P represent line bundles and Q represent the Novikov variables associated to P.

$\psi^k(Pq^{Q\partial_Q}) = P^k q^{Q\partial_Q}$

$\psi^k(Pq^{kQ\partial_Q}) = P^k q^{kQ\partial_Q}$

We denote by $\mathcal{P}$ the group generated by operators above. $\mathcal{L}^X$ is preserved by $\mathcal{P}$.

Recall our questions:

- Permutation-invariant big $\mathcal{J}$-function? ✓
- Pseudo-finite-difference operators?? ✓
- Why $\widetilde{J}$???

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One-line answer:

- Abelian/Non-Abelian Correspondence ("Non-abelian localization")

We obtain $\widetilde{J}$, the "starting" point to generate the overruled cone $\mathcal{L}^X$ of the flag variety (*the non-abelian quotient*), from a twisted quantum K-theory of Y, the (*abelian quotient*) associated to X.

The abelian quotient Y

We regard the flag variety X as a GIT quotient of vector space

$$X = R//G = \underline{\mathrm{Hom}(\mathbb{C}^{v_1}, \mathbb{C}^{v_2}) \oplus \cdots \oplus \mathrm{Hom}(\mathbb{C}^{v_n}, \mathbb{C}^{N})//GL(v_1) \times \cdots \times GL(v_n)}.$$

Then the associated **abelian quotient** Y is defined as

$$Y = R//S = \underline{\mathrm{Hom}(\mathbb{C}^{v_1}, \mathbb{C}^{v_2}) \oplus \cdots \oplus \mathrm{Hom}(\mathbb{C}^{v_n}, \mathbb{C}^{N})//(\mathbb{C}^{\times})^{v_1} \times \cdots \times (\mathbb{C}^{\times})^{v_n}}.$$

Here $S \subset G$ is the maximal torus.

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Here $S \subset G$ is the maximal torus.

The torus $T = (\mathbb{C}^{\times})^N$ acts naturally on both X and Y by acting on $\mathbb{C}^N$.
We denote the characters by $\Lambda_1, \cdots, \Lambda_N$.

Example

When $X = \mathrm{Fl}(1,2;3)$,

$$Y = \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \oplus \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^3) // \mathbb{C}^\times \times (\mathbb{C}^\times)^2.$$

$$\begin{array}{ccc} \mathbb{C}P^1 & \hookrightarrow & Y \\ \uparrow p_{11} & & p_{21} \downarrow \quad \downarrow \quad \downarrow p_{22} \\ & & \mathbb{C}P^2 \times \mathbb{C}P^2. \end{array}$$

We denote by P_{11}, P_{21}, P_{22} the tautological bundles of Y. These bundles generate the K-ring of Y.

In general, the picture of Y is a tower of fiber bundles

$$
\begin{array}{ccc}
(\mathbb{C}P^{v_2-1})^{v_1} & \hookrightarrow & Y \\
 & & \downarrow \\
 & & \cdots \\
 & & \downarrow \\
(\mathbb{C}P^{v_{n-1}-1})^{v_{n-2}} & \hookrightarrow & F_{n-2} \\
 & & \downarrow \\
(\mathbb{C}P^{v_n-1})^{v_{n-1}} & \hookrightarrow & F_{n-1} \\
 & & \downarrow \\
 & & (\mathbb{C}P^{N-1})^{v_n}.
\end{array}
$$

We denote by P_{ij} the **tautological bundle** $\mathcal{O}(-1)$ on the j-th copy of $\mathbb{C}P^{v_{i+1}-1}$ in the i-th level $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant v_i)$.
We denote by $\{Q_{ij}\}_{i=1,j=1}^{n \quad v_i}$ the corresponding **Novikov variables** of Y.

The abelian quotient Y

$$\begin{array}{ccc} R^s(G)/S & \overset{\iota}{\hookrightarrow} & Y = R^s(S)/S \\ \downarrow q & & \\ X = R^s(G)/G & & \end{array}$$

P_{ij}

$V_i = P_{i1} + P_{i2} + \cdots + P_{iv_i}$

where $R^s(G)$ and $R^s(S)$ stands for the stable locus of the G- and S-action respectively.

Fact

We have the following relations of the tautological bundles

$$\iota^* \bigoplus_{k=1}^{v_i} P_{ik} = q^* V_i.$$

Grassmannian case: main result (rigorous formulation)

For the case of grassmannian $X = \mathrm{Gr}(v, N)$, we simplify our notations as follows.

