Instructions:

- There are 6 questions on this exam.
- Please write your NAME and UNI on top of EVERY page.
- In order to get full credit you need to answer the first 5 questions correctly.
- The last question is a bonus question, and you do not have to answer it.
- Unless otherwise is explicitly stated SHOW YOUR WORK in every question.
- Please write neatly, and put your final answer in a box.
- No calculators, cell phones, books, notebooks, notes or cheat sheets are allowed.
- Some useful identities:
  - \( \sin^2(\theta) + \cos^2(\theta) = 1 \)
  - \( \tan^2(\theta) + 1 = \sec^2(\theta) \)
  - \( \sin(2\theta) = 2\sin(\theta)\cos(\theta) \)
  - \( \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \)
1. (6 points) For which values of $p$ does the following integral converge? (Take $p \in \mathbb{R}$)

$$
\int_{1}^{\infty} \frac{dx}{x(\ln(x))^p}
$$

*Hint*: There analyze 1 and $\infty$ separately.

**Solution:**

Following the hint we break the integral into two parts,

$$
\int_{1}^{\infty} \frac{dx}{x(\ln(x))^p} = \int_{1}^{2} \frac{dx}{x(\ln(x))^p} + \int_{2}^{\infty} \frac{dx}{x(\ln(x))^p} = I_1 + I_2
$$

Note that by substituting $u = \ln(x)$ we get

$$
\int \frac{dx}{x(\ln(x))^p} = \int \frac{du}{u^p} = \begin{cases}
\frac{(\ln(x))^{p+1}}{p+1} & p \neq 0, 1 \\
\ln(x) & p = 0 \\
\ln(\ln(x)) & p = 1
\end{cases}
$$

We will analyze the three cases $p = 0, 1$ and $p \neq 0, 1$ separately.

- $p = 0$. In this case,

$$
I_2 = \int_{1}^{2} \frac{dx}{x} = \lim_{T \to \infty} \ln(x)|_{2}^{T} = \lim_{T \to \infty} (\ln(T) - \ln(2)) = \infty
$$

Hence the series diverges.

- $p = 1$. In this case,

$$
I_1 = \int_{1}^{2} \frac{dx}{x(\ln(x))^p} = \lim_{T \to 1^+} \ln(\ln(x))|_{2}^{T} = \lim_{T \to 1^+} (\ln(\ln(2)) - \ln(\ln(T))) = \infty
$$

Since $I_1$ diverges, the integral diverges.

- $p \neq 0, 1$. In this case,

$$
I_1 = \int_{1}^{2} \frac{dx}{x(\ln(x))^p} = \lim_{T \to 1^+} \frac{(\ln(2))^{p+1}}{p+1} - \frac{(\ln(T))^{p+1}}{p+1} = \begin{cases}
\frac{(\ln(2))^{p+1}}{p+1} & p > 1 \\
-\infty & p < 1
\end{cases}
$$
Therefore $I_1$ converges if and only if $p > 1$.

On the other hand,

\[
I_2 = \int_2^\infty \frac{dx}{x(\ln(x))^p}
\]

\[
= \lim_{T \to \infty} \frac{x^{p+1}}{p+1} \bigg|_2^T
\]

\[
= \lim_{T \to \infty} \left( \frac{(\ln(T))^{p+1}}{p+1} - \frac{(\ln(2))^{p+1}}{p+1} \right)
\]

\[
= \begin{cases} 
\infty & p > 1 \\
-\frac{(\ln(2))^{p+1}}{p+1} & p < 1
\end{cases}
\]

Therefore we see that $I_2$ converges if and only if $p < 1$.

Combining the results for $I_1$ and $I_2$ we get that for $p \neq 1$ the integral never converges.

