THE IWASAWA MAIN CONJECTURES FOR GL_2

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1. Introduction

In this paper we prove the Iwasawa-Greenberg Main Conjecture for a large class of elliptic curves and modular forms.

1.1. **The Iwasawa-Greenberg Main Conjecture.** Let p be an odd prime. Let $\overline{\mathbf{Q}} \subset \mathbf{C}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. We fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. For simplicity we also fix an isomorphism $\overline{\mathbf{Q}}_p \cong \mathbf{C}$ compatible with the inclusion of $\overline{\mathbf{Q}}$ into both. We let $\mathbf{Q}_\infty$ be the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$ and $\Gamma_{\mathbf{Q}} := \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ its Galois group. The reciprocity map of class field theory identifies $1 + p\mathbf{Z}_p$ with $\Gamma_{\mathbf{Q}}$; we let $\gamma \in \Gamma_{\mathbf{Q}}$ be the topological generator identified with $1 + p$.

Suppose $f \in S_k(N, \chi)$ is a weight $k \geq 2$ newform of level N and Nebentypus χ. The Hecke eigenvalues of f (equivalently, the Fourier coefficients of f) generate a finite extension $\mathbf{Q}(f)$ of $\mathbf{Q}$ in $\mathbf{C}$. Suppose f is ordinary; that is, $a(p, f)$ is a p-adic unit, $a(p, f)$ being the pth Fourier coefficient of f. Let L be any finite extension of $\mathbf{Q}_p$ containing $\mathbf{Q}(f)$ and the roots of $x^2 - a(p, f)x + \chi(p)p^{k-1}$. In this setting, Amice-Vélu [AV75] and Vishik [Vi76] (see also [MTT86]) have constructed a p-adic L-function for f. This is a power series $\mathcal{L}_f \in \Lambda_{\mathbf{Q},O_L} := O_L[\![\Gamma_{\mathbf{Q}}]\!]$, O_L the ring of integers of L, with the property that if $\phi : \Lambda_{\mathbf{Q},O_L} \to \overline{\mathbf{Q}}_p$ is a continuous O_L-homomorphism such that $\phi(\gamma) = \zeta(1 + p)^m$ with ζ a primitive $p^{t_\phi - 1}$th root of unity and $0 \leq m \leq k - 2$ an integer, then

$$\mathcal{L}_f(\phi) := \phi(\mathcal{L}_f) = e(\phi)\frac{p^{t'_\phi(m+1)} m! L(f, \chi_\phi^{-1}\omega^{-m}, m + 1)}{(-2\pi i)^m G(\chi_\phi^{-1}\omega^{-m})\Omega_f^{\mathrm{sgn}((-1)^m)}},$$

where χ_ϕ is the primitive Dirichlet character modulo p^{t_ϕ} of p-power order such that $\chi_\phi(1+p) = \zeta^{-1}$; ω is the cyclotomic character modulo p (normalized as in §2); $p^{t'_\phi}$ is the conductor of $\omega^m\chi_\phi$; $G(\tau)$ denotes the usual Gauss sum for a Dirichlet character τ; $\Omega_f^\pm$ are the canonical periods of f; and $e(\phi)$ is an interpolation factor that involves $\omega^m\chi_\phi(p)$, $\chi(p)$, k, m, and the roots of the aforementioned polynomial. The p-adic L-function $\mathcal{L}_f$ is one of the two main ingredients of the Iwasawa-Greenberg Main Conjecture for f.

The other main ingredient is the characteristic ideal of the p-adic Selmer group of f over $\mathbf{Q}_\infty$. Recall that there exists a continuous p-adic Galois representation $\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}_L(V_f)$, V_f a two-dimensional L-space, such that the L-function of ρ_f is the L-function of f (we take geometric conventions for all Galois representations). Furthermore, if f is ordinary, then it is known that there exists a unique G_p-stable unramified line $V_f^+ \subset V_f$; here G_p is a decomposition group at p. Let $T_f \subset V_f$ be a $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-stable O_L-lattice and let $T_f^+ := T_f \cap V_f^+$. Let $T := T_f(\det \rho_f^{-1})$ and $T^+ := T_f^+(\det \rho_f^{-1})$ be their respective twists by $\det \rho_f^{-1}$. We define a p-adic Selmer group of f over $\mathbf{Q}_\infty$ to be the subgroup

$$Sel_{\mathbf{Q}_\infty,L}(f) \subset \ker\left\{H^1(\mathbf{Q}_\infty, T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p) \to H^1(\mathbf{Q}_{\infty,p}, T/T^+ \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p)\right\}$$

