COLUMBIA UNIVERSITY

Math V1102
Calculus II
Fall 2014

Midterm I
09.30.2014

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A

Name and UNI: __________________________

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Instructions:

• There are 7 questions on this exam.
• Please write your NAME and UNI on top of EVERY page.
• In order to get full credit you need to answer the first 6 questions correctly.
• The last question is a bonus question, and you do not have to answer it.
• Unless otherwise is explicitly stated SHOW YOUR WORK in every question.
• Please write neatly, and put your final answer in a box.
• No calculators, cell phones, books, notebooks, notes or cheat sheets are allowed.
• Some useful identities:
  - \( \sin^2(\theta) + \cos^2(\theta) = 1 \)
  - \( \tan^2(\theta) + 1 = \sec^2(\theta) \)
  - \( \sin(2\theta) = 2\sin(\theta)\cos(\theta) \)
  - \( \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \)
1. (6 points)

\[ \int \frac{3x^2 + 2x + 1}{x^3 + x^2 + x} \, dx \]

**Solution:**

By partial fractions we have,

\[ \frac{3x^2 + 2x + 1}{x^3 + x^2 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 1} \]

Equating the denominators and solving for \( A, B, C \) gives \( A = 1, B = 2, C = 1 \). Hence,

\[ \int \frac{3x^2 + 2x + 1}{x^3 + x^2 + x} \, dx = \int \frac{1}{x} \, dx + \int \frac{2x + 1}{x^2 + x + 1} \, dx \]

\[ = \ln |x| + \ln |x^2 + x + 1| + C \]

Where we used \( u = x^2 + x + 1 \) in the second integral.
2. (6 points)

\[ \int \sin^4(x) \cos^2(x) dx \]

**Hint:** Double angle formulas.

**Solution:**

By the double angle formulas we have \( \sin^2(x) = \frac{1 - \cos(2x)}{2} \) and \( \cos^2(x) = \frac{1 + \cos(2x)}{2} \). Therefore,

\[
\int \sin^4(x) \cos^2(x) dx = \int \left( \frac{1 - \cos(2x)}{2} \right)^2 \left( \frac{1 + \cos(2x)}{2} \right) dx
\]

\[
= \frac{1}{8} \int (1 - \cos(2x)) (1 - \cos^2(2x)) dx
\]

\[
= \frac{1}{8} \int (1 - \cos(2x)) \sin^2(2x) dx
\]

\[
= \frac{1}{8} \int \sin^2(2x) dx - \frac{1}{8} \int \cos(2x) \sin^2(2x) dx
\]

\[
= \frac{1}{8} \int \left( \frac{1 - \cos(4x)}{2} \right) dx - \frac{1}{8} \int \cos(2x) \sin^2(2x) dx
\]

\[
= \frac{1}{16} \int dx - \frac{1}{16} \int \cos(4x) dx - \frac{1}{8} \int \cos(2x) \sin^2(2x) dx
\]

\[
= \frac{1}{16} \int dx - \frac{1}{16} \int \cos(4x) dx - \frac{1}{16} \int u^2 du
\]

\[
= \frac{x}{16} - \frac{\sin(4x)}{64} - \frac{u^3}{48} + C
\]

\[
= \frac{x}{16} - \frac{\sin(4x)}{64} - \frac{\sin^3(2x)}{48} + C
\]

Where we used \( u = \sin(2x) \) substitution in evaluating the last integral.
3. (6 points)

$$\int_{0}^{\frac{\pi}{2}} \sin(2x) \cos(6x) \, dx$$

**Hint:** Try integration by parts (most probably more than once).

**Solution:** We use integration by parts. Let \( u = \sin(2x) \) and \( dv = \cos(6x) \, dx \), then \( du = 2 \cos(2x) \) and \( v = \frac{\sin(6x)}{6} \). Integration by parts then gives,

$$\int \sin(2x) \cos(6x) \, dx = \frac{\sin(2x) \sin(6x)}{6} - \frac{1}{3} \int \cos(2x) \sin(6x) \, dx$$

In the second integral we use integration by parts again. Let \( u = \cos(2x) \) and \( dv = \sin(6x) \, dx \), then \( du = -2 \sin(2x) \) and \( v = -\frac{\cos(6x)}{6} \). Hence,

$$\int \cos(2x) \sin(6x) \, dx = -\frac{\cos(2x) \cos(6x)}{6} - \frac{1}{3} \int \sin(2x) \cos(6x) \, dx$$

