Instructions:

• There are 8 questions on this exam.
• Please write your NAME and UNI on top of EVERY page.
• In order to get full credit you need to answer the first 7 questions correctly.
• The last question is a bonus question, and you do not have to answer it.
• Unless otherwise is explicitly stated SHOW YOUR WORK in every question.
• Please write neatly, and put your final answer in a box.
• No calculators, cell phones, books, notebooks, notes or cheat sheets are allowed.
• Some useful identities:

  \[ \sin^2(\theta) + \cos^2(\theta) = 1 \]

  \[ \tan^2(\theta) + 1 = \sec^2(\theta) \]

  \[ \sin(2\theta) = 2\sin(\theta)\cos(\theta) \]

  \[ \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \]
1. (4 points)
\[ \int \frac{\ln(x)}{\sqrt{x}} \, dx \]

*Hint:* Integration by parts.

**Solution:**
We will use integration by parts. Let \( u' = \frac{1}{\sqrt{x}} \) and \( v = \ln(x) \). Then, \( u = 2\sqrt{x} \) and \( v' = \frac{1}{x} \). Therefore,

\[
\int \frac{\ln(x)}{\sqrt{x}} \, dx = 2\sqrt{x} \ln(x) - 2 \int \frac{\sqrt{x}}{x} \, dx \\
= 2(\sqrt{x} \ln(x) - 2\sqrt{x}) + C
\]
2. (4 points)
\[ \int \frac{dx}{\sqrt{x} + x} \]

*Hint:* Try substituting \( x = u^2 \).

**Solution:**
Let \( x = u^2 \), then \( dx = 2udu \). Hence,
\[
\int \frac{dx}{\sqrt{x} + x} = 2 \int \frac{udu}{u + u^3} = 2 \int \frac{du}{1 + u^2} = 2 \arctan(u) + C = 2 \arctan(\sqrt{x}) + C
\]
3. (6 points)

\[ \int \cos^4(x) \, dx \]

**Solution:** We use the double angle formula \( \cos(2x) = 2\cos^2(x) + 1 \). This gives,

\[
\cos^2(x) = \frac{\cos(2x) + 1}{2}
\]

\[
\cos^2(2x) = \frac{\cos(4x) + 1}{2}
\]

Substituting these in we get,

\[
\int \cos^4(x) \, dx = \int \frac{(\cos(2x) + 1)^2}{4} \, dx
\]

\[
= \frac{1}{4} \int (\cos^2(2x) + 2\cos(2x) + 1) \, dx
\]

\[
= \frac{1}{4} \left( \int \cos^2(2x) \, dx + 2 \int \cos(2x) \, dx + \int \, dx \right)
\]

\[
= \frac{1}{4} \int \cos^2(2x) \, dx + \frac{1}{4} \sin(2x) + \frac{x}{4}
\]

\[
= \frac{1}{4} \int \frac{\cos(4x) + 1}{2} \, dx + \frac{\sin(2x)}{4} + \frac{x}{4}
\]

\[
= \frac{\sin(4x)}{32} + \frac{\sin(2x)}{4} + \frac{3x}{8} + C
\]
4. (5 points)

\[ \int \frac{x^2 + 1}{x^3 - x} \, dx \]

**Solution:**

By partial fractions we can rewrite the integral as,

\[ \frac{x^2 + 1}{x^3 - x} = \frac{x^2 + 1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{A_2}{x+1} + \frac{A_3}{x-1} \]

Equating the denominators gives,

\[ x^2 + 1 = A_1(x^2 - 1) + A_2(x^2 - x) + A_3(x^2 + x) \]

Therefore we need to solve,

\[ 1 = A_1 + A_2 + A_3 \]
\[ 0 = A_3 - A_2 \]
\[ 1 = -A_1 \]

Hence, \( A_1 = -1, A_2 = A_3 = 1 \), and we have

\[ \int \frac{x^2 + 1}{x^3 - x} \, dx = \int \left( \frac{-1}{x} + \frac{1}{x+1} + \frac{1}{x-1} \right) \, dx \]
\[ = -\ln |x| + \ln |x+1| + \ln |x-1| + C \]
\[ = \ln \left| \frac{x - 1}{x} \right| + C \]
5. (5 points) Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sin(x)$ and $\cos(x)$ between $x = 0$ and $x = \frac{\pi}{4}$ around the $y$-axis.