of classes unramified at all finite places not dividing p, where the map is that induced by restriction and where $\mathbf{Q}_{\infty,p}$ is the completion of $\mathbf{Q}_\infty$ at the unique prime above p. This is a discrete $\Lambda_{\mathbf{Q},O_L}$-module. Its Pontrjagin dual $X_{\mathbf{Q}_\infty,L}(f) := \mathrm{Hom}_{cont}(Sel_{\mathbf{Q}_\infty,L}(f), \mathbf{Q}_p/\mathbf{Z}_p)$ is a finitely-generated $\Lambda_{\mathbf{Q},O_L}$-module. The characteristic ideal $Ch_{\mathbf{Q}_\infty,L}(f)$ of $Sel_{\mathbf{Q}_\infty,L}(f)$ is defined to be the characteristic ideal in $\Lambda_{\mathbf{Q},O_L}$ of the module $X_{\mathbf{Q}_\infty,L}(f)$. The group $Sel_{\mathbf{Q}_\infty,L}(f)$ depends on the choice of the lattice T_f, but this dependency is reflected in the characteristic ideal $Ch_{\mathbf{Q}_\infty,L}(f)$ only at its valuation at the prime containing p. In particular, $Ch_{\mathbf{Q}_\infty,L}(f)$ is well-defined in $\Lambda_{\mathbf{Q},O_L} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Furthermore, if ρ_f is residually irreducible then the isomorphism class of the $\Lambda_{\mathbf{Q},O_L}$-module $Sel_{\mathbf{Q}_\infty,L}(f)$ is independent of the choice of T_f and so $Ch_{\mathbf{Q}_\infty,L}(f) \subseteq \Lambda_{\mathbf{Q},O_L}$ is well-defined. For more precise definitions and references regarding these Selmer groups and p-adic L-functions see 3.3 and 3.4 below.

Iwasawa-Greenberg Main Conjecture for f.

$$Ch_{\mathbf{Q}_\infty}(f) = (\mathcal{L}_f) \ \text{ in } \ \Lambda_{\mathbf{Q},O_L} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

and furthermore, if ρ_f is residually irreducible then this equality holds in $\Lambda_{\mathbf{Q},O_L}$.

Kato [Ka04, Theorem 17.4] has proven that $\mathcal{L}_f \in Ch_{\mathbf{Q}_\infty,L}(f)$ under certain hypotheses on f and ρ_f. The following theorem, establishing the main conjecture in many cases, is one of the main results of this paper.

Theorem 1 (Theorem 3.6.4). *Suppose*

- $\chi = 1$ *and* $k \equiv 2 \bmod p-1$*;*
- *the reduction* $\bar{\rho}_f$ *of* ρ_f *modulo the maximal ideal of* O_L *is irreducible;*
- *there exists a prime* $q \neq p$ *such that* $q || N$ *and* $\bar{\rho}_f$ *is ramified at* q*;*
- $p \nmid N$.

Then $Ch_{\mathbf{Q}_\infty,L}(f) = (\mathcal{L}_f)$ *in* $\Lambda_{\mathbf{Q},O_L} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. *If furthermore*

- *there exists an* O_L*-basis of* T_f *with respect to which the image of* ρ_f *contains* $\mathrm{SL}_2(\mathbf{Z}_p)$,

then the equality holds in $\Lambda_{\mathbf{Q},O_L}$*; that is, the Iwasawa-Greenberg Main Conjecture for* f *is true.*

This theorem is deduced by combining Kato's result with the main theorem of this paper (Theorem 3.6.1; see also Theorem 3 below) which proves one of the divisibilities ('p-adic L-function divides characteristic ideal') of the Iwasawa-Greenberg Main Conjecture for a Hida family of eigenforms and an imaginary quadratic field. This main theorem should be thought of as part of a three-variable main conjecture, one variable being the variable in the Hida family and the two other variables being cyclotomic and anti-cyclotomic characters of the maximal $\mathbf{Z}_p$-extension of the imaginary quadratic field. For more on the general Iwasawa-Greenberg Main Conjectures, the reader should consult §3 for the special cases of interest for this paper and the papers of Greenberg more generally, especially [Gr94].

The hypotheses of Theorem 1 intervene at various points in the proof, occasionally only to shorten an argument. Following the statement of our main theorems - Theorems 3.6.1 and 3.6.4 - we have attempted to indicate the places in the proof where these hypotheses have been used.

1.2. **Applications to elliptic curves.** When the f in Theorem 1 is the newform associated with an elliptic curve $E_{/\mathbf{Q}}$ the hypotheses of this theorem are frequently satisfied. For example they are always satisfied if E has semistable reduction and $p \geq 11$ is a prime of good ordinary reduction for E. In any case, the above theorem implies the main conjecture[1] for many elliptic curves (see Theorem 3.6.8). As shown by Greenberg, this has consequences for the Birch-Swinnerton-Dyer formula for E.

Theorem 2 (Theorem 3.6.11). *Let E be an elliptic curve over $\mathbf{Q}$ with conductor N_E. Suppose*

- *E has good ordinary reduction at p;*
- *$\bar{\rho}_{E,p}$ is irreducible;*
- *there exists a prime $q \neq p$ such that $q||N_E$ and $\bar{\rho}_{E,p}$ is ramified at q.*

(a) If $L(E,1) \neq 0$ and $\bar{\rho}_{E,p}$ is surjective then

$$\left|\frac{L(E,1)}{\Omega_E}\right|_p^{-1} = \#\text{Ш}(E/\mathbf{Q})_p \cdot \prod_{\ell | N_E} c_\ell(E).$$

(b) If $L(E,1) = 0$ then the corank of the Selmer group $Sel_{p^\infty}(E/\mathbf{Q})$ is at least one.

Here $\bar{\rho}_{E,p}$ is the representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $E[p]$, $\text{Ш}(E/\mathbf{Q})_p$ is the p-primary part of the Tate-Shafarevich group of $E/\mathbf{Q}$, $c_\ell(E) := |\#E(\mathbf{Q}_\ell)/E_0(\mathbf{Q}_\ell)|_p^{-1}$ is the maximal power of p that divides the Tamagawa number of E at the prime ℓ, and $Sel_{p^\infty}(E/\mathbf{Q})$ is the p^∞-Selmer group of $E/\mathbf{Q}$. Again we note that the hypotheses of the theorem are satisfied if E is semistable and $p \geq 11$ is a prime of good ordinary reduction.

