The Moduli of Maps Has a Canonical Obstruction Theory

arxiv.org/abs/2109.03377

Rachel Webb
UC Berkeley

$\mathfrak{M}(X)$ has a canonical obstruction theory

Fix a locally Noetherian base scheme S

$\mathfrak{M}(X)$ has a canonical obstruction theory

Fix a locally Noetherian base scheme S

Fix $X \to S$ an algebraic stack (with hypotheses)

$\mathfrak{M}(X)$ has a canonical obstruction theory

Fix a locally Noetherian base scheme S

Fix $X \to S$ an algebraic stack (with hypotheses)

Notation: $\mathcal{C}$ denotes a family of nodal (twisted, tame) curves

$\mathfrak{M}(X)$ has a canonical obstruction theory

Fix a locally Noetherian base scheme S

Fix $X \to S$ an algebraic stack (with hypotheses)

Notation: $\mathcal{C}$ denotes a family of nodal (twisted, tame) curves

$$\mathfrak{M}(X)(T) = \left\{ \begin{array}{ccc} \mathcal{C} & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array} \right\}$$

$\mathfrak{M}(X)$ has a canonical obstruction theory

Fix a locally Noetherian base scheme S

Fix $X \to S$ an algebraic stack (with hypotheses)

Notation: $\mathcal{C}$ denotes a family of nodal (twisted, tame) curves

$$\mathfrak{M}(X)(T) = \left\{ \begin{array}{ccc} \mathcal{C} & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array} \right\}$$

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \mathcal{C}_{\mathfrak{M}(X)} & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \\ \mathfrak{M} & \longleftarrow & \mathfrak{M}(X) & & \end{array}$$

$\mathfrak{M}(X)$ has a canonical obstruction theory

Fix a locally Noetherian base scheme S

Fix $X \to S$ an algebraic stack (with hypotheses)

Notation: $\mathcal{C}$ denotes a family of nodal (twisted, tame) curves

$$\mathfrak{M}(X)(T) = \left\{ \begin{array}{ccc} \mathcal{C} & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array} \right\}$$

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \mathcal{C}_{\mathfrak{M}(X)} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \pi & & \\ \mathfrak{M} & \longleftarrow & \mathfrak{M}(X) & & \end{array}$$

- Hall-Rydh: $\mathfrak{M}(X)$ is an algebraic stack

$\mathfrak{M}(X)$ has a canonical obstruction theory

Fix a locally Noetherian base scheme S

Fix $X \to S$ an algebraic stack (with hypotheses)

Notation: $\mathcal{C}$ denotes a family of nodal (twisted, tame) curves

$$\mathfrak{M}(X)(T) = \left\{ \begin{array}{ccc} \mathcal{C} & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array} \right\}$$

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \mathcal{C}_{\mathfrak{M}(X)} & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \\ \mathfrak{M} & \longleftarrow & \mathfrak{M}(X) & & \end{array}$$

- Hall-Rydh: $\mathfrak{M}(X)$ is an algebraic stack
- Can allow $\mathcal{C}$ to have marks

$\mathfrak{M}(X)$ has a canonical obstruction theory

Fix a locally Noetherian base scheme S

Fix $X \to S$ an algebraic stack (with hypotheses)

Notation: $\mathcal{C}$ denotes a family of nodal (twisted, tame) curves

$$\mathfrak{M}(X)(T) = \left\{ \begin{array}{ccc} \mathcal{C} & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array} \right\}$$

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \mathcal{C}_{\mathfrak{M}(X)} & \xrightarrow{f} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \\ \mathfrak{M} & \longleftarrow & \mathfrak{M}(X) & & \end{array}$$

- Hall-Rydh: $\mathfrak{M}(X)$ is an algebraic stack
- Can allow $\mathcal{C}$ to have marks
- In fact can use *moduli of sections*

Aside: moduli of sections

Let $\mathcal{M}$ locally Noetherian algebraic stack $\qquad \mathcal{M}_{g,n}$

$\mathcal{C} \to \mathcal{M}$ family of twisted curves $\qquad \mathcal{C}_{g,n}$

$\mathcal{Z} \to \mathcal{C}$ morphism of algebraic stacks st ... $\qquad \mathcal{Z} = \mathcal{C}_{g,n} \times X$

$$\mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})(T) = \left\{ \begin{array}{ccc} & & \mathcal{Z} \\ & \nearrow & \downarrow \\ \mathcal{C}_T & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathcal{M} \end{array} \right\} = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}_T, \mathcal{Z})$$

<u>Thm</u> (Hall - Rydh) $\qquad \mathrm{Sec}_{\mathcal{M}}(\mathcal{Z}/\mathcal{C})$ is algebraic