**Solution:**
Let $V$ denote the volume, then by the method of cylindrical shells,

$$V = 2\pi \int_0^{\frac{\pi}{4}} x(\cos(x) - \sin(x))dx$$

$$= 2\pi \int_0^{\frac{\pi}{4}} (x \cos(x) - x \sin(x))dx$$

- $\int x \sin(x)dx$.
  Let $u' = \sin(x)$ and $v = x$, then $u = -\cos(x)$ and $v' = 1$. Therefore by integration by parts,
  
  $$\int x \sin(x)dx = -x \cos(x) + \int \cos(x)dx$$
  $$= \sin(x) - x \cos(x)$$

- $\int x \cos(x)dx$.
  Let $u' = \cos(x)$ and $v = x$, then $u = \sin(x)$ and $v' = 1$. Therefore by integration by parts,
  
  $$\int x \cos(x)dx = x \sin(x) - \int \sin(x)dx$$
  $$= \cos(x) + x \sin(x)$$

By the above calculations we get,

$$\int (x \cos(x) - x \sin(x))dx = \cos(x) - \sin(x) + x(\sin(x) + \cos(x))$$

Finally,

$$V = 2\pi \int_0^{\frac{\pi}{4}} (x \cos(x) - x \sin(x))dx$$

$$= 2\pi \left( \cos(x) - \sin(x) + x(\sin(x) + \cos(x)) \right)_0^{\frac{\pi}{4}}$$

$$= 2\pi \left( \frac{\pi \sqrt{2}}{4} - 1 \right)$$
6. (6 points)

\[ \int \frac{3x + 1}{2x^2 + 3} \, dx \]

Solution:

\[
\int \frac{3x + 1}{2x^2 + 3} \, dx = \int \frac{3x}{2x^2 + 3} \, dx + \int \frac{1}{2x^2 + 3} \, dx
\]

Where,

\[ I_1 = \int \frac{3x}{2x^2 + 3} \, dx \quad , \quad I_2 = \int \frac{1}{2x^2 + 3} \, dx \]

- \( I_1 \).

Let \( u = 2x^2 + 3 \), then \( du = 4xdx \) hence \( xdx = du/4 \). Then,

\[
I_1 = \frac{3}{4} \int \frac{du}{u}
\]

\[
= \frac{3}{4} \ln |u| + C_1
\]

\[
= \frac{3}{4} \ln |2x^2 + 3| + C_1
\]

- \( I_2 \).

\[
I_2 = \frac{1}{2} \int \frac{1}{x^2 + \frac{3}{2}} \, dx
\]

Let \( x = u^{\frac{1}{2}} \), then \( dx = \sqrt{\frac{3}{2}} \, du \). Substituting these in the integral above we get

\[
I_2 = \frac{1}{2} \int \frac{1}{u^2 + \frac{3}{2}} \, du
\]

\[
= \frac{\sqrt{3}}{2\sqrt{2}} \int \frac{du}{\frac{3}{2}u^2 + \frac{3}{2}}
\]

\[
= \frac{\sqrt{3}}{2\sqrt{2}} \int \frac{du}{u^2 + 1}
\]

\[
= \frac{\sqrt{3}}{2\sqrt{3}} \arctan u + C_2
\]

\[
= \frac{\sqrt{2}}{2\sqrt{3}} \arctan \left( \sqrt{\frac{2}{3}} x \right) + C_2
\]

Combining the above we get,

\[
\int \frac{3x + 1}{2x^2 + 3} \, dx = I_1 + I_2
\]

\[
= \frac{3}{4} \ln |2x^2 + 3| + C_1 + \frac{\sqrt{2}}{2\sqrt{3}} \arctan \left( \sqrt{\frac{2}{3}} x \right) + C_2
\]

\[
= \frac{3}{4} \ln |2x^2 + 3| + \frac{\sqrt{2}}{2\sqrt{3}} \arctan \left( \sqrt{\frac{2}{3}} x \right) + C
\]
7. Determine if the following integrals converge or diverge.