1.3. **The nature of the proof.** Iwasawa's original Main Conjecture (cf. [Gr75]) identified the Kubota-Leopoldt p-adic L-function of an even Dirichlet character χ as a generator of the characteristic ideal of the p-adic Selmer group over $\mathbf{Q}_\infty$ of $\chi\epsilon$ (ϵ is the p-adic cyclotomic character; χ is identified with a Galois character with the same L-function as the Dirichlet character). This conjecture was first proved by Mazur and Wiles [MW84]

[1] In the case of elliptic curves this conjecture was first stated by Mazur and Swinnerton-Dyer [MSD74].

by analyzing the cuspidal subgroup of quotients of Jacobians of modular curves; we refer the interested reader to the original paper of Mazur and Wiles for more history regarding this conjecture and its proof. Then in [Wi90] Wiles proved the Iwasawa Main Conjecture for all totally real fields. The proof in [Wi90] involves an extensive generalization of the construction in [Ri76], replacing the analysis of cuspidal subgroups of Jacobians in [MW84] with congruences between p-adic families of Eisenstein series and p-adic families of cuspforms. The resulting relation between these congruences and the constant term of the Eisenstein family (essentially the p-adic L-function) is combined with the Galois representations associated with families of cuspforms to prove that the p-adic L-function divides the characteristic ideal. When combined with the analytic class number formula, this implies equality. Subsequent to the work of Mazur and Wiles, another proof of the Main Conjecture for $\mathbf{Q}$ was given by Rubin (based on work of Kolyvagin and Thaine) using an Euler system. The Euler system argument yields a result opposite to that obtained via congruences: the characteristic ideal contains the p-adic L-function. But again, together with the analytic class number formula this implies equality. Rubin also used Euler systems and the analytic class number formula to prove the one- and two-variable main conjectures for imaginary quadratic fields [Ru91]; this includes the Iwasawa-Greenberg Main Conjecture for CM forms. Using an Euler system constucted from elements in K-groups of modular curves, Kato proved what amounts to half of the Iwasawa-Greenberg Main Conjecture for modular forms (Theorem 3.5.6). Lacking an analog of the analytic class number formula, Kato's result does not imply the main conjecture in general.

The main result of this paper - Theorem 3 below (also Theorem 3.6.1) - is proved following the strategy used by Wiles in his proof of the main conjecture for totally real fields [Wi90]. In essence the result is an inclusion (divisibility) in the opposite direction from that in Kato's theorem. Combining the results then yields equality. The strategy of studying congruences between Eisenstein families and cuspidal families has also been employed by the second named author [Ur04] to prove many cases of the Iwasawa-Greenberg Main Conjecture for the adjoint of a modular form; there the class number formula is the one that appears in the theory of Galois deformations as in the work of Wiles [Wi95] and its various extensions to totally real fields. There have been other results proved in the direction of various main conjectures, too many to survey here. While most have made use of Euler systems, some, most notably [MTi90] and [HiTi94], have exploited congruences between cuspforms and various 'special' modular forms, analogously to the approach employed in this paper.

In this paper we work in the context of automorphic forms on the unitary group $G := GU(2,2)_{/\mathbf{Q}}$ defined by a Hermitian pairing of signature $(2,2)$ on a four-dimensional space over an imaginary quadratic field $\mathcal{K}$. The connection with L-functions for elliptic modular forms comes through constant terms of Eisenstein series. Let P be the maximal $\mathbf{Q}$-parabolic subgroup of G fixing an isotropic line. Then P has Levi decomposition $P = MN$ with Levi subgroup $M \cong GU(1,1)_{/\mathbf{Q}} \times \mathrm{Res}_{\mathcal{K}/\mathbf{Q}}\mathbf{G}_m$. The Eisenstein series on $G(\mathbf{A}_\mathbf{Q})$ induced from cuspforms on $M(\mathbf{A}_\mathbf{Q})$ have constant terms along P that involve L-series of the form $L^S(BC(\pi)\otimes\xi, s)$ where π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$,

$BC(\pi)$ is its base change to $\mathrm{GL}_2(\mathbf{A}_\mathcal{K})$, and ξ is an idele class character of $\mathbf{A}_\mathcal{K}^\times$. When interpreted classically, for an appropriate choice of inducing data this yields holomorphic scalar-valued Hermitian Eisenstein series E on $\mathbf{H} := \{Z \in M_2(\mathbf{C}) \ : \ -i(Z - {}^t\bar{Z}) > 0\}$ (the Hermitian upper half-space) whose singular Fourier coefficients are simple multiples of products of normalized L-values of the form $\mathfrak{L} := L^S(f,\xi,k-1)L^S(\chi^{-1}\xi', k-2)/\Omega$. Here ξ is a finite idele class character, $\xi' := \xi|_{\mathbf{A}_\mathbf{Q}^\times}$, f is a weight k eigenform of character χ, $L^S(f,\xi,k-1) := L^S(\mathrm{BC}(\pi(f))\otimes\xi,(k-1)/2)$ with $\pi(f)$ the usual unitary representation associated with f, and Ω is an explicit algebraic multiple of a product of periods. Also, $L^S(-)$ denote the partial L-function with the Euler factors at the places in S removed. One might hope that the Fourier coefficients of the Eisenstein series E so-constructed are p-adic integers and that the singular Fourier coefficients are divisible by $\mathfrak{L}$. If $\mathfrak{L}$ is not a p-adic unit, then one would expect that E is congruent modulo $\mathfrak{L}$ to a cusp form, provided some non-singular Fourier coefficient is a p-adic unit. If such were the case, then this congruence could be combined with the p-adic Galois representations associated with cuspidal eigenforms on $\mathbf{H}$ to construct classes in a Selmer group related to f and ξ. This last part is a generalization of the Galois arguements in [Ri76] and [Wi90]. Carrying this argument out for a p-adic family of Eisenstein series, where $\mathfrak{L}$ is replaced by a product of p-adic L-functions, leads to the main theorem of this paper. We do this as outlined in the following.