$\mathfrak{M}(X)$ has a canonical obstruction theory

Gromov-Witten invariants

$\mathfrak{M}(X)$ has a canonical obstruction theory

integrate on $\mathcal{M}(X) \subset \mathfrak{M}(X)$	$\Longrightarrow$	Gromov-Witten invariants

$\mathfrak{M}(X)$ has a canonical obstruction theory

$[\mathcal{M}(X)]^{\mathrm{vir}}$ in $A_*(\mathcal{M}(X))$	$\Longrightarrow$	integrate on $\mathcal{M}(X) \subset \mathfrak{M}(X)$	$\Longrightarrow$	Gromov-Witten invariants

$\mathfrak{M}(X)$ has a canonical obstruction theory

perfect obstruction theory				
$\Downarrow$				
$[\mathcal{M}(X)]^{\mathrm{vir}}$ in $A_*(\mathcal{M}(X))$	$\Longrightarrow$	integrate on $\mathcal{M}(X) \subset \mathfrak{M}(X)$	$\Longrightarrow$	Gromov-Witten invariants

$\mathfrak{M}(X)$ has a canonical obstruction theory

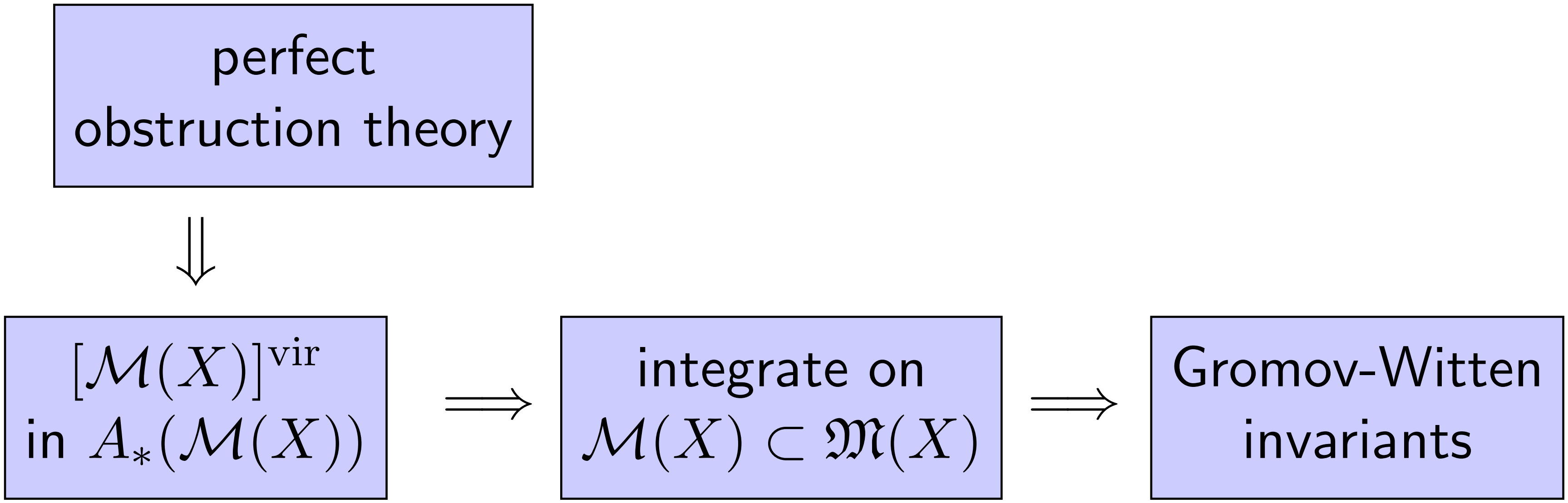
perfect obstruction theory

$\Downarrow$

$[\mathcal{M}(X)]^{\mathrm{vir}}$ in $A_*(\mathcal{M}(X))$ $\Longrightarrow$ integrate on $\mathcal{M}(X) \subset \mathfrak{M}(X)$ $\Longrightarrow$ Gromov-Witten invariants

What is an obstruction theory?