- V: the (only) tautological bundle of X;
- $P_1, \cdots, P_v$: the tautological bundles of $Y = (\mathbb{C}P^{N-1})^v$;
- Q and $Q_i (1 \leqslant i \leqslant v)$: the Novikov variables of X and Y respectively.

Theorem (Main theorem)

$W = S_v$

The orbit of $\widetilde{J}^{tw,Y}$ under the group $\mathcal{P}^W$ of Weyl-group-invariant pseudo-finite-difference operators covers $\mathcal{L}^X$ under the specialization $Q_i = Q$ and $y = 1$, where

$$\widetilde{J}^{tw,Y} = \sum_{0 \leqslant d_1, \ldots, d_v} \prod_{i=1}^{v} Q_i^{d_i} \frac{\prod_{i \neq j}^{1 \leqslant i,j \leqslant v} \prod_{m=1}^{d_i - d_j} (1 - yq^m P_i / P_j)}{\prod_{i=1}^{v} \prod_{m=1}^{d_i} (1 - q^m P_i)^N}.$$

Grassmannian case: main result (rigorous formulation)

In fact, we prove the T-equivariant version of the above theorem.

Theorem (Main theorem', Givental-X.Y.)

The orbit of $\widetilde{J}^{tw,Y}$ *under the group* $\mathcal{P}^W$ *of Weyl-group-invariant pseudo-finite-difference operators cover the image* $\mathcal{L}^X$ *of the* T-**equivariant** *permutation-invariant big* $\mathcal{J}$*-function of* X *under the specialization* $Q_i = Q$ *and* $y = 1$*, where*

$$\widetilde{J}^{tw,Y} = \sum_{0 \leqslant d_1, \dots, d_v} \prod_{i=1}^{v} Q_i^{d_i} \frac{\prod_{i \neq j}^{1 \leqslant i,j \leqslant v} \prod_{m=1}^{d_i - d_j} (1 - yq^m P_i / P_j)}{\prod_{i=1}^{v} \prod_{j=1}^{N} \prod_{m=1}^{d_i} (1 - q^m P_i / \Lambda_j)}.$$

Taking $\Lambda_i \to 1$ gives us the previous theorem back.

The theorem has two aspects:

- elements in the orbit of $\widetilde{J}^{tw,Y}$ lie on $\mathcal{L}^X$;
- all points on $\mathcal{L}^X$ appear in the orbit $\widetilde{J}^{tw,Y}$.

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- elements in the orbit of $\widetilde{J}^{tw,Y}$ lie on $\mathcal{L}^X$;
- all points on $\mathcal{L}^X$ appear in the orbit $\widetilde{J}^{tw,Y}$. $\rightarrow$ save for later

Idea: abelian/non-abelian correspondence

elements in the orbit of $\widetilde{J}^{tw,Y}$ lie on the image of big $\mathcal{J}$-function of X

$$\Uparrow$$

(after specializing $Q_i = Q$)

$\widetilde{J}^{tw,Y}$ lies on the image of big $\mathcal{J}$-function of X

$$\Uparrow$$

① $\widetilde{J}^{tw,Y}$ lies on the image of big $\mathcal{J}$-function of Y *twisted* by $\mathfrak{g}/\mathfrak{s}$

$Y = R/\!/S$; G, S

$$+$$

② big $\mathcal{J}$-function of Y *twisted* by $\mathfrak{g}/\mathfrak{s}$ "=" big $\mathcal{J}$-function of X

abel/non-abel correspondence

Idea: abelian/non-abelian correspondence

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$$+$$

big $\mathcal{J}$-function of Y *twisted* by $\mathfrak{g}/\mathfrak{s}$ "$=$" big $\mathcal{J}$-function of X

$$\Uparrow$$

(Fixed point localization)

(Classical) Abelian/non-abelian correspondence

$$
\begin{array}{ccc}
R^s(G)/S & \xhookrightarrow{\ \iota\ } & Y = R^s(S)/S \\
\Big\downarrow q & & \\
X = R^s(G)/G & &
\end{array}
$$

Fact ([10])

Let $\sigma \in H_T^*(X)$ and $\tilde{\sigma} \in H_T^*(Y)$ such that $\iota^*\tilde{\sigma} = q^*\sigma$. Then,

$$\frac{1}{|W|}\int_Y \omega\tilde{\sigma} = \int_X \sigma,$$

where $\omega = \mathrm{Eu}(\mathfrak{g}/\mathfrak{s})$.

