Virtual Coulomb branch and quantum K-theory

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3d $\mathcal{N} = 4$ theory

G: complex reductive group; N: G-representation

In physics, the pair (G, T^*N) defines a 3d $\mathcal{N} = 4$ supersymmetric gauge theory.

- The theory admits two interesting components of moduli space of vacua: *Higgs branch* and *Coulomb branch*.
- The theory is parameterized by two families of parameters: *FI parameters* and *mass parameters*.

3d $\mathcal{N} = 4$ Higgs branch

The Higgs branch is the holomorphic symplectic quotient:

$$X := \mu^{-1}(0)//_\theta G,$$

where $\mu : T^*N \to \mathfrak{g}^*$ is the moment map, and $\theta \in \mathrm{char}(G)$ is a stability condition.

When θ is generic, i.e. $\mu^{-1}(0)^{ss} = \mu^{-1}(0)^s$, X is smooth.

Usually, there is a flavor symmetry T acting on N, commuting with G. Equivariant parameters in $K_T(\mathrm{pt})$ are the mass parameters.

3d $\mathcal{N} = 2$ Higgs branch

The pair (G, N) (instead of T^*N) gives a 3d $\mathcal{N} = 2$ theory.

Its Higgs branch is the GIT quotient

$$Y := N /\!/_\theta G.$$

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$)

G acts on N by weights $(1, \cdots, 1)$, and on T^*N by $(1, \cdots, 1, -1, \cdots, -1)$.

- Higgs branch: $T^*\mathbb{P}^n$

  $\mu : T^*\mathbb{C}^{n+1} \to \mathbb{C}$ is $\mu(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}$. Choice $\theta > 0$ implies $\vec{x} \neq 0$.

- flavor symmetry $T = (\mathbb{C}^*)^{n+1}$, $K_T(\mathrm{pt}) = \mathbb{C}[a_1^{\pm 1}, \cdots, a_{n+1}^{\pm 1}]$.

  Another torus $\mathbb{C}^*_\hbar$ scales the cotangent fiber.

- 3d $\mathcal{N} = 2$ Higgs branch: $\mathbb{P}^n$

Enumerative geometry: quasimaps and vertex function

Definition (Ciocan-Fontanine–Kim–Maulik)

- A quasimap from $\mathbb{P}^1$ to the Higgs branch $X = \mu^{-1}(0)//_\theta G$ is a map to the stacky quotient

$$f : \mathbb{P}^1 \to \mathfrak{X} = [\mu^{-1}(0)/G]$$

which maps generically into the stable locus X.

- Alternatively, it consists of a principal G-bundle $\mathcal{P}$ over $\mathbb{P}^1$, together with a section s of the bundle $\mathcal{P} \times_G T^*N$, which satisfies the moment map equation $\mu(s) = 0$, and takes values generically in the stable locus $\mu^{-1}(0)^s$.
- Quasimaps to 3d $\mathcal{N} = 2$ Higgs branch $Y = N//_\theta G$ are similar.

$\mathrm{QM}^{\circ}_d(X)$: open substack where $\infty \in \mathbb{P}^1$ is not a base point.

$\mathrm{ev}_\infty : \mathrm{QM}^{\circ}_d(X) \to X,\ f \mapsto f(\infty)$.
$\mathrm{ev}_0 : \mathrm{QM}^{\circ}_d(X) \to \mathfrak{X} = [\mu^{-1}(0)/G]$.

Let $\mathbb{C}^*_q$ scales $\mathbb{P}^1$, $q := T_0\mathbb{P}^1 \in K_{\mathbb{C}^*_q}(\mathrm{pt})$.

Definition (Ciocan-Fontanine–Kim, A. Okounkov)

Descendent vertex function

$$V^{(\tau(s))}(Q) := \sum_\beta Q^\beta \,\mathrm{ev}_{\infty *}(\widehat{\mathcal{O}}_{\mathrm{vir}} \cdot \mathrm{ev}_0^* \,\tau(s)) \in K_{T\times\mathbb{C}^*_\hbar\times\mathbb{C}^*_q}(X)_{loc}[[Q]],$$

(K-theoretic big I-function, for 3d $\mathcal{N}=2$ Higgs branch Y)

$\tau(s) \in K_{T\times\mathbb{C}^*_\hbar}(\mathfrak{X}) = K_{G\times T\times\mathbb{C}^*_\hbar}(\mathrm{pt})$; Q: Kähler parameters;

loc: pass to fraction field of $K_{T\times\mathbb{C}^*_\hbar\times\mathbb{C}^*_q}(\mathrm{pt})$.