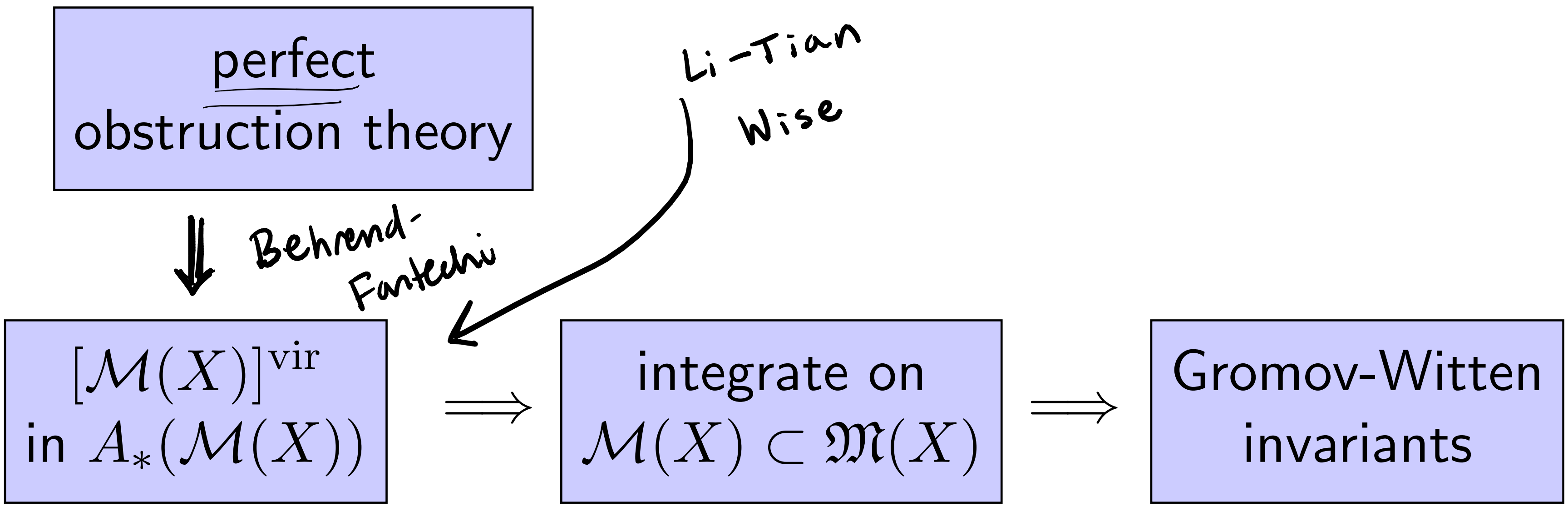
$\mathfrak{M}(X)$ has a canonical obstruction theory



What is an obstruction theory?

- $\phi : E^\bullet \to L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$ in $\mathrm{D_{qc}}^{\leq 1}(\mathfrak{M}(X))$ such that . . .

$\mathfrak{M}(X)$ has a canonical obstruction theory



What is an obstruction theory?

- $\phi : E^\bullet \to L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$ in $\mathrm{D_{qc}}^{\leq 1}(\mathfrak{M}(X))$ such that . . .
- $E^\bullet$ sees deformation theory of $\mathfrak{M}(X)$ encoded in $L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$
- $\phi_{\mathcal{M}(X)}$ is perfect if E is perfect in $[-1,0]$

$\mathfrak{M}(X)$ has a canonical obstruction theory

perfect obstruction theory

$\Downarrow$

$[\mathcal{M}(X)]^{\mathrm{vir}}$ in $A_*(\mathcal{M}(X))$ $\Longrightarrow$ integrate on $\mathcal{M}(X) \subset \mathfrak{M}(X)$ $\Longrightarrow$ Gromov-Witten invariants

What is an obstruction theory?

- $\phi : E^\bullet \to L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$ in ${\mathrm{D}_{\mathrm{qc}}}^{\leq 1}(\mathfrak{M}(X))$ such that . . .
- $E^\bullet$ sees deformation theory of $\mathfrak{M}(X)$ encoded in $L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$
- *Perfect:* $E^\bullet$ is perfect of amplitude $[-1, 1]$