Substituting this back into the first integration by parts we see that

$$\int \sin(2x) \cos(6x) \, dx = \frac{\sin(2x) \sin(6x)}{6} + \frac{\cos(2x) \cos(6x)}{18} + \frac{1}{9} \int \sin(2x) \cos(6x) \, dx$$

Solving for \( \int \sin(2x) \cos(6x) \, dx \) gives,

$$\int \sin(2x) \cos(6x) \, dx = \frac{3 \sin(2x) \sin(6x)}{16} + \frac{\cos(2x) \cos(6x)}{16}$$

Finally substituting the boundary value gives,

$$\int_{0}^{\frac{\pi}{2}} \sin(2x) \cos(6x) \, dx = 0$$
4. (6 points) 

\[ \int x^7 \sqrt{1 + x^4} \, dx \]

**Hint:** Start with \( u = x^2 \) substitution.

**Solution:**

Substituting \( u = x^2 \) gives,

\[ \int x^7 \sqrt{1 + x^4} \, dx = \frac{1}{2} \int u^{\frac{3}{2}} \sqrt{1 + u^2} \, du \]

We then use the substitution \( u = \tan(\theta) \) and get

\[ \int u^{\frac{3}{2}} \sqrt{1 + u^2} \, du = \int \tan^3(\theta) \sec^3(\theta) \, d\theta \]

\[ = \int \tan^2(\theta) \sec^2(\theta) \tan(\theta) \sec(\theta) \, d\theta \]

\[ = \int (\sec^2(\theta) - 1) \sec^2(\theta) \tan(\theta) \sec(\theta) \, d\theta \]

\[ = \int \sec^4(\theta) \tan(\theta) \sec(\theta) \, d\theta - \int \sec^2(\theta) \tan(\theta) \sec(\theta) \, d\theta \]

\[ = \frac{\sec^5(\theta)}{5} - \frac{\sec^3(\theta)}{3} + C \]

Where we used \( u = \sec(\theta) \) substitution to evaluate the last integrals. We then substitute the variables back: Since \( u = \tan(\theta), \sec(\theta) = \sqrt{1 + u^2}. \) Then since \( u = x^2 \) we get that \( \sec(\theta) = \sqrt{1 + x^4}. \) Therefore the final answer is,

\[ \int x^7 \sqrt{1 + x^4} \, dx = \frac{(1 + x^4)^{\frac{3}{2}}}{10} - \frac{(1 + x^4)^{\frac{5}{2}}}{6} + C \]
5. (6 points)

\[ \int \frac{1}{x + \sqrt{1 + x^2}} \, dx \]

Hint: Try \( x = \frac{1}{2}(e^u - e^{-u}) \) substitution.

Solution:
We will use the hint. Let \( x = \frac{1}{2}(e^u - e^{-u}) \), then \( dx = \frac{1}{2}(e^u + e^{-u}) \). Substituting these back into the integral gives,

\[
\int \frac{1}{x + \sqrt{1 + x^2}} \, dx = \int \frac{e^u + e^{-u}}{e^u - e^{-u} + 2\sqrt{1 + \frac{1}{4}(e^{2u} - 2 + e^{-2u})}} \, du
\]

\[
= \int \frac{e^u + e^{-u}}{e^u - e^{-u} + 2\sqrt{\frac{1}{4}(e^{2u} + 2 + e^{-2u})}} \, du
\]

\[
= \int \frac{e^u + e^{-u}}{e^u - e^{-u} + 2\sqrt{\frac{1}{4}(e^u + e^{-u})^2}} \, du
\]

\[
= \frac{1}{2} \int \frac{e^u + e^{-u}}{e^u} \, du + \frac{1}{2} \int e^{-2u} \, du
\]

\[
= \frac{u}{2} - \frac{e^{-2u}}{4} + C
\]

Now in order to solve for \( x \) in terms of \( u \) we note the following: Let \( e^u = y \). Then \( y > 0 \) and the equation \( x = \frac{1}{2}(e^u - e^{-u}) \) becomes \( 2x = y - \frac{1}{y} \). This in particular implies \( y^2 - 2xy - 1 = 0 \). Therefore \( y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1} \). Since \( y > 0 \) we need to have \( y = x + \sqrt{1 + x^2} \). Therefore we get,

\[ e^u = x + \sqrt{1 + x^2} \]

Substituting this back into the answer then gives

\[
\int \frac{1}{x + \sqrt{1 + x^2}} \, dx = \frac{\ln |x + \sqrt{1 + x^2}|}{2} - \frac{1}{4(x + \sqrt{1 + x^2})^2} + C
\]
6. Determine if the following integrals converge or diverge.