Quantum q-difference module

Descendent vertex functions

$$\widetilde{V}^{(\tau(s))}(Q) := e^{\frac{\langle \ln S, \ln Q\rangle}{\ln q}} \cdot V^{(\tau(s))}(Q), \quad \tau(s) \in K_{T\times\mathbb{C}^*_\hbar}(\mathfrak{X})$$

form a quantum q-difference module of rank $\mathrm{rk}\, K(X)$.

Lemma

$$q^{\chi Q\partial_Q}\widetilde{V}^{(\tau(s))}(Q) = \widetilde{V}^{(s^\chi\cdot\tau(s))}(Q).$$

$q^{\chi Q\partial_Q} Q^d = q^{\langle \chi, d\rangle} Q^d$, $\chi \in \mathrm{char}(G)$, $d \in \mathrm{cochar}(G)$,

S^χ: tautological line bundle associated with s^χ (image under Kirwan surjection $K_{T\times\mathbb{C}^*_\hbar}(\mathfrak{X}) \to K_{T\times\mathbb{C}^*_\hbar}(X)$).

Bethe algebra / quantum K-ring

- $q \to 1$ limit of quantum q-difference module gives the Bethe algebra / quantum K-ring.

  (analogous to Givental's quantum K-theory)

- This is a deformation of the usual K-ring $K_{T\times\mathbb{C}^*_\hbar}(X)$ over $\mathbb{C}[[Q^{\mathrm{Eff}(X)}]]$.

- It can be defined in terms of certain 3-point functions counting relative quasimaps.

Physics

- Vertex functions: partition functions on $S^1 \times_q D$; holomorphic blocks; vortex partition function
- Desendents: line operators
- Bethe algebra/quantum K-ring: Wilson loop algebra; chiral algebra
- Operators r_d in quantized Coulomb branch: monopole operators

BFN construction

$\mathcal{K} = \mathbb{C}((z))$, $\mathcal{O} = \mathbb{C}[[z]]$, $D = \operatorname{Spec} \mathcal{O}$, $D^* = \operatorname{Spec} \mathcal{K}$.

- Affine Grassmannian

$$\begin{aligned} Gr_G &= \{(P, \varphi) \mid P : G\text{-bundle over } D,\ \varphi : P|_{D^*} \cong D^* \times G\}/\sim \\ &= G_{\mathcal{K}}/G_{\mathcal{O}} \end{aligned}$$

- Gr_G admits a $G_{\mathcal{O}}$-action from the left.
- There is a convolution product $m : Gr_G \widetilde{\times} Gr_G \to Gr_G$, defined by composing the trivializations.

BFN construction

- Moduli of triples

$$\mathcal{T} := \{(P, \varphi, s) \mid (P, \varphi) \in Gr_G,\ s \in H^0(D, N_{\mathcal{O}})\}$$

$$\mathcal{R} := \{(P, \varphi, s) \in \mathcal{T} \mid \varphi(s|_{D^*}) \text{ extend over } D\},$$

where $N_{\mathcal{O}} := P \times_G N$ is the associated bundle.

- $\mathcal{T}$ is a (∞-rank) vector bundle over Gr_G, and hence smooth over Gr_G.

  $\mathcal{R}$ is not smooth over Gr_G, unless G abelian.

Convolution diagram

Intuitively, $\mathcal{R}$ "acts" on $\mathcal{T}$ from the right.

$$\begin{array}{ccccccc}
\mathcal{R}\times\mathcal{R} & \longleftarrow & p^{-1}(\mathcal{R}\times\mathcal{R}) & \longrightarrow & q(p^{-1}(\mathcal{R}\times\mathcal{R})) & \longrightarrow & \mathcal{R} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{T}\times\mathcal{R} & \xleftarrow{p} & G_{\mathcal{K}}\times\mathcal{R} & \xrightarrow{q} & G_{\mathcal{K}}\times_{G_{\mathcal{O}}}\mathcal{R} & \xrightarrow{m} & \mathcal{T}
\end{array}$$

A convolution product can be defined via

$$m_* \circ (q^*)^{-1} \circ p^!$$

Theorem (Braverman–Finkelberg–Nakajima)

The equivariant K-theory $K_0^{G_{\mathcal{O}} \rtimes \mathbb{C}_q^*}(\mathcal{R})$ admits a convolution product $*$, which is associative, and $K_{G \times \mathbb{C}_q^*}(\mathrm{pt})$-linear in the first variable. It is commutative when $q \to 1$.