1.4. **An outline of the proof.** The proof of the main result of this paper can be divided into two parts. The first part, comprising §§2-7, explains how the index of a certain ideal - the Eisenstein ideal - in the cuspidal p-ordinary Hecke algebra for the unitary group $GU(2,2)$ divides the characteristic ideal of a certain three-variable Selmer group and how this implies the Iwasawa-Greenberg Main Conjecture if the index is divisible by a certain three-variable p-adic L-function. The second part, comprising §§8-13, constructs a p-adic family of Eisenstein series for $GU(2,2)$ with singular Fourier coefficients divisible by the three-variable p-adic L-function, and this family is then used to relate the index of the Eisenstein ideal to the p-adic L-function.

1.4.1. *Selmer groups.* After introducing in §2 the notation and conventions necessary to describe many of the objects studied in this paper and the main results about them, in §3 we develop the theory of Selmer groups as used in this paper, particularly the relations between Selmer groups over various fields and rings and the corresponding relations between characteristic ideals. This includes the results that allow the deduction of Theorem 1 from the main theorem of this paper (in combination with Kato's theorem). In §3 we state the main results of this paper about Selmer groups and their connections with p-adic L-functions and explain how they follow from the main theorem.

The Selmer groups that appear in the main results of this paper are associated with Hida families of ordinary cuspidal eigenforms of tame level N and character χ (a Dirichlet character modulo Np). Let L be a finite extension of $\mathbf{Q}_p$ containing the values of χ. Such a family is a formal q-expansion $\mathbf{f} = \sum_{n=1}^\infty \mathbf{a}(n)q^n \in \mathbb{I}[\![q]\!]$, $\mathbb{I}$ a local reduced finite integral extension of $\Lambda_{W,O_L} := O_L[\![W]\!]$, with the property that for a continuous O_L-homomorphism $\phi : \mathbb{I} \to \overline{\mathbf{Q}}_p$ with $\phi(1+W) = (1+p)^{\kappa_\phi - 2}$ and $\kappa_\phi \geq 2$ an integer, $\mathbf{f}_\phi := \sum_{n=1}^\infty \phi(\mathbf{a}(n))q^n$ is a p-ordinary cuspidal eigenform of weight κ_ϕ, level Np, and character $\chi\omega^{\kappa_\phi - 2}$. Associated with a Hida family $\mathbf{f}$ are continuous two-dimensional semisimple Galois representations $\rho_\mathbf{f} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_{F_\mathbb{I}}(V_\mathbf{f})$, $F_\mathbb{I}$ being the fraction field of $\mathbb{I}$, and $\bar{\rho}_\mathbf{f} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_\mathbf{F}(V)$, $\mathbf{F}$ being the residue field of $\mathbb{I}$, such that for all primes $\ell \nmid Np$ the trace of these representations on a geometric frobenius element for ℓ is the image of the Fourier coefficient $\mathbf{a}(\ell)$. Assume that

$$(\mathbf{irred})_\mathbf{f} \qquad \bar{\rho}_\mathbf{f} \text{ is irreducible.}$$

Then there is a basis such that $\rho_\mathbf{f}$ takes values in $\mathrm{GL}_\mathbb{I}(T_\mathbf{f})$ with $T_\mathbb{I}$ a free $\mathbb{I}$-module of rank two. The image of $T_\mathbf{f}$ under a homomorphism ϕ as above is just a lattice $T_{\mathbf{f}_\phi}$ in the usual p-adic Galois representation $\rho_{\mathbf{f}_\phi}$ associated with the ordinary eigenform $\mathbf{f}_\phi$. If we also assume that

$$(\mathbf{dist})_\mathbf{f} \qquad \bar{\rho}_\mathbf{f} \text{ is } G_p\text{-distinquished,}$$

meaning that the semisimplification of $\bar{\rho}_\mathbf{f}|_{G_p}$, G_p being a decomposition group at p, is a sum of two distinct characters, then there exists an unramified G_p-stable rank-one $\mathbb{I}$-summand $T_\mathbf{f}^+ \subset T_\mathbf{f}$.