(a) (3 points) 
\[ \int_1^\infty \frac{1}{x^2 + 2} \, dx \]

(b) (3 points) 
\[ \int_0^1 x \ln(x) \, dx \]

**Hint:** For evaluating limits of the form \( \lim_{t \to 0^+} x \ln(x) \) you can use L’Hospital’s rule.

**Solution:**

(a) By definition, 
\[
\int_1^\infty \frac{1}{x^2 + 2} \, dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^2 + 2} \, dx
\]
\[
= \lim_{t \to \infty} \frac{1}{2} \int_1^t \frac{1}{(x/\sqrt{2})^2 + 1} \, dx
\]
\[
= \lim_{t \to \infty} \frac{1}{\sqrt{2}} \int_{1/\sqrt{2}}^{t/\sqrt{2}} \frac{1}{u^2 + 1} \, du
\]
\[
= \frac{1}{\sqrt{2}} \lim_{t \to \infty} \arctan(u) \bigg|_{1/\sqrt{2}}^{t/\sqrt{2}}
\]
\[
= \frac{1}{\sqrt{2}} \lim_{t \to \infty} (\arctan(t/\sqrt{2}) - \arctan(1/\sqrt{2}))
\]
\[
= \frac{\pi}{2} - \arctan(1/\sqrt{2})
\]

Therefore the integral is convergent.

(b) The integrand is continuous on (0, 1) and the only discontinuity is due to \( \ln(x) \) and is at \( x = 0 \). The integral is then defined by,

\[ \int_0^1 x \ln(x) \, dx = \lim_{t \to 0^+} \int_t^1 x \ln(x) \]

We calculate the latter integral by integration by parts. Let \( u = \ln(x) \) and \( dv = x \, dx \), then

\[
\int_t^1 x \ln(x) = \frac{x^2 \ln(x)}{2} \bigg|_t^1 - \frac{1}{2} \int_t^1 x \, dx
\]
\[
= \frac{2x^2 \ln(x) - x^2}{4} \bigg|_t^1
\]
\[
= \frac{1}{4} - \frac{2t^2 \ln(t) - t^2}{4}
\]

Substituting this back into the improper integral we get,

\[
\int_0^1 x \ln(x) \, dx = \lim_{t \to 0^+} \left( -\frac{1}{4} - \frac{2t^2 \ln(t) - t^2}{4} \right)
\]
\[
= -\frac{1}{4} - \frac{1}{2} \lim_{t \to 0^+} t^2 \ln(t)
\]
To calculate this last limit we use L’Hospital’s rule.

\[
\lim_{t \to +} t^2 \ln(t) = \lim_{t \to +} \frac{\ln(t)}{1/t^2} = \\
\lim_{t \to +} t^2 \ln(t) \cdot \frac{1/t}{2/t^3} = \\
\frac{-1}{2} \lim_{t \to +} t^2 \ln(t) \cdot x^2 = 0
\]

Hence we get,

\[
\int_0^1 x \ln(x) dx = -\frac{1}{4}
\]

and in particular the integral converges.
7. (5 points (bonus)) Let $n \geq 0$ be fixed. Calculate
\[
\int_0^{\pi/2} \frac{\sin^n(x)}{\sin^n(x) + \cos^n(x)} \, dx
\]

*Hint:* Try coming up with a substitution that flips the boundary. This will result in an integral that looks very similar to the original integral. You may then try to take a combination of the two forms of the same integral. *Hint-2:* The result is independent of the specific choice of $n$.

**Solution:** The substitution that the hint is referring to is $x = \pi/2 - u$. This transforms the integral to
\[
I = \int_0^{\pi/2} \frac{\sin^n(x)}{\sin^n(x) + \cos^n(x)} \, dx
\]
\[
= \int_0^{\pi/2} \frac{\sin^n(\pi/2 - u)}{\sin^n(\pi/2 - u) + \cos^n(\pi/2 - u)} \, du
\]
\[
= \int_0^{\pi/2} \frac{\cos^n(u)}{\cos^n(u) + \sin^n(u)} \, du
\]

Therefore we get,
\[
2I = \int_0^{\pi/2} \frac{\sin^n(x)}{\sin^n(x) + \cos^n(x)} \, dx + \int_0^{\pi/2} \frac{\cos^n(x)}{\sin^n(x) + \cos^n(x)} \, dx
\]
\[
= \int_0^{\pi/2} \, dx
\]
\[
= \frac{\pi}{2}
\]

Hence $I = \frac{\pi}{4}$. 