Quantum abelian/non-abelian correspondence

Previous works using this idea:

- Bertram-Ciocan-Fontanine-Kim *Two proofs of a conjecture of Hori and Vafa*, *Gromov-Witten invariants for abelian and nonabelian quotients*
- Ciocan-Fontanine-Kim-Sabbah *The abelian/nonabelian correspondence and Frobenius manifolds*
- Webb *The abelian-nonabelian correspondence for I-functions*
- Wen *K-theoretic I-functions of* $V//_\theta G$ *and applications*
- González-Woodward *Quantum Witten localization and abelianization for qde solutions*, *Quantum Kirwan for quantum K-theory*

Idea: fixed point localization

① $\widetilde{J}^{tw,Y}$ lies on the image of big $\mathcal{J}$-function of Y *twisted* by $\mathfrak{g}/\mathfrak{s}$

$$+$$

② big $\mathcal{J}$-function of Y *twisted* by $\mathfrak{g}/\mathfrak{s}$ "=" big $\mathcal{J}$-function of X

Idea: fixed point localization

$\widetilde{J}^{tw,Y}$ lies on the image of big $\mathcal{J}$-function of Y *twisted* by $\mathfrak{g}/\mathfrak{s}$

$+$

big $\mathcal{J}$-function of Y *twisted* by $\mathfrak{g}/\mathfrak{s}$ "$=$" big $\mathcal{J}$-function of X

This may be proved by a **recursive characterization** of big $\mathcal{J}$-functions based on fixed point localization.

Fixed point localization

Assume that M has isolated fixed points under a torus action by T, and that the fixed points are connected by isolated one-dimensional T-orbits. Any q-rational function $\mathbf{f} \in \mathcal{K}$ has the expansion

$$\mathbf{f} = \sum_{a \in \mathcal{F}} \mathbf{f}_a \phi^a$$

q-rational functions with coeff. in $K_T(pt)[[Q]]$

basis of $K_T(M)$

where $\{\phi^a\}_{a \in \mathcal{F}}$ are fixed point classes. Then, the following characterization of big $\mathcal{J}$-function holds [2]:

Fact

$\mathbf{f}$ *represents a value of* $\mathcal{L}^M$ *if and only if it satisfies Conditions (i) and (ii).*

Fixed point localization

(i) $\mathbf{f}_a$, when expanded as meromorphic functions with **poles only at roots of unity**, lies in $\mathcal{L}^{pt}$, the cone of the permutation-invariant quantum K-theory for point target space with coefficient ring $K(M)[[Q]]$.

(ii) Outside $0, \infty$ and roots of unity, $\mathbf{f}_a$ has **poles only at values of the form** $\lambda^{1/m}$ with λ a T-character of the tangent space T_aM and m a positive integer, and the residues satisfy the recursion relations

$V \quad \mu_1 \cdots \mu_k$

$\mathrm{Eu}(V) = (1-\mu_1^{-1}) \cdots (1-\mu_k^{-1})$

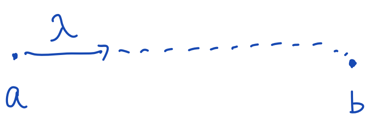
$$\mathrm{Res}_{q=\lambda^{1/m}} \mathbf{f}_a(q) \frac{dq}{q} = \frac{Q^{mD}}{m} \frac{\mathrm{Eu}(T_aM)}{\mathrm{Eu}(T_\phi \overline{\mathcal{M}}_{0,2}(M, mD))} \mathbf{f}_b(\lambda^{1/m}).$$

$D = \deg \overline{ab}$

$\phi: \mathbb{CP}^1 \longrightarrow \overline{ab}$

$z \longmapsto z^m$

$0, \infty \longmapsto a, b$



$\widetilde{J}^{tw,Y}$ is on twisted theory of Y

Proposition

$\widetilde{J}^{tw,Y}$ represents a value of the $(\mathrm{Eu}, y^{-1}\mathfrak{g}/\mathfrak{s})$-twisted big $\mathcal{J}$-function of the abelian quotient Y.

$\widetilde{J}^{tw,Y}$ is on twisted theory of Y

Proposition

$\widetilde{J}^{tw,Y}$ represents a value of the $(\mathrm{Eu}, y^{-1}\mathfrak{g}/\mathfrak{s})$-twisted big $\mathcal{J}$-function of the abelian quotient Y.

- One can directly check the recursion relations needed by the twisted theory.
- Alternatively, one could use the Quantum Adams-Riemann-Roch theorem [4] which describes the twisted big $\mathcal{J}$-function in terms of the untwisted big $\mathcal{J}$-function.