Let $\mathcal{K}$ be an imaginary quadratic field in which the prime p splits. Let $\mathcal{K}_\infty$ be the composite of all $\mathbf{Z}_p$-extensions of $\mathcal{K}$ and $\Gamma_\mathcal{K}$ its Galois group over $\mathcal{K}$ (so $\Gamma_\mathcal{K} \cong \mathbf{Z}_p^2$). Then to $\mathbf{f}$ and any finite set of primes Σ we attach a Selmer group $Sel_{\mathcal{K}_\infty}^\Sigma(\mathbf{f})$, defined analogously to the Selmer group for an eigenform. This is a discrete module over $\mathbb{I}_\mathcal{K} := \mathbb{I}[\![\Gamma_\mathcal{K}]\!]$. We explain how it is a consequence of Kato's work and and the relation of $Sel_{\mathcal{K}_\infty}^\Sigma(\mathbf{f})$ with other Selmer groups that the pontrjagin dual $X_{\mathcal{K}_\infty}^\Sigma(\mathbf{f}) := \mathrm{Hom}_{cont}(Sel_{\mathcal{K}_\infty}^\Sigma(\mathbf{f}), \mathbf{Q}_p/\mathbf{Z}_p)$ is a finitely-generated torsion $\mathbb{I}_\mathcal{K}$-module. We can therefore define the characteristic ideal $Ch_{\mathcal{K}_\infty}^\Sigma(\mathbf{f})$ in $\mathbb{I}_\mathcal{K}$ of $X_{\mathcal{K}_\infty}^\Sigma(\mathbf{f})$, and having done so we can then state the main result of this paper:

Theorem 3 (Theorem 3.6.1)**.** *Let* $\mathbf{f}$ *be an* $\mathbb{I}$*-adic ordinary eigenform of tame level* N *and trivial character. Assume that* L *contains* $\mathbf{Q}[\boldsymbol{\mu}_{Np}, i, D_\mathcal{K}^{1/2}]$. *Suppose* $N = N^+N^-$ *with* N^+ *divisible only by primes that split in* $\mathcal{K}$ *and* N^- *divisible only by primes inert in* $\mathcal{K}$. *Suppose also*

- $(\mathbf{irred})_\mathbf{f}$ *and* $(\mathbf{dist})_\mathbf{f}$ *hold;*
- N^- *is square-free and has an odd number of prime factors;*
- *the reduction* $\bar{\rho}_\mathbf{f}$ *of* $\rho_\mathbf{f}$ *modulo the maximal ideal of* $\mathbb{I}$ *is ramified at all* $\ell | N^-$.

Let Σ *be a finite set of primes containing all those that divide* $ND_\mathcal{K}$ *and some prime* $\ell \neq p$ *that splits in* $\mathcal{K}$. *Then*

$$Ch_{\mathcal{K}_\infty}^\Sigma(\mathbf{f}) \subseteq (\mathcal{L}_{\mathbf{f},\mathcal{K}}^\Sigma).$$

Here $\mathcal{L}_{\mathbf{f},\mathcal{K}}^\Sigma \in \mathbb{I}_\mathcal{K}$ is a certain three-variable p-adic L-function (described in 3.4.5 but constructed later on). We explain how to deduce Theorem 1 from this theorem and

Kato's theorem using the relations between $Sel^{\Sigma}_{\mathcal{K}_\infty}(\mathbf{f})$ and the Selmer groups for the various $\mathbf{f}_\phi$ together with the corresponding relations between $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$ and the usual p-adic L-functions $\mathcal{L}_{\mathbf{f}_\phi}$. From this we are also able to deduce that the inclusion in Theorem 3 is often an equality (see Theorem 3.6.6). We also explain other consquences, including Theorem 2.

We follow the discussion of Selmer groups in §3 with an exposition in §4 of an abstract set-up and concomitant construction of subgroups of group cohomology classes. This construction provides a means for relating orders of certain congruence ideals to orders of characteristic ideals; the setting abstracts that obtained from the Eisenstein ideals studied later. This generalizes and formalizes the construction of Selmer classes using Galois representations and congruences that was alluded to in 1.3.

1.4.2. *Hida theory, the Eisenstein ideal, and Galois representations.* In §5 and §6 we describe Hida theory for p-adic modular forms for $G = GU(2,2)$. In particular we explain the surjectivity of certain Λ-adic Siegel operators. This last point is the key to connecting the divisibility properties of singular Fourier coefficients of the p-adic families of Eisenstein series to congruences with cusp forms. Here $\Lambda = \mathbf{Z}_p[\![T(\mathbf{Z}_p)]\!]$ with $T \subset G$ a certain maximal torus. Our exposition of Hida theory generally follows [Hi04], but whereas Hida restricts attention to cuspidal forms, we require a theory for modular forms with non-zero constant terms. The explanation of this augmentation of Hida's results necessitates that we review (or sketch proofs of) some facts about the construction and nature of the arithmetic toroidal and minimal compactifications of the Shimura varieties associated with the unitary similitude groups $GU(n,n)$. After defining the spaces of ordinary Λ-adic forms (Hida families) for the groups $GU(n,n)$, we define an ideal in a Hida Hecke algebra for G and explain how, given the existence of a suitable Hida family of Eisenstein series, its index - the Eisenstein ideal - is related to p-adic L-functions.

