COLUMBIA UNIVERSITY

Math V1102
Calculus II
Fall 2014

Final
12.12.2014

Instructor: S. Ali Altuğ

B

Name and UNI: _______________________________________________________________

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points:</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>11</td>
<td>10</td>
<td>6</td>
<td>0</td>
<td>64</td>
</tr>
</tbody>
</table>

Score: __________________________

Instructions:

- There are 9 questions on this exam.
- Please write your NAME and UNI on top of EVERY page.
- In order to get full credit you need to answer the first 8 questions correctly.
- The last question is a bonus question, and you do not have to answer it.
- Unless otherwise is explicitly stated SHOW YOUR WORK in every question.
- Please write neatly, and put your final answer in a box.
- No calculators, cell phones, books, notebooks, notes or cheat sheets are allowed.
- Some useful identities:
  
  - \( \sin^2(\theta) + \cos^2(\theta) = 1 \)
  
  - \( \tan^2(\theta) + 1 = \sec^2(\theta) \)
  
  - \( \sin(2\theta) = 2\sin(\theta)\cos(\theta) \)
  
  - \( \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \)
1. Determine if the following statements are True (T) or False (F). You DO NOT need to justify your answers for this question.

(a) (1 point) If \( \sum_{n=1}^{\infty} a_n \) is convergent then \( \sum_{n=1}^{\infty} (-1)^n a_n \) is also convergent.

(b) (1 point) If \( \sum_{n=1}^{\infty} a_n \) diverges then \( \sum_{n=1}^{\infty} a_n^2 \) diverges.

(c) (1 point) If \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) converges then \( \sum_{n=1}^{\infty} a_n b_n \) converges.

(d) (1 point) If \( \int_1^{\infty} f(x) \, dx \) converges then \( \sum_{n=1}^{\infty} f(n) \) converges.

(e) (1 point) Every convergent sequence is bounded.

Solution:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>True. Ex. ( a_n = \frac{(-1)^n}{n} ).</td>
</tr>
<tr>
<td>(b)</td>
<td>False. Ex. ( a_n = \frac{1}{n^{1/2}} ).</td>
</tr>
<tr>
<td>(c)</td>
<td>False. Ex. ( a_n = b_n = \frac{(-1)^n}{\sqrt{n}} ).</td>
</tr>
<tr>
<td>(d)</td>
<td>False. Ex. ( f(x) = \frac{\sin(2\pi x)}{x} ).</td>
</tr>
<tr>
<td>(e)</td>
<td>True.</td>
</tr>
</tbody>
</table>
2. (6 points) For what values of \( p \in \mathbb{R} \) does the following integral converge?

\[
\int_1^\infty x^p \ln(x) \, dx
\]

Solution:
We first integrate by parts.

\[
\int_1^\infty x^p \ln(x) \, dx = \left. \frac{x^{p+1}}{p+1} \ln(x) \right|_1^\infty - \frac{1}{p+1} \int_1^\infty x^p \, dx
\]

\[
= \frac{x^{p+1}}{p+1} \ln(x) \bigg|_1^\infty - \frac{x^{p+1}}{(p+1)^2} \bigg|_1^\infty
\]

\[
= -\frac{1}{p+1} \left[ \frac{1}{p+1} + \lim_{t \to \infty} t^{p+1} \left( \ln(t) - \frac{1}{p+1} \right) \right]
\]

For the integral to converge the limit above should be finite. Note that this limit is indeterminate of the form \( 0 \cdot \infty \) is \( p < -1 \) and is \( \infty \cdot \infty \rightarrow \infty \) if \( p > -1 \). Hence we immediately conclude that \( p \leq -1 \). Moreover if \( p = -1 \) again the limit is \( \infty \). Therefore we necessarily have \( p > -1 \).

To analyze the limit we rewrite the integral and use L’Hospital’s rule.

\[
\lim_{t \to \infty} t^{p+1} \left( \ln(t) - \frac{1}{p+1} \right) = \lim_{t \to \infty} \frac{\ln(t) - \frac{1}{p+1}}{t^{-(p+1)}}
\]

\[
= \lim_{t \to \infty} \frac{t^{-1}}{(p+1)t^{-(p+2)}}
\]

\[
= \frac{-1}{p+1} \lim_{t \to \infty} t^{p+1}
\]

\[
= 0
\]

Where we have used that \( p+1 < 0 \) in the last line. Therefore we conclude that the integral converges for \( p < -1 \).
3. (6 points) Solve the following second degree differential equation.

\[ \frac{d^2 y}{dx^2} = \frac{dy}{dx}, \quad y(0) = 0, \quad y(1) = e \]

Hint: Note that \( \frac{d^2 y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) \). Using this observation you may first try to solve for \( \frac{dy}{dx} \) and then solve for \( y \).