Therefore we conclude that the integral does not converge for any value of $p$. 
2. (5 points) Let \( n \in \mathbb{R} \) denote an arbitrary constant, and \(-1 \leq y \leq 1\). Find the orthogonal trajectories to the family of curves given by
\[
y = \sin(x + n)
\]

**Solution:** For each \( n \) let us denote the slope of the tangent line to the curve \( y = \sin(x + n) \) at the point \((x, y)\) by \( m_n(x, y) \). Differentiating, we get
\[
m_n(x, y) = \cos(x + n)
\]
Then the slopes of the orthogonal trajectory, \( m_{\text{orth}}(x, y) \), must satisfy,
\[
m_{\text{orth}}(x, y) = -\frac{1}{\cos(x + n)}
\]
Since \( y = \sin(x + n) \), \( \cos(x + n) = \sqrt{1 - y^2} \). Therefore we get the differential equation
\[
\frac{dy}{dx} = -\frac{1}{\sqrt{1 - y^2}}
\]
Note that this equation is separable. Separating the variables gives
\[
\int \sqrt{1 - y^2} \, dy = -\int dx
\]
\[
\Rightarrow \int \cos^2(u) \, du = -x + C
\]
\[
\Rightarrow \frac{\sin(2u)}{4} + \frac{u}{2} = -x + C
\]
\[
\Rightarrow y\sqrt{1 - y^2} + \arcsin(y) = -x + C
\]
Where in integrating \( \sqrt{1 - y^2} \) we used the substitution \( y = \sin(u) \).
3. (5 points) Solve the following differential equation initial value problem, where \(0 < x \leq \frac{\pi}{2}\).

\[
sin(x)y' + 2\cos(x)y - 1 = 0, \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{4}
\]

**Solution:** We start by rewriting the equation by dividing both sides by \(\sin(x)\).

\[
y' + 2\cot(x)y - \frac{1}{\sin(x)} = 0
\]

Note that this equation is linear and first order, therefore we can use an integrating factor to solve the equation. An integrating factor is given by

\[
e^{2\int\cot(x)dx} = e^{2\ln|\sin(x)|}
\]

Since \(0 < x \leq \frac{\pi}{4}\), \(\sin(x)\) is positive hence \(|\sin(x)| = \sin(x)\). Therefore the integrating factor we calculated is \(e^{2\ln(\sin(x))} = \sin^2(x)\). Now multiplying the equation by \(\sin^2(x)\) then gives,

\[
\sin^2(x)y' + 2\sin(x)\cos(x)y - \sin(x) = 0
\]

\[
\Rightarrow (\sin^2(x)y)' = \sin(x)
\]

\[
\Rightarrow \sin^2(x)y = -\cos(x) + C
\]

Finally since \(y\left(\frac{\pi}{2}\right) = \frac{\pi}{4}\) we see that \(C = \frac{\pi}{4}\), and the solution is

\[
y = -\cot(x)\csc(x) + \frac{\pi}{4}\csc^2(x)
\]
4. Determine if the following series converge absolutely, conditionally or diverge.

(a) (4 points)
\[ \sum_{n=1}^{\infty} \frac{n \cos(n \pi)}{2^n} \]

(b) (4 points)
\[ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \]

(c) (4 points)
\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)} \]

Solution:

(a) We use ratio test.
\[
\lim_{n \to \infty} \left| \frac{(n+1) \cos((n+1)\pi) / n \cos(n \pi)}{2^{n+1} / 2^n} \right|
= \lim_{n \to \infty} \left| \frac{(n+1)(-1)^{n+1} / n(-1)^n}{2^{n+1} / 2^n} \right|
= \lim_{n \to \infty} \frac{n+1}{2n}
= \frac{1}{2}
\]

Since \(1/2 < 1\) we conclude that the series converges absolutely.

(b) Once again we use the ratio test.
\[
\lim_{n \to \infty} \left| \frac{(n+1)!^2 / (2(n+1))!}{(2n)!} \right|
= \lim_{n \to \infty} \frac{(n+1)! / (2n+1)!}{n!}(2n+2)n!
= \lim_{n \to \infty} \frac{(n+1)^2}{(2n+2)(2n+1)}
= \frac{1}{4}
\]

Since \(1/4 < 1\) we conclude that the series converges absolutely.