A Hida family $\mathbf{f}$ and a set of primes Σ as in Theorem 3 gives rise to a tuple $\mathbf{D}$ that we term a p-adic Eisenstein datum. We put $\Lambda_{\mathbf{D}} := \mathbb{I}_{\mathcal{K}}[\![\Gamma^-_{\mathcal{K}}]\!]$, $\Gamma^-_{\mathcal{K}}$ the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension of $\mathcal{K}$; this has the structure of a finite Λ-algebra. A Hida family over $\Lambda_{\mathbf{D}}$ (of prescribed tame level $K_{\mathbf{D}} \subset G(\mathbf{A}^p_f)$) is then a collection of formal series $\mathbf{F} = (\mathbf{F}_x)$ (indexed by certain cusps x) with $\mathbf{F}_x \in \Lambda_{\mathbf{D}}[\![q^\beta]\!]$, where β runs over a lattice of Hermitian matrices in $M_2(\mathcal{K})$ that depends on x and $K_{\mathbf{D}}$, such that for a certain class of continuous $\mathcal{O}_L$-homomorphisms $\phi : \Lambda_{\mathbf{D}} \to \overline{\mathbf{Q}}_p$, the specialization of $\mathbf{F}$ at ϕ - the collection of formal series obtained from applying ϕ to the coefficients of the $\mathbf{F}_x$ - is the collection of q-expansions at the cusps x of a p-ordinary holomorphic Hermitian modular form on $\mathbf{H} \times G(\mathbf{A}_f)$. The set of such forms is a finite free $\Lambda_{\mathbf{D}}$-module and there is a natural Hecke action on this space. We let $\mathbf{h}_{\mathbf{D}}$ be the $\Lambda_{\mathbf{D}}$-algebra generated by the Hecke operators acting on the $\Lambda_{\mathbf{D}}$-cuspforms; this is a finite $\Lambda_{\mathbf{D}}$-algebra. We define an ideal $I_{\mathbf{D}} \subseteq \mathbf{h}_D$ - determined by $\mathbf{D}$ - and consider the quotient $\mathbf{h}_{\mathbf{D}}/I_{\mathbf{D}}$. By the definition of $I_{\mathbf{D}}$, this is a quotient of $\Lambda_{\mathbf{D}}$ via the structure map. The Eisenstein ideal $\mathcal{E}_{\mathbf{D}} \subseteq \Lambda_{\mathbf{D}}$ is the kernel of this surjection: $\Lambda_{\mathbf{D}}/\mathcal{E}_{\mathbf{D}} \xrightarrow{\sim} \mathbf{h}_D/I_{\mathbf{D}}$. The ideal $I_{\mathbf{D}}$ is defined with the expectation that there exists a $\Lambda_{\mathbf{D}}$-eigenform $\mathbf{E}_{\mathbf{D}}$ such that $I_{\mathbf{D}}$ is generated by the image in $\mathbf{h}_{\mathbf{D}}$ of the annihilator of $\mathbf{E}_{\mathbf{D}}$ in the abstract Hecke algebra and such that the singular coefficients of $\mathbf{E_D}$ are divisible by the three-variable p-adic L-function $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}} \in \mathbb{I}_{\mathcal{K}}$. Assuming such a form $\mathbf{E_D}$ exists, from the aforementioned surjectivity of the $\Lambda_{\mathbf{D}}$-Siegel operators we conclude that if $P \subset \mathbb{I}_{\mathcal{K}}$ is a height one prime containing $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$ but $\mathbf{E_D}$ is non-zero modulo P, then $\mathrm{ord}_P(\mathcal{E}_{\mathbf{D}}) \geq \mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}})$.

In §7 we recall the Galois representations associated with cuspidal representations of G and explain the existence of Galois representations associated with Hida eigen-families and particulary with components of $\mathbf{h_D}$. Using this we show that the ring $\mathbf{h_D}$, the ideal $I_{\mathbf{D}}$, and a prime $P \subset \mathbb{I}_{\mathcal{K}}$ as in the preceding paragraph give rise to a set-up as formalized in §4. Assuming the existence of the Hida family $\mathbf{E_D}$, the main theorem then follows from the abstract construction given there and the inequality relating the orders of $\mathcal{E}_{\mathbf{D}}$ and $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$.

The rest of the paper is taken up with proving the existence of the three-variable p-adic L-function $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$ and the Hida family $\mathbf{E_D}$.

1.4.3. *Eisenstein series.* After introducing more notation in §8, in §9 we define the Eisenstein series that belong to the family $\mathbf{E_D}$ and describe their Hecke eigenvalues and constant terms. More specifically, for certain of the homomorphisms ϕ we define an Eisenstein datum $\mathfrak{D}_\phi$ and thence an Eisenstein series $E_{\mathfrak{D}_\phi}$. This series is induced from from a cuspform on $M(\mathbf{A})$ associated with $\mathbf{f}_\phi$. The singular Fourier coefficients turn out to be essentially Fourier coefficients of $\mathbf{f}_\phi$.

In §10 and §11 we recall some auxiliary functions - theta functions and Siegel Eisenstein series - that show-up in our analysis of the p-adic properties of the $E_{\mathfrak{D}_\phi}$. In particular, in §11 we recall a formula of Garrett and Shimura that essentially expresses a multiple $G_{\mathfrak{D}_\phi}$ of $E_{\mathfrak{D}_\phi}$ as an inner-product of $\mathbf{f}_\phi$ and the pull-back to $\mathfrak{h} \times \mathbf{H}$ of a Siegel Eisenstein series on $GU(3,3)$. The multiple $G_{\mathfrak{D}_\phi}$ is essentially $L^{\Sigma}(\mathbf{f}_\phi, \xi_\phi, k_\phi - 1)L^{\Sigma}(\chi^{-1}_{\mathbf{f}_\phi}\xi'_\phi, k_\phi - 2)E_{\mathfrak{D}_\phi}$, with ξ_ϕ a finite idele class character of $\mathcal{K}$. We use the pull-back formula to express the Fourier coefficients of $G_{\mathfrak{D}_\phi}$ as inner-products of modular forms - the inner-products of $\mathbf{f}_\phi$ with the restrictions of Fourier-Jacobi coefficients of the Siegel Eisenstein series (these Fourier-Jacobi coefficients are essentially products of theta functions and Eisenstein series on $\mathfrak{h}$). Similarly, the L-function $L^{\Sigma}(\mathbf{f}_\phi, \xi_\phi, k_\phi - 1)$ is realized as an inner product of $\mathbf{f}_\phi$ with the pull-back to $\mathfrak{h} \times \mathfrak{h}$ of a Siegel Eisenstein series on $GU(2,2)$. Much of §11 is taken up with the definitions of the ramified local sections used to define the Siegel Eisenstein series and the computations of the Fourier-Jacobi coefficients of these series.