Definition (BFN)

- The algebra $\mathcal{A}(G, N) = K_0^{G_{\mathcal{O}} \rtimes \mathbb{C}_q^*}(\mathcal{R})$ is defined as the quantized K-theoretic Coulomb branch.
- $\mathrm{Spec}\, K_0^{G_{\mathcal{O}}}(\mathcal{R})$ is defined as the classical K-theoretic Coulomb branch.

The commutative subalgebra $K_{G \times \mathbb{C}_q^*}(\mathrm{pt})$ is called the Cartan subalgebra.

Abelian case

When $G = (\mathbb{C}^*)^k$ is abelian, there is an explicit presentation.

- $Gr_G = \mathbb{Z}^k = \{[z^d], d \in \mathrm{cochar}(G)\}$, $K_{G\times\mathbb{C}_q^*}(\mathrm{pt}) = \mathbb{C}[q^{\pm 1}, s^\chi, \chi \in \mathrm{char}(G)]$.
- Let r_d be the structure sheaf of $\mathcal{R}$ over $[z^d]$.
- $\mathcal{A}(G, N)$ is generated by r_d and s^χ over $\mathbb{C}[q^{\pm 1}]$.
- $r_d s^\chi = q^{-\langle \chi, d\rangle} s^\chi r_d$.
- There is a grading $\mathcal{A} = \bigoplus_{d\in\mathrm{cochar}(G)} \mathcal{A}^d$, where $\mathcal{A}^0 = K_{G\times\mathbb{C}_q^*}(\mathrm{pt})$, $\mathcal{A}^d = K_{G\times\mathbb{C}_q^*}(\mathrm{pt}) \cdot r_d$.
- One can add flavor symmetry $T = (\mathbb{C}^*)^n$, if $\dim N = n$.

BFN construction of Coulomb branch

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$)

- $Gr_G = \mathbb{C}((z))/\mathbb{C}[[z]]^* = \{[z^d] \mid d \in \mathbb{Z}\}$.
- Convolution product $[z^{d_1}] * [z^{d_2}] = [z^{d_1+d_2}]$.
- $\mathcal{T} = \bigsqcup_d [z^d] \times N[[z]]/\mathbb{C}[[z]]^*$.
- $\mathcal{R} = \bigsqcup_d [z^d] \times (N[[z]] \cap z^d N[[z]]/\mathbb{C}[[z]]^*$.
- Convolution product.

  Fibers for $d > 0$, $\mathcal{R}_d = [z^d] \times z^d N[[z]]$, $\mathcal{R}_{-d} = [z^{-d}] \times N[[z]]$.

  Apply convolution and intersection, $\rightsquigarrow [z^0] \times z^d N[[z]]$.

  Compare with $\mathcal{R}_0 = [z^0] \times N[[z]]$.

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$)

- Quantized Coulomb branch: generated by $s^{\pm 1}, r_1, r_{-1}$, such that $r_{\pm d} = r_{\pm 1}^d$ for $d > 0$, $r_d s = q^{-d} s r_d$, and

$$r_{-d} \cdot r_d = \prod_{i=1}^{n+1} (1 - q a_i s) \cdots (1 - q^d a_i s), \qquad d \geq 0$$

$K_{G \times \mathbb{C}_q^*}(\mathrm{pt}) = \mathbb{C}[s^{\pm 1}, q^{\pm 1}]$.

- $r_{-d} \cdot r_d$ is essentially computing the K-theoretic Euler class of $\mathcal{R}$ over $[z^d]$.
- Classical Coulomb branch:

$$\operatorname{Spec} \mathbb{C}[a_i^{\pm 1}, s^{\pm 1}, r_1, r_{-1}] / \langle r_{-1} \cdot r_1 - \textstyle\prod_{i=1}^{n+1} (1 - a_i s) \rangle.$$

Deformation of A_n-singularity $\mathbb{C}^2 / \mathbb{Z}_{n+1}$ (singular when some a_i's are equal).

Motivation

A. Braverman's work.

- There is a $\mathfrak{g}$-action on the intersection cohomology of moduli spaces of (Drinfeld's) quasimaps into G/B.
- The resulting representation is a Verma module of $\mathfrak{g}$.
- J-function of G/B can be expressed as Whittaker function.
- Whittaker function $\rightsquigarrow$ quantum Toda system.
- There's also a K-theoretic version.