$\mathfrak{M}(X)$ has a canonical obstruction theory

Definition

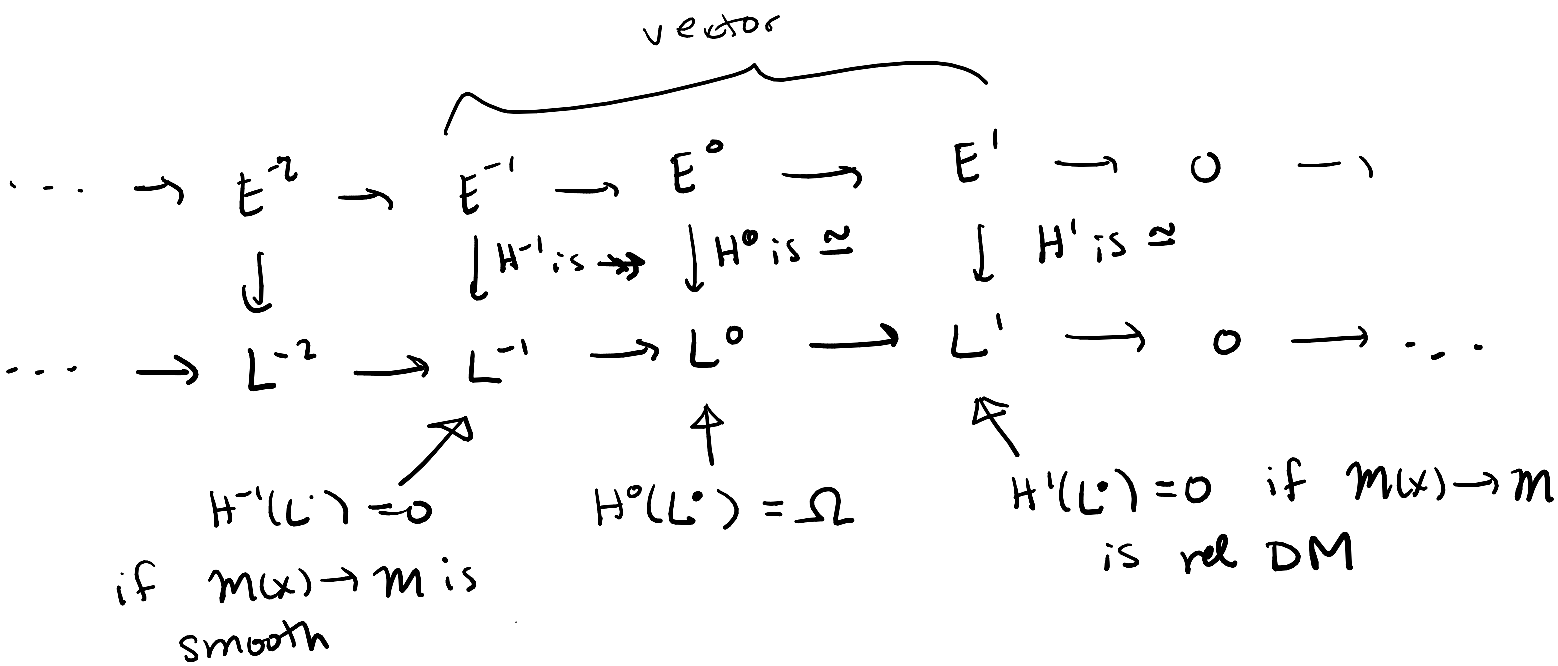
An *obstruction theory* on $\mathfrak{M}(X)$ is a morphism

$$\phi : E^\bullet \to L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$$

in $\mathrm{D}^{\leq 1}_{\mathrm{qc}}(\mathfrak{M}(X))$ such that $H^1(\phi)$ and $H^0(\phi)$ are isomorphisms, and $H^{-1}(\phi)$ is surjective

cotangent complex $\mathfrak{M}(X) \to \mathfrak{M}$

ϕ perfect $\leftrightarrow$ E perfect in [-1,1]



$\mathfrak{M}(X)$ has a canonical obstruction theory

Definition

An *obstruction theory* on $\mathfrak{M}(X)$ is a morphism

$$\phi : E^\bullet \to L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$$

in $\mathrm{D}^{\leq 1}_{\mathrm{qc}}(\mathfrak{M}(X))$ such that $H^1(\phi)$ and $H^0(\phi)$ are isomorphisms, and $H^{-1}(\phi)$ is surjective

Theorem (Behrend-Fantechi) If E is *perfect* and $\mathcal{M} \subset \mathfrak{M}(X)$ is an open Deligne-Mumford substack, separated, and finite type, then it defines a virtual fundamental class $[\mathcal{M}]^{\mathrm{vir}} \in A_*(\mathcal{M})$.

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem

The algebraic stack $\mathfrak{M}(X)$ has a canonical obstruction theory (relative to $\mathfrak{M}$). It is functorial in every way you might hope.

$\mathfrak{M}(X)$ has a canonical obstruction theory

X ↗ BG

Theorem

The algebraic stack $\mathfrak{M}(X)$ has a canonical obstruction theory (relative to $\mathfrak{M}$). It is functorial in every way you might hope.