To help orient the reader amidst the notationally dense calculations in §§9,10, and 11, we have included a more detailed summary of their contents at the start of each of these sections.

The use of the pull-back formula to express the Fourier coefficients of cuspidal Eisenstein series like $G_{\mathfrak{D}_\phi}$ as inner-products of modular forms was also exploited in [Ur04], and the idea goes back at least to [BSP]. The work [Ur04] contains a similar analysis of the ramified local sections defining the Siegel-Eisenstein series. In [Zh07], a similar use was made of the pull-back formula to analyze the Fourier-Jacobi expansions of cuspidal Eisenstein series on $GU(3,1)$.

The non-singular Fourier coefficients of $G_{\mathfrak{D}_\phi}$ are indexed by positive-definite Hermitian matrices $\beta \in \mathrm{GL}_2(\mathcal{K})$, and the coefficients for a given β define automorphic functions on the unitary group $U(\beta)$ of the Hermitian pairing on $\mathcal{K}^2$ defined by β. Pairing these functions with a suitable automorphic form on $U(\beta)$ - which amounts to taking a linear combination of Fourier coefficients of $G_{\mathfrak{D}_\phi}$ - results in a Rankin-Selberg convolution of $\mathbf{f}_\phi$ with a theta lift to GL_2 of the form on $U(\beta)$ (this lift has weight 2). If all the data has been chosen well (and one is lucky) the resulting formulas are essentially products of L-functions whose arithmetic properties are sufficiently understood.

1.4.4. *p-adic interpolations.* In §12 we use the formulas from §11 to construct the $\Lambda_{\mathbf{D}}$ Hida families $\mathbf{E}_{\mathbf{D}}$ (which essentially specialize to $G_{\mathfrak{D}_\phi}$) and the three-variable p-adic L-function $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$ (which we relate to the p-adic L-functions $\mathcal{L}_{\mathbf{f}_\phi}$ and other p-adic L-functions such as the anticyclotomic L-functions of the $\mathbf{f}_\phi$). That the singular coefficients of $\mathbf{E}_{\mathbf{D}}$ are divisible by $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$ is immediate. The key to this interpolation is that the Fourier coefficients of Siegel Eisenstein are particularly simple; they p-adically interpolate by inspection. Via the pull-back formulas, this p-adic interpolation carries over to the L-functions and to the Fourier coefficients of the induced Eisenstein series.

1.4.5. *Co-primality of the p-adic L-functions and Eisenstein series.* Finally, in §13 we show that the Fourier coefficients of the family $\mathbf{E}_{\mathbf{D}}$ have the needed properties; essentially, the non-singular coefficients are prime to the p-adic L-function. Our proof of this involves some of the formulas from §11 and appeals to the mod p non-vanishing results of Vatsal [Va03] and Finis [Fi06]. The appeal to the former in particular is responsible for some of the hypotheses in the main theorem as well as Theorems 1 and 2. An appeal to Vatsal's non-vanishing result is also made in [Ur04]. There the resulting formula for the Fourier coefficient of the Eisenstein series is essentially the Rankin-Selberg convolution of an eigenform with a weight one theta function (as opposed to the weight two theta functions that show up in the formulas in this paper), and so a suitable linear combination of Fourier coefficients is a special value of the twist of the L-function of an eigenform by a finite Hecke character; Vatsal's theorem applies directly to this last L-value. The situation in this paper is less straightforward.

Using our formulas for linear combinations of Fourier coefficients of $G_{\mathfrak{D}_\phi}$, we show that there is a p-adic family $\mathbf{g}$ of CM forms such that a suitable $\Lambda_{\mathbf{D}}$-combination of non-singular coefficients of $\mathbf{E}_{\mathbf{D}}$ factors as $\mathcal{A}_{\mathbf{D},\mathbf{g}}\mathcal{B}_{\mathbf{D},\mathbf{g}}$ with $\mathcal{A}_{\mathbf{D},\mathbf{g}} \in \mathbb{I}[\![\Gamma^+_{\mathcal{K}}]\!] \subset \mathbb{I}_{\mathcal{K}}$, $\Gamma^+_{\mathcal{K}}$ the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of $\mathcal{K}$, and $\mathcal{B}_{\mathbf{D},\mathbf{g}} \in \mathbb{I}_{\mathcal{K}}$. The factor $\mathcal{A}_{\mathbf{D},\mathbf{g}}$ interpolates Rankin-Selberg convolutions of the $\mathbf{f}_\phi$ with weight two specializations of $\mathbf{g}$ and so is easily observed to be non-zero. That $\mathcal{A}_{\mathbf{D},\mathbf{g}}$ is co-prime to $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$ under the hypotheses of the main theorem then follows from Vatsal's result on the vanishing of the anticyclotomic μ-invariant (and the relation of $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$ with the p-adic anticyclotomic L-functions). The factor $\mathcal{B}_{\mathbf{D},\mathbf{g}}$ specializes under some ϕ to a convolution of a specialization of $\mathbf{g}$ and a weight one Eisenstein series, and we show by appeal to the results of Finis that $\mathbf{g}$ can be chosen so that this convolution - and hence $\mathcal{B}_{\mathbf{D},\mathbf{g}}$ - is a unit.