Motivation from physics

Bullimore–Dimofte–Gaiotto–Hilburn–Kim ('16), "Vortices and Vermas":

- monopole operators (quantized (homological) Coulomb branch) acts on the homology of the vortex moduli space (quasimaps), with target space $N//_\theta G$;

  (3d $\mathcal{N} = 2$ Higgs branch)
- the resulting representation is a Verma module of the quantized Coulomb branch;
- generating function of quasimap counting into $N//_\theta G$ can be expressed as generalized characters of the Verma module;
- quantum differential equation can be obtained.

Quasimaps to 3d $\mathcal{N}=2$ Higgs branch

Example ($G=\mathbb{C}^*$, $N=\mathbb{C}^{n+1}$, 3d $\mathcal{N}=2$)

- 3d $\mathcal{N}=2$ Higgs branch: $N//_{\theta>0}G=\mathbb{P}^n$.
- A quasimap f from $\mathbb{P}^1$ to $\mathbb{P}^n$ is (L,s), where L is a line bundle on $\mathbb{P}^1$, and s is a section of $L^{\oplus(n+1)}$, such that $s\neq 0$ generically on $\mathbb{P}^1$.
- Moduli space of quasimaps of degree d is $\mathrm{QM}(\mathbb{P}^n,d)=\mathbb{P}H^0(\mathbb{P}^1,\mathcal{O}(d)^{\oplus(n+1)})$.
- At a point f, its tangent space is the deformation space of quasimaps

$$H^0(\mathbb{P}^1,\mathcal{O}(d)^{\oplus(n+1)})-H^0(\mathbb{P}^1,\mathcal{O}).$$

No obstruction, since $H^1(\mathcal{O}(d))=0$ always as $d\geq 0$.

Action on quasimaps

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$)

- Apply $\mathbb{C}_q^*$-action. $\mathcal{T}^{\mathbb{C}_q^*} = \bigsqcup_d [z^d] \times N$
- If we restrict to stable locus N^s and $d \geq 0$, these are the $\mathbb{C}_q^*$-equivariant quasimaps.
- r_d acts by "changing the quasimap locally at 0 by degree d".
- $\bigoplus_{d \geq 0} K(\mathrm{QM}_d(\mathbb{P}^n))$ is a "Verma module" of

$$\mathcal{A} = \bigoplus_{d \in \mathbb{Z}} \mathcal{A}^d = \bigoplus_{d \in \mathbb{Z}} K_{G \times \mathbb{C}_q^* \times T}(\mathrm{pt}) \cdot r_d.$$

Action on quasimaps

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$, 3d $\mathcal{N} = 2$)

- The 3d $\mathcal{N} = 2$ I-function is

$$\sum_{d \geq 0} \frac{1}{\prod_{i=1}^{n+1} (1 - q a_i S) \cdots (1 - q^d a_i S)} \cdot Q^d$$

  S: tautological line bundle on $\mathbb{P}^n$ (image of s under the Kirwan surjection);

- The denominator comes from K-theoretic Euler class of the deformation space, which resembles RHS of $r_{-d} \cdot r_d$.

Idea

- Now: quantized Coulomb branch acts on K-theory of 3d $\mathcal{N} = 2$ Higgs branch.
- Question: why 3d $\mathcal{N} = 2$?
- The Coulomb branch comes from a 3d $\mathcal{N} = 4$ theory. We may expect it acts on the K-theory of the original 3d $\mathcal{N} = 4$ Higgs branch.
- For moduli spaces: same.

  For vertex functions/I-functions: different.

Quasimaps into 3d $\mathcal{N} = 4$ Higgs branch

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$, 3d $\mathcal{N} = 4$ theory)

- 3d $\mathcal{N} = 4$ Higgs branch: $\mu^{-1}(0)//_{\theta>0}G = T^*\mathbb{P}^n$.
- Quasimaps: same as 3d $\mathcal{N} = 2$ theory (for $d \neq 0$)!
  Moduli space is still $\mathrm{QM}_d(T^*\mathbb{P}^n) = \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}(d)^{\oplus(n+1)})$.
- However, in enumerative geometry we count virtually. The deformation-obstruction theory is now (for $d > 0$)

$$H^\bullet(\mathbb{P}^1, \mathcal{O}(d)^{\oplus(n+1)} \oplus \hbar^{-1}\mathcal{O}(-d)^{\oplus(n+1)}) - H^\bullet(\mathbb{P}^1, \mathcal{O} \oplus \hbar^{-1}\mathcal{O})$$

  i.e. deformation $H^0(\mathbb{P}^1, \mathcal{O}(d)^{\oplus(n+1)}) - H^0(\mathbb{P}^1, \mathcal{O} \oplus \hbar^{-1}\mathcal{O})$,
  obstruction $H^1(\mathbb{P}^1, \hbar^{-1}\mathcal{O}(-d)^{\oplus(n+1)})$.