(c) This series is alternating. Note that since \(\ln(x)\) is increasing \(\frac{1}{\ln(x)}\) is decreasing, and \(\lim_{n \to \infty} \frac{1}{\ln(n)} = 0\). Therefore the alternating series test tells us that the series converges.

To answer if it converges absolutely or conditionally we check if the series \(\sum_{n=2}^{\infty} \frac{1}{\ln(n)}\) converges. Note that the terms of this series are positive and decreasing, and \(\frac{1}{\ln(x)}\) is continuous. Therefore the integral test says that the series converges if and only if \(\int_{2}^{\infty} \frac{dx}{\ln(x)}\) converges. Now note that for \(x > 2\)
\[
\ln(x) < x \ln(x) \Rightarrow \frac{1}{\ln(x)} > \frac{1}{x \ln(x)}
\]

Hence,
\[
\int_{2}^{\infty} \frac{dx}{\ln(x)} > \int_{2}^{\infty} \frac{1}{x \ln(x)}
\]

However by problem 1 we know that the right hand side diverges, hence the left hand side diverges and hence the series, \(\sum_{n=2}^{\infty} \frac{1}{\ln(n)}\) diverges.

Therefore \(\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}\) converges conditionally.
5. Determine if the following series converge or diverge.

(a) (4 points)
\[ \sum_{n=3}^{\infty} \frac{1}{n \ln(n) \ln(\ln(n))} \]

(b) (4 points)
\[ \sum_{n=1}^{\infty} \tan \left( \frac{1}{n} \right) \]

(c) (4 points)
\[ \sum_{n=1}^{\infty} \left( 1 - \frac{1}{n^2} \right)^n \]

Solution:

(a) The function \( x \ln(x) \ln(\ln(x)) \) is increasing and continuous (non-zero) and positive for \( x > 3 \)
therefore \( \frac{1}{x \ln(x) \ln(\ln(x))} \) is decreasing, continuous and positive. Hence we can use the integral test.
\[ \int_{3}^{\infty} \frac{1}{x \ln(x) \ln(\ln(x))} \, dx = \ln(\ln(\ln(x))) \bigg|_{3}^{\infty} = \infty \]

Hence our series diverges.

(b) For \( n > 1 \tan \left( \frac{1}{n} \right) > 0 \), therefore we can use limit comparison test. We compare the series with \( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \) (which we know from class that diverges).
\[ \lim_{n \to \infty} \frac{\tan \left( \frac{1}{n} \right)}{\sin \left( \frac{1}{n} \right)} = \lim_{n \to \infty} \frac{1}{\cos \left( \frac{1}{n} \right)} = 1 \]

Since \( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \) diverges, so does \( \sum_{n=1}^{\infty} \tan \left( \frac{1}{n} \right) \).

(c) Since \( 1 - \frac{1}{n^2} \geq 1 - \frac{1}{n} \) for \( n \geq 1 \), we get
\[ \left( 1 - \frac{1}{n^2} \right)^n \geq \left( 1 - \frac{1}{n} \right)^n \]

Both \( \left( 1 - \frac{1}{n^2} \right)^n \) and \( \left( 1 - \frac{1}{n} \right)^n \) are also positive. Moreover we know that the series \( \sum_{n=1}^{\infty} \left( 1 - \frac{1}{n} \right)^n \) diverges (we did this in class, and it is also in the practice midterm), by comparison test we see that the series diverges.
6. (5 points (bonus)) Determine if the following series converges or diverges.

\[ \sum_{n=1}^{\infty} \frac{1}{\ln(n) \ln(\ln(n))} \]

*Hint:* \( \ln(x) < \sqrt{x} \).

**Solution:**

Using the hint we see that \( \ln(\ln(x)) < \sqrt{\ln(x)} \). Therefore \( \ln(x)^{\ln(\ln(x))} = e^{(\ln(\ln(x)))} < e^{(\sqrt{\ln(x)})^2} = e^{\ln(x)} = x \). Therefore \( \frac{1}{\ln(x) \ln(\ln(x))} > \frac{1}{x} \). Hence by the comparison theorem the series diverges.