Application to quasimaps:

Functoriality of DTs: $X_1 \to X_2$ yields $\mathfrak{M}(X_1) \xrightarrow{F} \mathfrak{M}(X_2)$ and a morphism

$$\begin{array}{ccccccc} F^* E_{\mathfrak{M}(X_2)/\mathfrak{M}} & \to & E_{Q/\mathfrak{M}} & \to & E_{Q/\mathrm{Bun}_G} & \to \\ \wr\downarrow \text{OT} & & \downarrow \text{OT} & & \downarrow \text{OT} & \\ F^* L_{\mathfrak{M}(X_2)/\mathfrak{M}} & \to & L_{Q/\mathfrak{M}} & \to & L_{Q/\mathrm{Bun}_G} & \to \end{array}$$

OT rel to curves — OT in the literature

Application: $X_1 = [Y/G]$, $X_2 = BG$

$\mathfrak{M}([Y/G]) \supset$ moduli of quasimaps $= Q$

$\mathfrak{M}(BG) =$ moduli of prin G-bundles on $C \to T$ $= \mathrm{Bun}_G$

$X = BG$

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem
The algebraic stack $\mathfrak{M}(X)$ has a canonical obstruction theory (relative to $\mathfrak{M}$). It is functorial in every way you might hope.

Proof credits:

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem

The algebraic stack $\mathfrak{M}(X)$ has a canonical obstruction theory (relative to $\mathfrak{M}$). It is functorial in every way you might hope.

Proof credits:

- (Behrend-Fantechi, Abramovich-Graber-Vistoli) X and $\mathcal{M}(X) \subset \mathfrak{M}(X)$ are Deligne-Mumford over k characteristic 0

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem
The algebraic stack $\mathfrak{M}(X)$ has a canonical obstruction theory (relative to $\mathfrak{M}$). It is functorial in every way you might hope.

Proof credits:

- (Behrend-Fantechi, Abramovich-Graber-Vistoli) X and $\mathcal{M}(X) \subset \mathfrak{M}(X)$ are Deligne-Mumford over k characteristic 0
- (Schürg, Toën, Vezzosi) Same result, via derived AG

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem
The algebraic stack $\mathfrak{M}(X)$ has a canonical obstruction theory (relative to $\mathfrak{M}$). It is functorial in every way you might hope.

Proof credits:

- (Behrend-Fantechi, Abramovich-Graber-Vistoli) X and $\mathcal{M}(X) \subset \mathfrak{M}(X)$ are Deligne-Mumford over k characteristic 0
- (Schürg, Toën, Vezzosi) Same result, via derived AG
- (Poma) $\mathcal{M}(X) \subset \mathfrak{M}(X)$ is algebraic over a Dedekind domain

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem

The algebraic stack $\mathfrak{M}(X)$ has a canonical obstruction theory (relative to $\mathfrak{M}$). It is functorial in every way you might hope.

Proof credits:

- (Behrend-Fantechi, Abramovich-Graber-Vistoli) X and $\mathcal{M}(X) \subset \mathfrak{M}(X)$ are Deligne-Mumford over k characteristic 0
- (Schürg, Toën, Vezzosi) Same result, via derived AG
- (Poma) $\mathcal{M}(X) \subset \mathfrak{M}(X)$ is algebraic over a Dedekind domain
- (Webb) X and $\mathfrak{M}(X)$ are algebraic over locally Noetherian S

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem

The algebraic stack $\mathfrak{M}(X)$ has a canonical obstruction theory (relative to $\mathfrak{M}$). It is functorial in every way you might hope.

Proof credits:

- (Behrend-Fantechi, Abramovich-Graber-Vistoli) X and $\mathcal{M}(X) \subset \mathfrak{M}(X)$ are Deligne-Mumford over k characteristic 0
- (Schürg, Toën, Vezzosi) Same result, via derived AG
- (Poma) $\mathcal{M}(X) \subset \mathfrak{M}(X)$ is algebraic over a Dedekind domain
- (Webb) X and $\mathfrak{M}(X)$ are algebraic over locally Noetherian S
  - Rigorously construct the dualizing sheaf
  - Clarify why the "obvious" isomorphism $H^i(E^\bullet) \simeq H^i(L^\bullet)$ for $i = 0, 1$ is induced by ϕ

$\mathfrak{M}(X)$ has a canonical obstruction theory

Definition

An *obstruction theory* on $\mathfrak{M}(X)$ is $\phi : E^\bullet \to L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$ inducing an isomorphism on cohomology in degrees 0, 1 and a surjection in degree -1.

$\mathfrak{M}(X)$ has a canonical obstruction theory

Definition

An *obstruction theory* on $\mathfrak{M}(X)$ is $\phi : E^\bullet \to L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$ inducing an isomorphism on cohomology in degrees 0, 1 and a surjection in degree -1.