Quasimaps into 3d $\mathcal{N} = 4$ Higgs branch

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$, 3d $\mathcal{N} = 4$ theory)

The vertex function is now (after some extra modification)

$$\sum_{d \geq 0} (-q^{1/2}\hbar^{-1/2})^{(n+1)d} \prod_{i=1}^{n+1} \frac{(1-\hbar a_i S)\cdots(1-\hbar q^{d-1} a_i S)}{(1-q a_i S)\cdots(1-q^d a_i S)} \cdot Q^d.$$

- Question: how does the numerator (i.e. obstruction part) emerge from the Coulomb branch?
- Idea: use the same moduli space of triples $\mathcal{R}$; introduce nontrivial obstruction theory, and apply virtual intersection in convolution product.

Recall: Gysin pullback and intersection theory

- Given a regular embedding $i : X \hookrightarrow Y$, there is a Gysin pullback $i^! : K(Y) \to K(X)$.
- $i^! [\mathcal{O}_Y] = [\mathcal{O}_X]$.
- $i^! i_* = \bigwedge^\bullet (N^\vee_{X/Y})$.
- Given a smooth variety X, the intersection product is defined via the Gysin pullback of the diagonal embedding $\Delta : X \hookrightarrow X \times X$.

Virtual Gysin pullback

- Introduce obstruction theories $E_X^\bullet = \Omega_X \oplus \hbar\Omega_X^\vee[1]$, and $E_Y^\bullet$ similarly.
- The complex $E_i^\bullet = N_{X/Y}[1] \oplus \hbar N_{X/Y}^\vee[2]$ is a relative obstruction theory of the morphism i, which form a compatible triple with $E_X^\bullet$, $E_Y^\bullet$, but not perfect (it lies in $[-2,-1]$).
- Define the virtual Gysin pullback as $i^!_{\mathrm{vir}} := \dfrac{i^!}{\bigwedge^\bullet(\hbar^{-1}N_{X/Y})}$.
- $i^!_{\mathrm{vir}}\mathcal{O}_Y^{\mathrm{vir}} = \mathcal{O}_X^{\mathrm{vir}}$.
- This is beyond the usual virtual pullback [C. Manolache][F. Qu].

Virtual convolution product

Recall the diagram in [BFN]:

$$\begin{array}{ccccccc}
\mathcal{R}\times\mathcal{R} & \longleftarrow & p^{-1}(\mathcal{R}\times\mathcal{R}) & \longrightarrow & q(p^{-1}(\mathcal{R}\times\mathcal{R})) & \longrightarrow & \mathcal{R} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{T}\times\mathcal{R} & \longleftarrow & G_{\mathcal{K}}\times\mathcal{R} & \longrightarrow & G_{\mathcal{K}}\times_{G_{\mathcal{O}}}\mathcal{R} & \longrightarrow & \mathcal{T} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{T}\times Gr_G\times N_{\mathcal{O}} & \xleftarrow{p} & G_{\mathcal{K}}\times Gr_G\times N_{\mathcal{O}} & \xrightarrow{q} & G_{\mathcal{K}}\times_{G_{\mathcal{O}}}(Gr_G\times N_{\mathcal{O}}) & \xrightarrow{m} & Gr_G\times N_{\mathcal{K}}
\end{array}$$

The map p factorizes as

$$\mathcal{T}\times Gr_G\times N_{\mathcal{O}} \xleftarrow{p'} G_{\mathcal{K}}\times N_{\mathcal{O}}\times Gr_G\times N_{\mathcal{O}} \xleftarrow{\Delta} G_{\mathcal{K}}\times Gr_G\times N_{\mathcal{O}}$$

where p' is smooth and Δ is a regular embedding.

Virtual convolution product

The virtual convolution product is defined by the following steps.

- For the 3rd row of the diagram, where each space is smooth over Gr_G, replace the usual Ω by the perfect obstruction theory $\Omega \oplus \hbar\Omega^\vee[1]$ (all relative over Gr_G).
- Replace smooth pullback $(p')^*$, q^* by the usual virtual pullback [C. Manolache] [F. Qu].
- Replace the Gysin pullback $\Delta^!$ by the virtual Gysin pullback. Some localization of coefficients is needed.