Twisted big $\mathcal{J}$-function of Y "=" big $\mathcal{J}$-function of X

Quantum K-theory of $Y = R//S$ *twisted* by $y^{-1}\mathfrak{g}/\mathfrak{s}$

$$\mathrm{Res}_{q=\lambda^{1/m}} \mathbf{f}_a \frac{dq}{q} = \frac{\prod_i Q_i^{mD_i}}{m} \frac{\mathrm{Eu}(T_a Y)}{\mathrm{Eu}(y^{-1}\mathfrak{g}/\mathfrak{s})|_a} \frac{\mathrm{Eu}((y^{-1}\mathfrak{g}/\mathfrak{s})_{0,2,mD})|_\phi}{\mathrm{Eu}(T_\phi Y_{0,2,mD})} \mathbf{f}_b(\lambda^{1/m}).$$

$$\Downarrow$$

Rachel Webb

Under the limit $Q_i = Q, y = 1$

$\mathcal{L}^{tw,Y}$ "=" $\mathcal{L}^X$

$$\Downarrow$$

$J^{tw,Y}(t)$ "=" $J^X(t')$

Quantum K-theory of $X = R//G$

$$\mathrm{Res}_{q=\lambda^{1/m}} \mathbf{f}_a(q) \frac{dq}{q} = \frac{Q^{m\sum_i D_i}}{m} \frac{\mathrm{Eu}(T_a X)}{\mathrm{Eu}(T_\phi X_{0,2,m\sum_i D_i})} \mathbf{f}_b(\lambda^{1/m}).$$

In other words, we check the recursion coefficients of the two theories coincide, under the specialization $Q_i = Q, y = 1$.

Remarks

Generating functions of quantum K-theory invariants of *symplectic* quiver varieties defined by quasi-map compactifications appear in the study of quantum integrable systems and representation theory. One often needs such functions to be **balanced** [11, 9] in order to apply rigidity arguments.

For the case of $T^*Gr(v,N)$, one may consider $I =$

$$\sum_{0\leqslant d_i} Q^{\sum_i d_i} \frac{\prod_{i\neq j}^{1\leqslant i,j\leqslant v}\prod_{m=1}^{d_i-d_j}(1-q^m P_i/P_j)}{\prod_{i\neq j}^{1\leqslant i,j\leqslant v}\prod_{m=0}^{d_i-d_j-1}(1-\hbar q^m P_i/P_j)} \frac{\prod_{i=1}^{v}\prod_{j=1}^{N}\prod_{m=0}^{d_i-1}(1-\hbar q^m P_i/\Lambda_j)}{\prod_{i=1}^{v}\prod_{j=1}^{N}\prod_{m=1}^{d_i}(1-q^m P_i/\Lambda_j)},$$

where $\hbar$ denotes the equivariant parameter of an extra fiberwise $\mathbb{C}^\times$-action on $T^*Gr(v,N)$.

Remarks

- Question: Can I be realized in terms of the language we introduced earlier?

Remarks

- Question: Can I be realized in terms of the language we introduced earlier?
- Yes, but after certain *twistings*.

Fact

Let $X = Gr(v, N)$. Then $I/\mathrm{Eu}(TX)$ lies on the image $\mathcal{L}^{Eu,TX}$ of the big $\mathcal{J}$-function of X twisted by its tangent bundle.

This may be proved using the same method.

Remarks

$\in \mathcal{L}^{X,TX}$

Note however that $I =$

$$\sum_{0\leqslant d_i} Q^{\sum_i d_i} \frac{\prod_{i\neq j}^{1\leqslant i,j\leqslant v}\prod_{m=1}^{d_i-d_j}(1-q^m P_i/P_j)}{\prod_{i\neq j}^{1\leqslant i,j\leqslant v}\prod_{m=0}^{d_i-d_j-1}(1-\hbar q^m P_i/P_j)} \frac{\prod_{i=1}^{v}\prod_{j=1}^{N}\prod_{m=0}^{d_i-1}(1-\hbar q^m P_i/\Lambda_j)}{\prod_{i=1}^{v}\prod_{j=1}^{N}\prod_{m=1}^{d_i}(1-q^m P_i/\Lambda_j)},$$

is not the small J-function in the twisted theory. In other words, under the polarization $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, $I = 1 - q + \mathbf{t} + \mathcal{K}_-$ with $\mathbf{t} \neq 0$.

This is due to the possible $\hbar$-terms in the denominator.