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \mathcal{C}_{\mathfrak{M}(X)} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \pi & & \\ \mathfrak{M} & \longleftarrow & \mathfrak{M}(X) & & \end{array}$$

$$f^* \mathbb{L}_X \longrightarrow \mathbb{L}_{\mathcal{C}_{\mathfrak{M}(X)}/\mathcal{C}} \xleftarrow{\sim} \pi^* L_{\mathfrak{M}(X)/\mathfrak{M}}$$

$X_1 \to X_2$, $\mathfrak{M}(X_2)$, $\mathbb{L}_{X_1/X_2}$

$\mathcal{F}$

Need to define: Functorial (ω, tr)

$$R\pi_*(f^*\mathbb{L}_X \otimes \omega) \to R\pi_*(\pi^* L_{\mathfrak{M}(X)/\mathfrak{M}} \otimes \omega) \simeq L_{\mathfrak{M}(X)/\mathfrak{M}} \otimes R\pi_*\omega \to L_{\mathfrak{M}(X)/\mathfrak{M}}$$

What we wish we could do

$\mathcal{F} \longrightarrow \pi^* L_{\mathfrak{M}(X)/\mathfrak{M}}$ $\quad (*)$

- $R\pi_*$ has a right adjoint $\pi^!$ $\quad \omega := \pi^! \mathcal{O}_{\mathfrak{M}}$
- So π^* has a left adjoint $R^\omega_{\pi_*} := R\pi_*(- \otimes \omega)$

$R\pi^\omega_* \mathcal{F} \xrightarrow{\phi^{naive}} L_{\mathfrak{M}(X)/\mathfrak{M}}$ adjoint to $(*)$

works when $\mathfrak{M}(X)$ is not too "big"

also: can't show ϕ^{naive} is functorial b/c $\pi^!$ isn't known to be preserved by arbitrary basechange

$tr : R\pi_* \omega \to \mathcal{O}$

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem (W.)

For every tame curve $\pi : \mathcal{C} \to \mathcal{M}$ there is a pair (ω, tr) where ω is locally free in degree -1 and $tr : R\pi_*\omega \to \mathcal{O}_{\mathcal{M}}$, such that

1. the pair is preserved by arbitrary base change ✓
2. if $\mathcal{M}$ is a quasi-separated Noetherian algebraic space, then $\omega = \pi^!\mathcal{O}_{\mathcal{M}}$ and tr is the counit ✓

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem (W.)

For every tame curve $\pi : \mathcal{C} \to \mathcal{M}$ there is a pair (ω, tr) where ω is locally free in degree -1 and $tr : R\pi_*\omega \to \mathcal{O}_{\mathcal{M}}$, such that

1. the pair is preserved by arbitrary base change
2. if $\mathcal{M}$ is a quasi-separated Noetherian algebraic space, then $\omega = \pi^!\mathcal{O}_{\mathcal{M}}$ and tr is the counit

Proof contents:

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem (W.)

For every tame curve $\pi : \mathcal{C} \to \mathcal{M}$ there is a pair (ω, tr) where ω is locally free in degree -1 and $tr : R\pi_*\omega \to \mathcal{O}_{\mathcal{M}}$, such that

1. the pair is preserved by arbitrary base change
2. if $\mathcal{M}$ is a quasi-separated Noetherian algebraic space, then $\omega = \pi^!\mathcal{O}_{\mathcal{M}}$ and tr is the counit

Proof contents:

- When $\mathcal{M}$ is an affine scheme, prove that $\pi^!\mathcal{O}_{\mathcal{M}}$ is locally free and $\pi^!$ has basechange

$\mathfrak{M}(X)$ has a canonical obstruction theory

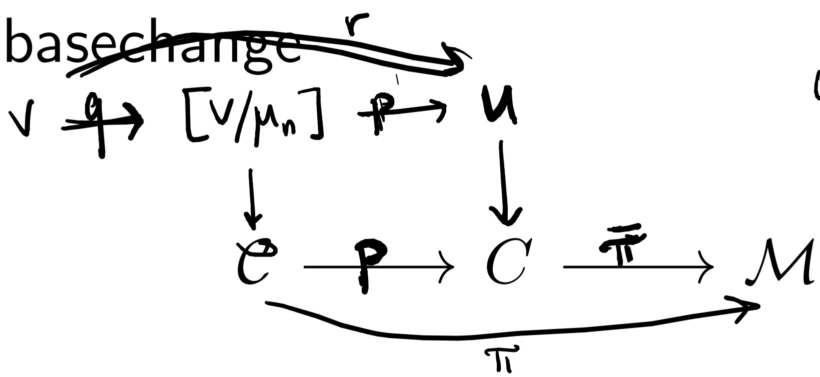
Theorem (W.)