Virtual Coulomb branch

Theorem (Z. '21)

The virtual convolution product is associative and $K_{G\rtimes\mathbb{C}_q^*}(\mathrm{pt})$-linear in the first variable. It is commutative when $q\to 1$.

Definition

The *K-theoretic quantized virtual Coulomb branch* is defined as $K_0^{G\rtimes\mathbb{C}_q^*\times\mathbb{C}_\hbar^*\times T}(\mathcal{R})$, with the virtual convolution product (with some modification).

When G is abelian, there exists explicit presentation of the generators and relations.

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$)

- Quantized virtual Coulomb branch: generated by $s^{\pm 1}, r_1, r_{-1}$, such that $r_{\pm d} = r_{\pm 1}^d$ for $d > 0$, $r_d s = q^{-d} s r_d$, and

$$r_{-d} \cdot r_d = \prod_{i=1}^{n+1} (-q^{1/2}\hbar^{-1/2})^{-d} \frac{(1 - qa_i s) \cdots (1 - q^d a_i s)}{(1 - \hbar a_i s) \cdots (1 - q^{d-1} \hbar a_i s)}$$

  $d \geq 0$, $K_{G \times \mathbb{C}_q^*}(\mathrm{pt}) = \mathbb{C}[s^{\pm 1}, q^{\pm 1}]$.

- r_d is the virtual structure sheaf of $\mathcal{R}$ over $[z^d]$. The relation $r_{-d} \cdot r_d$ is essentially computing the virtual tangent bundle of K-theoretic Euler class of $\mathcal{R}$ over $[z^d]$.

- Need to invert $1 - q^{\mathbb{Z}} \hbar a_i s$.

Verma module

G: abelian (Higgs branch X is a hypertoric variety);

$\mathrm{p} \in X^T$ defines a character of the Cartan $\mathcal{A}^0$, $s^\chi \mapsto S^\chi|_\mathrm{p}$.

$\mathrm{Eff}(\mathrm{p})$: effective cone of quasimaps into p.

$\mathcal{A}_\mathrm{p} = \bigoplus_d \mathcal{A}_\mathrm{p}^d$: certain localized version of virtual Coulomb branch.

The Verma module $M(\mathrm{p})$ of $\mathcal{A}_\mathrm{p}$ is generated by $\mathcal{A}_\mathrm{p}^d$ for $d \in \mathrm{Eff}(\mathrm{p})$, acting on a highest weight vector v:

$$s^\chi \cdot v = S^\chi|_\mathrm{p} \cdot v, \qquad \mathcal{A}_\mathrm{p}^{-d} \cdot v = 0, \qquad d \in \mathrm{Eff}(\mathrm{p}).$$

Theorem (Z. 21')

$\mathcal{A}_\mathrm{p}$ acts on $\bigoplus_{d \in \mathrm{Eff}(\mathrm{p})} K_{\mathsf{T} \times \mathbb{C}^*_\hbar \times \mathbb{C}^*_q}(\mathrm{QM}_d(X;\mathrm{p})^\circ)_{loc}$, realizing it as the Verma module $M(\mathrm{p})$.

Whittaker function

A Whittaker vector $w_{\mathrm{p}}(Q) \in M(\mathrm{p})[[Q^{1/2\,\mathrm{Eff}(\mathrm{p})}]]$ of $\mathcal{A}_{\mathrm{p}}$ is defined as

$$\mathfrak{r}_{-d} w_{\mathrm{p}}(Q) = Q^{d/2} w_{\mathrm{p}}(Q), \qquad d \in \mathrm{Eff}(\mathrm{p}).$$

$\mathfrak{r}_d$: generators in $\mathcal{A}_{\mathrm{p}}^d$, modified by "polarizations".

Proposition (Vertex function = Whittaker function)

$$V^{(\tau(s))}(Q)|_{\mathrm{p}} = \langle w_{\mathrm{p}}(Q), \tau(s) w_{\mathrm{p}}(Q) \rangle.$$

$\langle\ ,\ \rangle$: invariant bilinear form on $M(\mathrm{p})$, s.t. $\mathfrak{r}_{\pm d}$ are adjoint to each other.

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$)

- $X = T^*\mathbb{P}^n$, $\mathrm{p} = \mathrm{p}_k$, $s \mapsto S|_{\mathrm{p}_k} = a_k^{-1}$.

  Highest weight vector v, $sv = a_k^{-1} v$, $r_{-d} v = 0$, $d > 0$.