(a) (3 points)
\[ \int_{e}^{\infty} \frac{\ln(x)}{x} \, dx \]

*Hint:* You can use the inequality \( \ln(x) > 1 \) which is valid for \( x > e \).

(b) (3 points)
\[ \int_{0}^{\infty} \frac{dx}{x^2 + 1} \]

**Solution:**

(a) By the hint,
\[ \int_{e}^{\infty} \frac{\ln(x)}{x} \, dx > \int_{e}^{\infty} \frac{1}{x} \, dx \]

The integral on the right hand side diverges, hence the original integral also diverges.

(b) By definition,
\[ \int_{0}^{\infty} \frac{dx}{x^2 + 1} = \lim_{t \to \infty} \int_{0}^{t} \frac{dx}{x^2 + 1} = \lim_{t \to \infty} \arctan(x) \bigg|_{0}^{t} = \lim_{t \to \infty} \arctan(t) = \frac{\pi}{2} \]

Hence the integral converges and moreover its value is \( \frac{\pi}{2} \).
8. (5 points (bonus)) For \(a, b > 0\) evaluate the following integral.

\[
\int_0^\infty \frac{\ln(x)}{ax^2 + b} \, dx
\]

*Hint:* You may first use a simple substitution to put the denominator in a standard form. This will result in two integrals, one you can evaluate, and one that looks similar to the original integral. Then, in the integral that looks like the original one, try using a substitution that flips the boundary.

**Solution:** First we use the substitution \(x = \sqrt{\frac{b}{a}}u\) to put the integral in a standard form.

\[
\int_0^\infty \frac{\ln(x)}{ax^2 + b} \, dx = \frac{1}{\sqrt{ab}} \int_0^\infty \frac{\ln\left(\sqrt{\frac{b}{a}}u\right)}{u^2 + 1} \, du
\]

\[
= \frac{\ln\left(\sqrt{\frac{b}{a}}\right)}{\sqrt{ab}} \int_0^\infty \frac{1}{u^2 + 1} \, du + \frac{1}{\sqrt{ab}} \int_0^\infty \frac{\ln(u)}{u^2 + 1} \, du
\]

\[
= \frac{\ln\left(\sqrt{\frac{b}{a}}\right)}{\sqrt{ab}} \left[\arctan(u)\right]_0^\infty + \frac{1}{\sqrt{ab}} \int_0^\infty \frac{\ln(u)}{u^2 + 1} \, du
\]

\[
= \frac{\pi(\ln(b) - \ln(a))}{4\sqrt{ab}} + \frac{1}{\sqrt{ab}} \int_0^\infty \frac{\ln(u)}{u^2 + 1} \, du
\]

Now let,

\[
I = \int_0^\infty \frac{\ln(u)}{u^2 + 1} \, du
\]

Let \(x = \frac{1}{u}\), then \(dx = -\frac{du}{u^2}\). Note that \(x \to 0^+\) implies \(u \to \infty\) and \(x \to \infty\) implies \(u \to 0^+\) (i.e. This substitution flips the boundary!). Using this we have,

\[
I = \int_0^\infty \frac{\ln(x)}{x^2 + 1} \, dx
\]

\[
= -\int_0^\infty \frac{\ln\left(\frac{1}{u}\right)}{u^2 + 1} \, du
\]

\[
= -\int_\infty^0 \left(-\frac{\ln(u)}{u^2 + 1}\right) \, du
\]

\[
= \int_0^\infty \frac{\ln(u)}{u^2 + 1} \, du
\]

\[
= -I
\]

Which shows \(2I = 0\) and hence \(I = 0\). Therefore we conclude that,

\[
\int_0^\infty \frac{\ln(x)}{ax^2 + b} \, dx = \frac{\pi(\ln(b) - \ln(a))}{4\sqrt{ab}}
\]