For every tame curve $\pi : \mathcal{C} \to \mathcal{M}$ there is a pair (ω, tr) where ω is locally free in degree -1 and $tr : R\pi_*\omega \to \mathcal{O}_{\mathcal{M}}$, such that

1. the pair is preserved by arbitrary base change
2. if $\mathcal{M}$ is a quasi-separated Noetherian algebraic space, then $\omega = \pi^!\mathcal{O}_{\mathcal{M}}$ and tr is the counit

Proof contents:

- When $\mathcal{M}$ is an affine scheme, prove that $\pi^!\mathcal{O}_{\mathcal{M}}$ is locally free and $\pi^!$ has basechange



$\omega = \pi^!\mathcal{O}_{\mathcal{M}} = p^!\bar{\pi}^!\mathcal{O}_{\mathcal{M}}$

$= p^*(\bar{\pi}^!\mathcal{O}_{\mathcal{M}}) \otimes p^!\mathcal{O}_C$

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem (W.)

For every tame curve $\pi : \mathcal{C} \to \mathcal{M}$ there is a pair (ω, tr) where ω is locally free in degree -1 and $tr : R\pi_*\omega \to \mathcal{O}_{\mathcal{M}}$, such that

1. the pair is preserved by arbitrary base change
2. if $\mathcal{M}$ is a quasi-separated Noetherian algebraic space, then $\omega = \pi^!\mathcal{O}_{\mathcal{M}}$ and tr is the counit

Proof contents:

- When $\mathcal{M}$ is an affine scheme, prove that $\pi^!\mathcal{O}_{\mathcal{M}}$ is locally free and $\pi^!$ has basechange

$$\begin{array}{ccccccc} V & \longrightarrow & [V/\mu_n] & \longrightarrow & U & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{C} & \longrightarrow & C & \longrightarrow & \mathcal{M} \end{array}$$

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem (W.)

For every tame curve $\pi : \mathcal{C} \to \mathcal{M}$ there is a pair (ω, tr) where ω is locally free in degree -1 and $tr : R\pi_*\omega \to \mathcal{O}_{\mathcal{M}}$, such that

1. the pair is preserved by arbitrary base change
2. if $\mathcal{M}$ is a quasi-separated Noetherian algebraic space, then $\omega = \pi^!\mathcal{O}_{\mathcal{M}}$ and tr is the counit

Proof contents:

- When $\mathcal{M}$ is an affine scheme, prove that $\pi^!\mathcal{O}_{\mathcal{M}}$ is locally free and $\pi^!$ has basechange

$$\begin{array}{ccccccc} V & \longrightarrow & [V/\mu_n] & \longrightarrow & U & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{C} & \longrightarrow & C & \longrightarrow & \mathcal{M} \end{array}$$

- Glue these smooth-local objects on the algebraic stacks $\mathcal{C}, \mathcal{M}$

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem (W.)

The algebraic stack $\mathfrak{M}(X)$ has a canonical obstruction theory given by $\phi : R\pi_*(f^*L_X^\bullet \otimes \omega) \to L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$.

To show that $H^i(\phi)$ is an isomorphism for $i = 0, 1$:

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem (W.)

The algebraic stack $\mathfrak{M}(X)$ has a canonical obstruction theory given by $\phi : R\pi_*(f^*L_X^\bullet \otimes \omega) \to L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$.

To show that $H^i(\phi)$ is an isomorphism for $i = 0, 1$:

- Step 1: reduce to the following local statement:
- For $T \xrightarrow{g} \mathfrak{M}(X)$ and I defining a square-zero extension

$\mathfrak{M}(X)$ has a canonical obstruction theory

Theorem (W.)

The algebraic stack $\mathfrak{M}(X)$ has a canonical obstruction theory given by $\phi : R\pi_*(f^* L_X^\bullet \otimes \omega) \to L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}$.

also H^{-1} is surj…

To show that $H^i(\phi)$ is an isomorphism for $i = 0, 1$:

- Step 1: reduce to the following local statement:
- For $T \xrightarrow{g} \mathfrak{M}(X)$ and I defining a square-zero extension
- $\phi : \mathrm{Ext}^i(g^* L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}, I) \underset{\phi}{\xrightarrow{\sim}} \mathrm{Ext}^i(g^* E^\bullet, I)$ for $i = 0, -1$

Step 2: For $i = 0, -1$, interpret

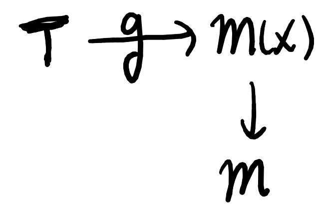
$$\phi : \mathrm{Ext}^i(g^* L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}, I) \xrightarrow{\sim} \mathrm{Ext}^i(g^* E^\bullet, I)$$

as a morphism of deformation categories.