  Verma module is spanned by $r_d v$, $d \geq 0$.

- Recall ($d \geq 0$)

  $$r_{-d} \cdot r_d = \prod_{i=1}^{n+1} (-q^{1/2} \hbar^{-1/2})^{-d} \frac{(1 - q a_i s) \cdots (1 - q^d a_i s)}{(1 - \hbar a_i s) \cdots (1 - q^{d-1} \hbar a_i s)}.$$

- Whittaker vector $w_{\mathrm{p}_k}(Q) = \sum_{d \geq 0} \frac{r_d v}{(r_{-d} r_d)|_{\mathrm{p}_k}} Q^{d/2}$.

- Whittaker fucntion

  $$\langle w_{\mathrm{p}_k}(Q), \tau(s) w_{\mathrm{p}_k}(Q) \rangle = \sum_{d \geq 0} \frac{\tau(q^d s)|_{\mathrm{p}_k}}{(r_{-d} r_d)|_{\mathrm{p}_k}} Q^d = V^{(\tau(s))}(Q).$$

Quantum q-difference module

G: abelian; $\mathcal{A}_{\mathrm{T}}(G,N)_X$: certain localized version; $d \in \mathrm{Eff}(X)$.

$$q^{\chi Q\partial_Q}\widetilde{V}^{(\tau(s))}(Q) = \widetilde{V}^{(s^\chi \tau(s))}(Q)$$

$$Q^d\widetilde{V}^{(\tau(s))}(Q) = \widetilde{V}^{(\mathfrak{r}_d \tau(s)\mathfrak{r}_{-d})}(Q).$$

Theorem (Z. '21)

- *q-difference module generated by $\widetilde{V}^{(1)}(Q)$ is isomorphic to*

  $$\mathbb{C}[[Q^{\mathrm{Eff}(X)}]] \otimes_{\mathbb{C}} \mathcal{A}^0_{\mathrm{T}}(G,N)_X / \langle 1 \otimes \mathfrak{r}_d \tau(s)\mathfrak{r}_{-d} - Q^d \otimes \tau(s)\rangle$$

  where $d \in \mathrm{Eff}(X)$, $\tau(s) \in \mathcal{A}^0_X$.
- *The Bethe algebra of X can be obtained from the q-difference module by setting $q \mapsto 1$.*

Quantum q-difference module/equation

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$)

Take $\tau(s) = 1$ in the theorem. We can determine the q-difference equations that $\widetilde{V}^{(1)}(Q)$ satisfies as follows.

We have $Q\widetilde{V}^{(1)}(Q) = \widetilde{V}^{(r_1 r_{-1})}(Q)$, $q^{Q\partial_Q}\widetilde{V}^{(1)}(Q) = \widetilde{V}^{(s)}(Q)$, where (omit constant factor for simplicity)

$$r_1 r_{-1} = \prod_{i=1}^{n+1} \frac{(1 - a_i s)}{(1 - q^{-1}\hbar a_i s)}.$$

We get

$$\prod_{i=1}^{n+1}(1 - a_i q^{Q\partial_Q})\widetilde{V}^{(1)}(Q) = Q\prod_{i=1}^{n+1}(1 - \hbar a_i q^{Q\partial_Q})\widetilde{V}^{(1)}(Q).$$

Nonabelian case

Abelianization: $X^{ab} = \mu^{-1}(0)//_\theta K$; $K \subset G$: maximal torus.

Vertex function can be written in terms of X^{ab} with extra descendent coming from roots of G.

Theorem (Z. '21)

- *q-difference module generated by all $\widetilde{V}^{(\tau(s))}(Q)$ is*

$$\frac{\mathbb{C}[[Q^{\mathrm{Eff}(X)}]] \otimes_{\mathbb{C}} \mathcal{A}^0_{\mathrm{T}}(K,N)^W_{X^{ab},loc}}{\left\langle 1 \otimes \mathfrak{r}_{wd}\tau(s)\mathfrak{r}_{-wd} \cdot \prod_\alpha \frac{(qs^\alpha)_{-\langle\alpha,wd\rangle}}{(\hbar s^\alpha)_{-\langle\alpha,wd\rangle}} - Q^{\bar{d}} \otimes \tau(s) \right\rangle}$$

  where $d \in \mathrm{Eff}(X^{ab}) \cap \mathrm{cochar}(G)_+$, $w \in W$, $\tau(s) \in \mathcal{A}^0_X$.