Flag variety case: notations

Recall that

$$X = \mathrm{Hom}(\mathbb{C}^{v_1}, \mathbb{C}^{v_2}) \oplus \cdots \oplus \mathrm{Hom}(\mathbb{C}^{v_n}, \mathbb{C}^{N}) // GL(v_1) \times \cdots \times GL(v_n)$$

$$Y = \mathrm{Hom}(\mathbb{C}^{v_1}, \mathbb{C}^{v_2}) \oplus \cdots \oplus \mathrm{Hom}(\mathbb{C}^{v_n}, \mathbb{C}^{N}) // (\mathbb{C}^{\times})^{v_1} \times \cdots \times (\mathbb{C}^{\times})^{v_n}$$

New flavor in the flag variety case

Recall that in grassmannian case,

- T-fixed points of X are the coordinate subspaces and are isolated;
- T-fixed points of $Y = (\mathbb{C}P^{N-1})^v$ are also isolated.

New flavor in the flag variety case

Recall that in grassmannian case,

- T-fixed points of X are the coordinate subspaces and are isolated;
- T-fixed points of $Y = (\mathbb{C}P^{N-1})^v$ are also isolated.

In flag variety case, however,

- T-fixed points of X are the standard flags and are still isolated;
- but T-fixed points of Y are no longer isolated!

- For simplicity of notations, we will mainly consider the case $X = \mathrm{Fl}(1, 2; 3)$. The method carries over to all partial flag varieties entirely.

New flavor in the flag variety case

$\mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \oplus \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^3)$

Example

For $X = Fl(1, 2; 3)$, Y *is a* $\mathbb{C}P^1$*-bundle over* $\mathbb{C}P^2 \times \mathbb{C}P^2$.

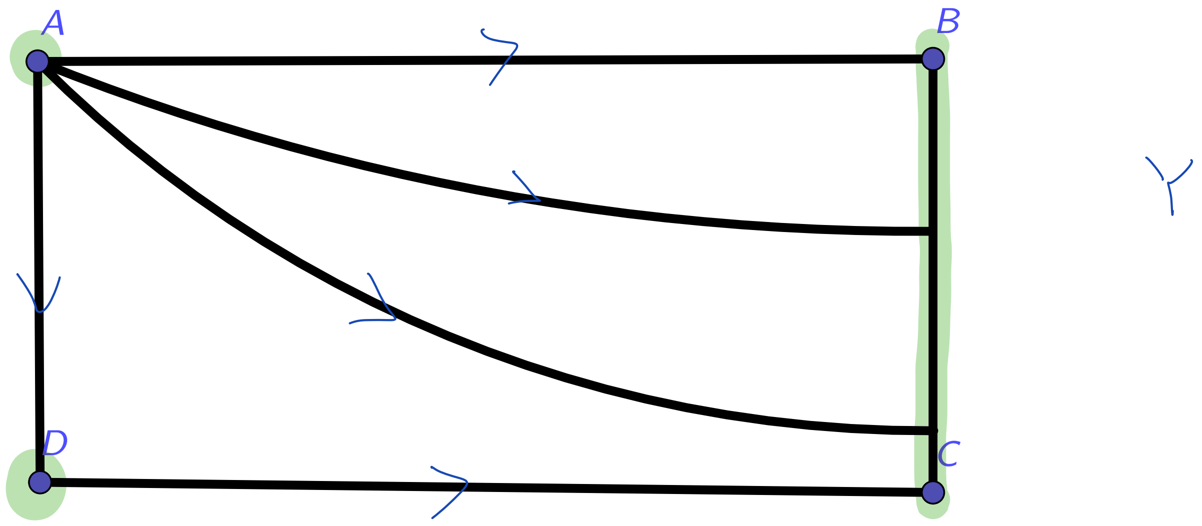
$\mathbb{C}\langle x_1\rangle \subset \mathbb{C}\langle x_1, x_2\rangle \subset \mathbb{C}^3$ $\qquad$ $\mathbb{C}\langle x_2\rangle \subset \mathbb{C}\langle x_1, x_2\rangle \subset \mathbb{C}^3$

- $A = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$ *and* $D = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$ *are isolated* T*-fixed points of* Y;
- $B = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \right)$ *and* $C = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \right)$ *are non-isolated* T*-fixed points of* Y. *In fact, the fixed point component containing* B *and* C *is isomorphic to* $\mathbb{C}P^1$:

$$\left\{ \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \right) \middle| a, b \in \mathbb{C} \text{ not both zero} \right\}.$$

New flavor in the flag variety case

A (very sketchy) picture is shown below:



New idea

We will have to address the issue of “non-isolated recursion”.