Step 2: For $i = 0, -1$, interpret

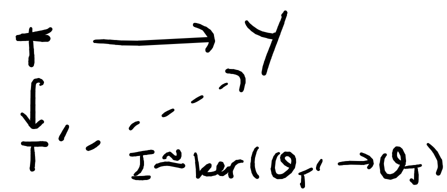
$$\phi : \mathrm{Ext}^i(g^* L^\bullet_{\mathfrak{M}(X)/\mathfrak{M}}, I) \xrightarrow{\sim} \mathrm{Ext}^i(g^* E^\bullet, I)$$

as a morphism of deformation categories.



For $T \longrightarrow Y$ representable

Theorem (Illusie, Olsson) $\quad \mathrm{Exal}_Y(T, I) \simeq \mathrm{Ext}^{0/-1}(L^\bullet_{T/Y}, I[1])$



objects = $\mathrm{Ext}^0(L^\bullet_{T/Y}, I[1])$

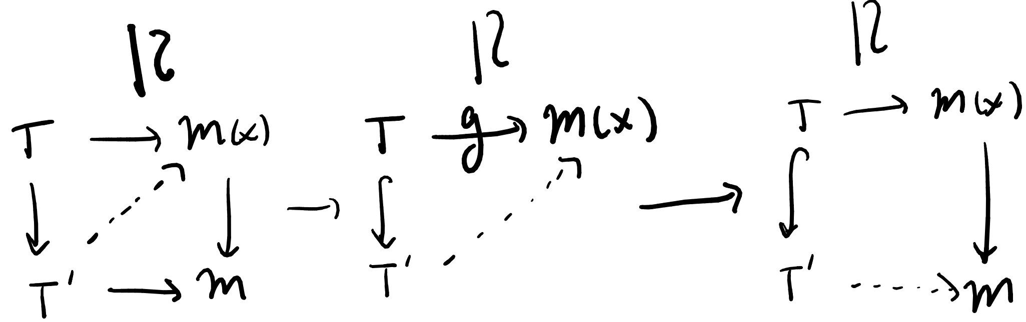
morphs = $\mathrm{Ext}^{-1}(L^\bullet_{T/Y}, I[1])$

Recall the DT

$g^* L_{\mathfrak{M}(X)/\mathfrak{M}} \to L_{T/\mathfrak{M}} \longrightarrow L_{T/\mathfrak{M}(X)} \longrightarrow$

yields

$E^{0/-1}(g^* L_{\mathfrak{M}(X)/\mathfrak{M}}, I) \to E^{0/-1}(L_{T/\mathfrak{M}(X)}, I[1]) \to E^{0/-1}(L_{T/\mathfrak{M}}, I[1])$

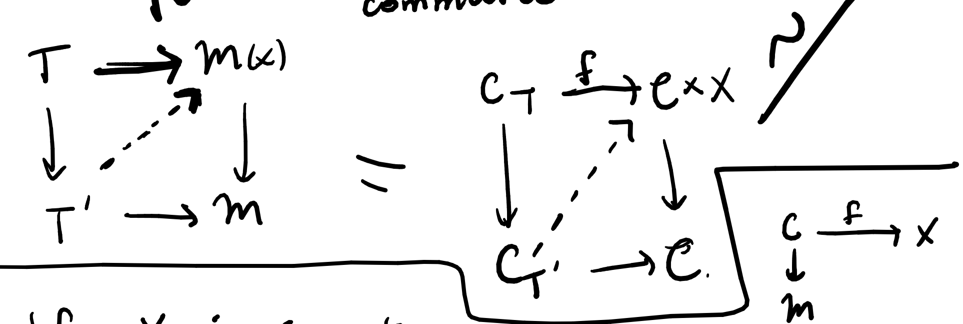


$E^{0/-1}(g^* L_{\mathfrak{M}(X)/\mathfrak{M}}, I) \xrightarrow{\phi} E^{0/-1}(g^* E, I)$

$R\pi^\omega_* f^* L_X$

" $E^{0/-1}(f^* L_X, \pi^* I)$

Thm (W) this commutes



If X is smooth variety:

$E = R\pi_*(f^* L_X \otimes \omega^\bullet)$ is perfect in $[-1, 0]$

(loc free in deg 0 under $f^* L_X$; loc free in deg 0 over $f^* L_X \otimes \omega^\bullet$)

Thank you.