- *The Bethe algebra of X can be obtained from the q-difference module by setting $q \mapsto 1$.*

Application: wall-crossing

Variation of GIT: change stability condition θ, $X' = \mu^{-1}(0)//_{\theta'} G$.

Restriction to fixed points are changed.

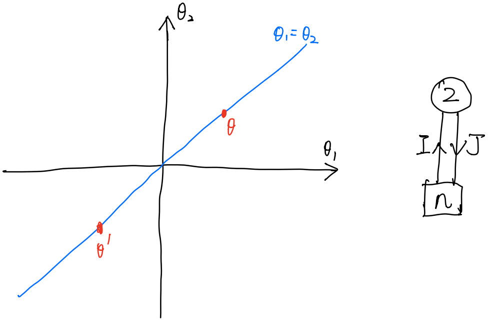
Effective cone is changed: for some reversing $d \in \mathrm{Eff}(X)$, we have $-d \in \mathrm{Eff}(X')$.

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$)

- $\theta > 0$, $X = \{(\vec{x}, \vec{y}) \mid \vec{x} \cdot \vec{y} = 0,\ \vec{x} \neq 0\}/\mathbb{C}^* = T^*\mathbb{P}^n$
  $\mathrm{Eff}(X) = \{d \mid d \geq 0\}$. $S|_{p_k} = a_k^{-1}$.
- $\theta' < 0$, $X' = \{(\vec{x}, \vec{y}) \mid \vec{x} \cdot \vec{y} = 0,\ \vec{y} \neq 0\}/\mathbb{C}^* = T^*\mathbb{P}^n$
  $\mathrm{Eff}(X') = \{d \mid d \leq 0\}$. $S|_{p_k} = \hbar^{-1} a_k^{-1}$.

Example (Nakajima quiver $(v, w) = (2, n)$)

- $\theta < 0$, $X = \{(I, J) \mid IJ = 0,\ \mathrm{rk}\, I = 2\}/GL(2) = T^*Gr(2, n)$
  $\mathrm{Eff}(X) = \{d \mid d \geq 0\}$. $S_i|_{\mathrm{p}} = a_{\mathrm{p}_i}^{-1}$.
- $\theta' > 0$, $X' = \{(I, J) \mid IJ = 0,\ \mathrm{rk}\, J = 2\}/GL(2) = T^*Gr(2, n)$
  $\mathrm{Eff}(X') = \{d \mid d \leq 0\}$. $S_i|_{\mathrm{p}} = \hbar^{-1} a_{\mathrm{p}_i}^{-1}$.
- $X^{ab} = (T^*\mathbb{P}^{n-1})^2$, same with $(X')^{ab}$.



Under wall-crossing, the virtual Coulomb branch is well-behaved: for those reversing curve classes d,

$$\mathfrak{r}'_{\pm d} = \mathfrak{r}_{\mp d}^{-1}.$$

Example ($G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$)

$$r_{-d} \cdot r_d = \prod_{i=1}^{n+1} (-q^{1/2}\hbar^{-1/2})^{-d} \frac{(1-qa_is)\cdots(1-q^d a_is)}{(1-\hbar a_is)\cdots(1-q^{d-1}\hbar a_is)}$$

$(d \geq 0)$ becomes $r'_{-d} r'_d = r_d^{-1} \cdot r_{-d}^{-1} =$ $\prod_{i=1}^{n+1} (-q^{1/2}\hbar^{-1/2})^{d} \frac{(1-\hbar a_is)\cdots(1-q^{d-1}\hbar a_is)}{(1-qa_is)\cdots(1-q^d a_is)}$, and then

$$r'_d r'_{-d} = \prod_{i=1}^{n+1} (-q^{1/2}\hbar^{-1/2})^{d} \frac{(1-q^{-1}\hbar a_is)\cdots(1-q^{-d}\hbar a_is)}{(1-a_is)\cdots(1-q^{1-d} a_is)}.$$

Application: wall-crossing

Observation: relations

$$1 \otimes \mathfrak{r}_{wd} \tau(s) \mathfrak{r}_{-wd} \cdot \prod_{\alpha} \frac{(qs^{\alpha})_{-\langle \alpha, wd \rangle}}{(\hbar s^{\alpha})_{-\langle \alpha, wd \rangle}} - Q^{\bar{d}} \otimes \tau(s)$$

in the quantum q-difference module are invariant under wall-crossing $\theta \mapsto \theta'$.

Theorem (Z. '21)

The quantum q-difference module (also the Bethe algebra) is invariant under wall-crossing.

Thank you!