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The most direct idea is to enlarge the torus action on Y. We enlarge $T \to \widetilde{T}$

- $\widetilde{T} = T \times (\mathbb{C}^\times)^2$ with the extra action scaling the two entries of $\mathrm{Hom}(\mathbb{C}, \mathbb{C}^2)$ (i.e. rotating the fibers $\mathbb{C}P^1$).
- We denote by Λ_4, Λ_5 the equivariant parameters of the extra action.
- Now, A, B, C, D are all isolated fixed points of $\widetilde{T}$-action.

Main result: flag variety case

We may follow the same idea of grassmannian case. Under the action of the enlarged torus $\widetilde{T}$, we have

Theorem (Main theorem, X.Y.)

The orbit of $\widetilde{J}^{tw,Y}$ under the group of Weyl-group-invariant pseudo-finite-difference operators covers the entire image $\mathcal{L}^X$ of the big $\mathcal{J}$-function of X under the specialization $\Lambda_4 = \Lambda_5 = 1$, $Q_{ij} = Q_i$ and $y = 1$, where

$$\widetilde{J}^{tw,Y} = (1-q)\sum_{d_{ij}\geqslant 0}\prod Q_{ij}^{d_{ij}} \cdot \frac{\prod_{l=1}^{d_{21}-d_{22}}(1 - y\frac{P_{21}}{P_{22}}q^l)\prod_{l=1}^{d_{22}-d_{21}}(1 - y\frac{P_{22}}{P_{21}}q^l)}{\prod_{s=1}^{2}\prod_{l=1}^{d_{11}-d_{2s}}(1 - \frac{P_{11}}{P_{2s}\Lambda_{s+3}}q^l)\cdot\prod_{r=1}^{2}\prod_{s=1}^{3}\prod_{l=1}^{d_{2r}}(1 - \frac{P_{2r}}{\Lambda_s}q^l)}.$$

Idea

Consider the 1-dim $\widetilde{T}$-orbit AD as an example.

Step 1: $\widetilde{J}^{tw,Y}|_A$ satisfies the recursion relations of the $\widetilde{T}$-equivariant $(\mathrm{Eu}, y^{-1}\mathfrak{g}/\mathfrak{s})$-twisted big $\mathcal{J}$-function of Y; ✓

Step 2: Under the specialization $\Lambda_4 = \Lambda_5 = 1$, $Q_i = Q$ and $y = 1$, the twisted recursion along AD of Y descends correctly to the expected recursion along AD of X. ✓

Idea

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Step 2: Under the specialization $\Lambda_4 = \Lambda_5 = 1$, $Q_i = Q$ and $y = 1$, the twisted recursion along AD of Y descends correctly to the expected recursion along AD of X.

- But are we done?