Solution: We follow the hint and first solve for the function \( \frac{dy}{dx} \). Let,

\[ g(x) := \frac{dy}{dx} \]

Then the equation \( \frac{d^2 y}{dx^2} \) is the same as \( \frac{d}{dx} g = g \). In other words,

\[ \frac{dg}{dx} = g \]

Note that this equation is separable. Separating the variables gives

\[ \frac{dg}{g} = dx \Rightarrow \int \frac{dg}{g} = \int dx \Rightarrow g(x) = C_0 e^x \]

For some constant \( C_0 \). Now that we have found \( g(x) \), let us substitute that in the equation \( \frac{dy}{dx} = g(x) = C_0 e^x \). In other words we need to solve the equation,

\[ \frac{dy}{dx} = C_0 e^x \]

Note, once again, that this equation is separable. Separating the variables gives,

\[ dy = C_0 e^x dx \Rightarrow \int dy = \int C_0 e^x dx \Rightarrow y = C_0 e^x + C_1 \]

For some constant \( C_1 \). All we are left with is to determine the constants, for which we use the initial conditions.

\[ y(0) = 0 \Rightarrow C_0 + C_1 = 0 \Rightarrow C_0 = -C_1 \]

\[ y(1) = e \Rightarrow C_0 e - C_0 = e \Rightarrow C_0 = \frac{e}{e - 1} \]

Hence the solution is,

\[ y = \frac{e^{x+1} - e}{e - 1} \]
4. Determine if the following series converges absolutely, conditionally or diverges.

(a) (3 points)
\[ \sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n}}{\ln(n)} \]

(b) (4 points)
\[ \sum_{n=1}^{\infty} \tan \left( \frac{1}{n} \right) \]

(Hint: You can use the fact that \( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \) diverges.)

(c) (3 points)
\[ \sum_{n=1}^{\infty} \frac{n^{3/2}}{n} \]

Solution:

(a) The series diverges by the divergence test. All we need to note is
\[ \lim_{n \to \infty} \frac{\sqrt{n}}{\ln(n)} = \lim_{n \to \infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{\sqrt{n}}{2} = \infty \]

(To get the second equality we used L’Hospital’s rule) Therefore,
\[ \lim_{n \to \infty} \frac{(-1)^n \sqrt{n}}{\ln(n)} = DNE \]

Since the terms are going to \( \infty \) and alternating.

(b) The series diverges. We use limit comparison test (note that we can do this since the terms are positive for large \( n \) as \( 1/n \) gets close to 0 and \( \tan(1/n) \) is positive for those \( n \).) We compare it with \( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \).
\[ \lim_{n \to \infty} \frac{\tan(1/n)}{\sin(1/n)} = \lim_{n \to \infty} \frac{1}{\cos(1/n)} = 1 \]

Since \( \sum_{n=1}^{\infty} \sin(1/n) \) diverges (by the hint) we conclude that the series diverges.

(c) The series diverges. Note that \( n^{3/2}/n \geq 1 \) since \( n \geq 1 \). Therefore
\[ \sum_{n=1}^{\infty} \frac{n^{3/2}}{n} > \sum_{n=1}^{\infty} \frac{1}{n} \]

Since the smaller series diverges (and everything is positive) our series diverges too by the comparison test.
5. Calculate the sum of the following series.

(a) (3 points)
\[ \sum_{n=1}^{\infty} \frac{1}{2^n} \]

(b) (3 points)
\[ \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n} \]

(c) (4 points)
\[ \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{4^n} \]

Solution:

(a) By the formula for the geometric series this is 1.

(b) Recall that the Taylor expansion for \( \ln(1 + x) \) is
\[ \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \]
with the interval of convergence \((-1, 1]\). Therefore,
\[ \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n} = -1 + 1 + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{2} + \ln(1 + 1) = \frac{1}{2} + \ln(2) \]

(c) Recall that the Taylor expansion of \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \), with the interval of convergence \((-1, 1]\). Therefore we have
\[ \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \]
Differentiating this once we get
\[ \frac{-1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1} \]
Multiplying by \( x \) and differentiating again gives
\[ \frac{x-1}{(x+1)^3} = \sum_{n=1}^{\infty} (-1)^n n^2 x^{n-1} \]
Finally multiplying by \( x \) we get the identity,
\[ \frac{x^2 - x}{(x+1)^3} = \sum_{n=1}^{\infty} (-1)^n n^2 x^n \]
Note that the radius of convergence has not changed since it stays the same under differentiation (multiplication by \( x \) also does not change it.). Therefore the series converges at \( x = \frac{1}{4} \).
Substituting this in gives,
\[ \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{4^n} = \frac{(1/4)(-3/4)}{(5/4)^3} = \frac{-12}{125} \]
6. Find the Taylor expansion, around \( x = 0 \), (i.e. Maclaurin series) of the following functions and determine their radius and interval of convergence. (Note: You need to find explicit formulas for coefficients to get full credit.)