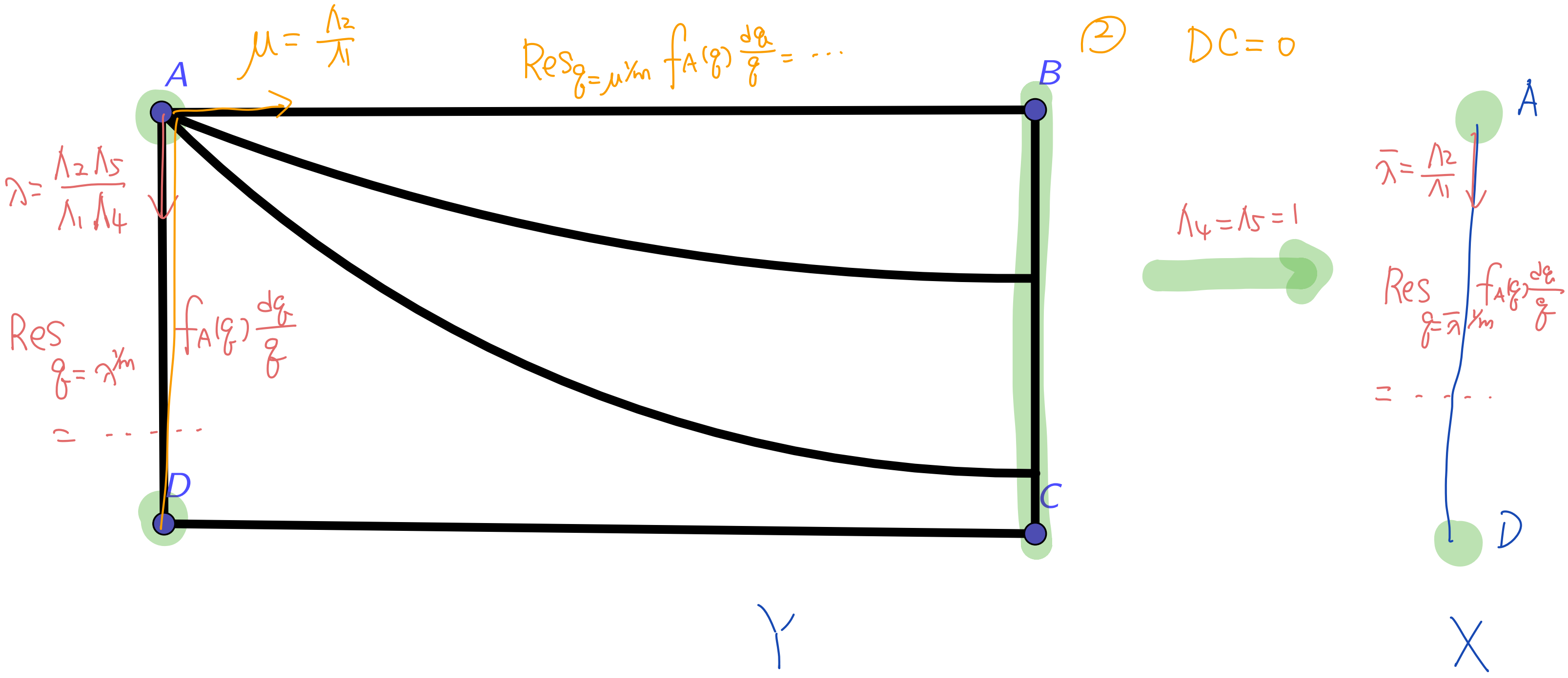
Idea

NO!! Both AD and AB contributes to the residue of of $\widetilde{J}^{tw,Y}|_A$ at the pole $q = (\frac{\Lambda_2}{\Lambda_1})^{1/m}$ as $\Lambda_4, \Lambda_5 \to 1$.

Prove contribution from AB = 0

① AB + ADC = 0

② DC = 0



Non-isolated recursion

Essentially, we are showing that the total **non-isolated recursion** from the component BC vanishes as $\Lambda_4 = \Lambda_5 = 1$, $Q_i = Q$ and $y = 1$.

Non-isolated recursion

Essentially, we are showing that the total **non-isolated recursion** from the component BC vanishes as $\Lambda_4 = \Lambda_5 = 1$, $Q_i = Q$ and $y = 1$.

For partial flag varieties in general, at a isolated fixed point (like A), we prove the vanishing of recursion from the "degenerate" orbits (like AB) following this same idea:

AB + ADC = 0

- complete it into non-isolated recursion from a fixed-point component (like BC) by taking **balanced broken orbits** (like ADC) into consideration;

  AB + ADC = 0

- prove that both the total non-isolated recursion and the added terms, which themselves are "lower non-isolated recursions", vanish.

DC = 0

A special case of the main theorem:

Corollary

$$J^X = (1-q)\sum_{d_{ij}\geqslant 0}\prod_i Q_i^{\sum_j d_{ij}} \frac{\prod_{i=1}^n \prod_{r\neq s}^{1\leqslant r,s\leqslant v_i} \prod_{l=1}^{d_{is}-d_{ir}} (1-\frac{P_{is}}{P_{ir}}q^l)}{\prod_{i=1}^n \prod_{1\leqslant r\leqslant v_{i+1}}^{1\leqslant s\leqslant v_i} \prod_{l=1}^{d_{is}-d_{i+1,r}} (1-\frac{P_{is}}{P_{i+1,r}}q^l)}$$

represents a value of the big $\mathcal{J}$-function of X.

This is actually the small J-function.

Remarks

Similar to the grassmannian case, we may consider balanced generating functions I^X of K-theoretic quasi-map invariants of T^*X.

Denote by J_d^X the coefficient of Q^d in the small J-function J^X. Then, $I^X = \sum_d Q^d I_d^X$ takes the form

$$I_d^X = J_d^X \cdot \frac{\prod_{i=1}^n \prod_{1 \leqslant r \leqslant v_{i+1}}^{1 \leqslant s \leqslant v_i} \prod_{l=0}^{d_{is}-d_{i+1,r}-1} (1 - \hbar \frac{P_{is}}{P_{i+1,r}} q^l)}{\prod_{i=1}^n \prod_{r \neq s}^{1 \leqslant r,s \leqslant v_i} \prod_{l=0}^{d_{is}-d_{ir}-1} (1 - \hbar \frac{P_{is}}{P_{ir}} q^l)}$$

In fact, we have

Fact

$I/\mathrm{Eu}(TX)$ represents a point on the image $\mathcal{L}^{\mathrm{Eu},TX}$ of the big $\mathcal{J}$-function of X twisted by its tangent bundle.

Surjectivity argument

Recall the theorem has two aspects:

- elements in the orbit of $\widetilde{J}^{tw,Y}$ lie on $\mathcal{L}^X$; ✓
- all points on $\mathcal{L}^X$ appear in the orbit of $\widetilde{J}^{tw,Y}$. $\rightarrow$ do this now

Surjectivity argument

Recall the theorem has two aspects:

- elements in the orbit of $\widetilde{J}^{tw,Y}$ lie on $\mathcal{L}^X$;
- all points on $\mathcal{L}^X$ appear in the orbit of $\widetilde{J}^{tw,Y}$. $\rightarrow$ do this now

Idea:

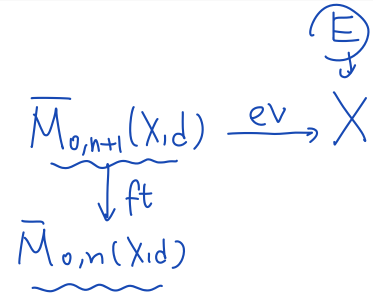
- We use the invariance of $\mathcal{L}^X$ under pseudo-finite-difference operators to generate a family on it from $\widetilde{J}^{tw,Y}$.
- We want to show that the projection of this family to $\mathcal{K}_+$ covers the entire $\mathcal{K}_+$: this is correct mod Q by quantum K-theory of point target space, and is thus correct with Q by Formal Implicit Function Theorem (Nakayama's Lemma).

Applications: level structures

Recently, the level structures are introduced to quantum K-theory, inspiring new progress in the field.

- Ruan-Zhang *The level structure in quantum K-theory and mock theta functions*
- Ruan-Wen-Zhou *Quantum K-theory of toric varieties, level structures, and 3d mirror symmetry*

Level structure



Definition

Let E be a vector bundle on X and l be an integer. The **level structure** (E, l) *is defined as the modification*

$$\mathcal{O}^{\text{virt}} \to \mathcal{O}^{\text{virt}} \otimes \det{}^{-l}(\text{ft}_* \, \text{ev}^* \, E)$$

to the virtual structure sheaf.

We consider the quantum K-theory of flag varieties with level structures.

Level structure

Using similar techinques as before, we can prove the following

Proposition

Write $J^X = \sum_{d \geqslant 0} Q^d J_d^X$ *as before, then the q-rational function*

$$J^{X,V_i,l} = \sum_{d \geqslant 0} Q^d \cdot \left[\prod_{s=1}^{v_i} P_{is}^{d_{is}} q^{\frac{d_{is}(d_{is}-1)}{2}} \right]^l \cdot J_d^X$$

represents a point on the overruled cone $\mathcal{L}^{X,V_i,l}$ *of* X *with level structure* (V_i, l).

Moreover, this is the small J-function as $|l|$ is small.

Level correspondence

A correspondence between level-twisted big $\mathcal{J}$-functions of dual grassmannians was observed in [1]. This may be generalized to the case of flag varieties as follows.

Consider the flag varieties

$$X = \mathrm{Flag}(v_1, v_2, \cdots, v_n; N)$$

and

$$X' = \mathrm{Flag}(N - v_n, N - v_{n-1}, \cdots, N - v_1; N).$$

There is a T-equivariant isomorphism which is explicitly given by

$$0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \mathbb{C}^N \longmapsto 0 \subset (V_n)^\perp \subset (V_{n-1})^\perp \subset \ldots \subset (V_1)^\perp \subset (\mathbb{C}^N)^*.$$

Both X and X' have n tautological bundles, and we name them V_i and V_i' respectively.

Level correspondence

The following fact is not hard to prove.

Fact

$$\mathcal{L}^{X,V_i,l} = \mathcal{L}^{X',(V_i')^\vee,-l}.$$

Therefore, combining the fact with what we have proved above, we have

Corollary

When $|l|$ is small,

$$J^{X,V_i,l} = J^{X',(V_i')^\vee,-l}.$$

Thank you!!!

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