(a) (5 points)
\[ f(x) = (4x - 1)^4 \]

(b) (6 points)
\[ f(x) = \cos^2(x) \]

**Solution:**
(a) We need to calculate the coefficients in the Maclaurin expansion.

\[
\begin{align*}
f(x) &= (4x - 1)^4 \Rightarrow f(0) = 1 \\
f'(x) &= 16(4x - 1)^3 \Rightarrow f'(0) = -16 \\
f''(x) &= 192(4x - 1)^2 \Rightarrow f''(0) = 192 \\
f'''(x) &= 1536(4x - 1) \Rightarrow f'''(0) = -1536 \\
f^{(4)}(x) &= 6144 \Rightarrow f^{(4)}(0) = 6144
\end{align*}
\]

Note that \( f^{(n)}(x) = 0 \) for every \( n > 4 \) since \( f \) is a polynomial of degree 4. Therefore the Maclaurin expansion is given by,

\[ f(x) = 1 - 16x + 96x^2 - 256x^3 + 256x^4 \]

Note that this is the same thing as expanding the product. Since this is a finite series the radius of convergence is \( \infty \) and interval of convergence is \( \mathbb{R} \).

(b) We use the identity \( \cos^2(x) = \frac{\cos(2x) + 1}{2} \). Recall that \( \cos(x) \) has the Maclaurin expansion

\[ \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \]

Substituting \( 2x \) instead of \( x \) gives

\[ \cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} \]

Hence,

\[ \cos^2(x) = \frac{\cos(2x) + 1}{2} = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!} \]

Since \( \cos(x) \) has radius \( \infty \) and interval \( \mathbb{R} \) the above series has the same.
7. Suppose the radius of convergence of the power series \( \sum_{n=0}^{\infty} a_n x^n \) is 5, and that \( a_n \neq 0 \) for any \( n \).

(a) (3 points) Does the following series converge?

\[
\sum_{n=0}^{\infty} a_n
\]

(b) (3 points) What is the radius of convergence of the following series?

\[
\sum_{n=0}^{\infty} \frac{x^n}{a_n}
\]

(c) (4 points) Suppose that the interval of convergence of \( \sum a_n x^n \) is \((-5, 5)\) and suppose \( \sum_{n=0}^{\infty} b_n x^n \) is another power series with interval of convergence \((-4, 4)\). What is the radius of convergence of the following series?

\[
\sum_{n=0}^{\infty} (a_n + b_n) x^n
\]

**Solution:**

(a) Yes because 1 is in the radius of convergence and \( \sum_{n=0}^{\infty} a_n \) is the value of the series at \( x = 1 \).

(b) The radius is 1/5. Start by the ratio test on the original series:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]

Since the radius of convergence is 5 this limit should be \( |x|/5 \), i.e. \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1/5 \).

Now applying the ratio test to \( \sum a_n / x^n \) gives

\[
\lim_{n \to \infty} \left| \frac{a_n x^{n+1}}{a_{n+1} x^n} \right| = |x| \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 5|x|
\]

Which implies that the series converges absolutely for \( 5|x| < 1 \), i.e. the radius is 1/5.

(c) The interval of convergence of \( \sum a_n x^n \) is 5 means that the series converges absolutely for \( |x| < 5 \) and diverges for \( |x| \geq 5 \). Similarly \( \sum b_n x^n \) converges absolutely for \( \geq 4 \). Therefore for \( 5 > |x| > 4 \) the second series diverges and the first series converges. Hence the sum, \( \sum (a_n + b_n) x^n \) can not converge for \( 5 > |x| \geq 4 \). On the other hand it does converge for \( |x| < 4 \) since both series converge there. Finally note the theorem saying that a power series converges in an interval. Putting everything together, since \( \sum (a_n + b_n) x^n \) converges in \((-4, 4)\) and diverges for \( 5 > |x| \geq 4 \), the radius of convergence should be 4.
8. In each of the parts below give an example of a series that satisfies the given conditions. (You need to verify that your series satisfies the conditions to get credit!)

(a) (3 points) A series with interval of convergence \((-3, 3)\).

(b) (3 points) A series with interval of convergence \([-1, 3]\).

Solution:

(a)

\[ \sum_{n=0}^{\infty} \frac{x^n}{3^n} \]

\[ \lim_{n \to \infty} \frac{|x|^{n+1}}{3^{n+1}} = \frac{|x|}{3} \]

(b)

\[ \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n n^2} \]
9. (5 points (bonus)) Give an example of two convergent series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) such that

\[
\sum_{n=1}^{\infty} a_n b_n
\]

is convergent. (Yo need to show that your series are satisfy the conditions to get any credit.)

**Solution:**

For example, take \( b_n = 1/n^2 \), \( a_n = e^{-n^4} \). Then \( a_n b_n = e^{-n^2} \) which obviously is convergent.