We have included an index to important notation not defined in §2 or §8

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Contents

2. Basic notations and conventions

In this section we collect the notation and conventions for fields, characters, and Galois representations needed to describe the basic framework and main results of this paper. For the most part these follow conventional practice. Additional notation will, of course, be introduced later in the paper; particularly significant notation or conventions will be given at the start of each section.

Throughout this paper p is a fixed odd prime number.

2.1. **Fields and Galois groups.**

2.1.1. *Number fields.* We fix algebraic closures $\overline{\mathbf{Q}}$ and $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}$ and $\mathbf{Q}_p$, respectively. For the purpose of p-adic interpolation we fix embeddings $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ and $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and a compatible isomorphism $\iota_p' : \overline{\mathbf{Q}}_p \stackrel{\sim}{\to} \mathbf{C}$ (so that $\iota_\infty = \iota_p' \circ \iota_p$).

The adeles of a number field F are denoted by $\mathbf{A}_F$ and the finite adeles are denoted by $\mathbf{A}_{F,f}$. When $F = \mathbf{Q}$ we will often drop it from our notation for the adeles. We let $\widehat{\mathbf{Z}} := \prod_\ell \mathbf{Z}_\ell \subset \mathbf{A}_f$ (so $\mathbf{A}_f = \widehat{\mathbf{Z}} \otimes \mathbf{Q}$). If v is a place of $\mathbf{Q}$ and $x \in \mathbf{A}_\mathbf{Q}$ then we write x_v for the v-component of x, and similarly for $\mathbf{A}_F$. For a place v of $\mathbf{Q}$, $|\cdot|_v$ will denote the usual absolute value on $\mathbf{Q}_v$ ($|\ell|_\ell = \ell^{-1}$) and $|\cdot|_\mathbf{Q}$ the corresponding absolute value on $\mathbf{A}$ ($|x|_\mathbf{Q} = \prod_v |x_v|_v$). We define an absolute value $|\cdot|_F$ on $\mathbf{A}_F$ by $|x|_F := |N_{F/\mathbf{Q}}(x)|_\mathbf{Q}$, where $N_{F/\mathbf{Q}}(x)$ is the norm from $\mathbf{A}_F$ to $\mathbf{A}$. We reserve $|\cdot|$ to denote the usual absolute value on $\mathbf{C}$ and $\mathbf{R}$.

We let $\mathcal{K} \subseteq \overline{\mathbf{Q}}$ be an imaginary quadratic extension of $\mathbf{Q}$ in which p splits and denote the ring of integers of $\mathcal{K}$ by $\mathcal{O}$. The absolute discriminant, class number, and different of $\mathcal{K}$ are denoted by $D_\mathcal{K}$, $h_\mathcal{K}$, and $\mathfrak{d}$, respectively. We let $\delta_\mathcal{K} := \sqrt{-D_\mathcal{K}}$ (so this generates $\mathfrak{d}$). The action of the nontrivial automorphism of $\mathcal{K}$ is often denoted by a 'bar' (thus $x \in \mathcal{K}$ is sent to $\bar{x}$ by this automorphism). For any $\mathbf{Z}$-algebra A this extends to $\mathcal{O} \otimes A$ and $\mathcal{K} \otimes A$ through its action on the first factor.

We let v_0 be the place of $\mathcal{K}$ over $\mathbf{Q}$ determined by the fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. We denote its conjugate place by $\bar{v}_0$. We let $\mathfrak{p}$ be the prime ideal of $\mathcal{O}$ corresponding to v_0 and let $\bar{\mathfrak{p}}$ be its conjugate ideal (so in $\mathcal{O}$, $(p) = \mathfrak{p}\bar{\mathfrak{p}}$).

If v is a finite place of $\mathbf{Q}$ then $\mathcal{K}_v := \mathcal{K} \otimes_\mathbf{Q} \mathbf{Q}_v$ and $\mathcal{O}_v := \mathcal{O} \otimes \mathbf{Z}_v$. If w is a finite place of $\mathcal{K}$ then $\mathcal{K}_w$ and $\mathcal{O}_w$ have their usual meanings. For a prime ℓ, D_ℓ is the absolute discriminant of $\mathcal{K}_\ell$ over $\mathbf{Q}_\ell$. If ℓ splits in $\mathcal{K}$ then we fix a $\mathbf{Z}_\ell$-algebra identification of $\mathcal{O}_\ell$ with $\mathbf{Z}_\ell \times \mathbf{Z}_\ell$ and hence of $\mathcal{K}_\ell$ with $\mathbf{Q}_\ell \times \mathbf{Q}_\ell$. With respect to these identifications, if $(a,b) \in \mathcal{K}_\ell$ then $\overline{(a,b)} = (b,a)$. We assume that the identification $\mathcal{K}_p = \mathbf{Q}_p \times \mathbf{Q}_p$ has been made so that $\mathfrak{p} = K \cap (p\mathbf{Z}_p \times \mathbf{Z}_p)$. We identify $\mathcal{K} \otimes \mathbf{R}$ with $\mathbf{C}$ by $x \otimes y \mapsto \iota_\infty(x)y$. We similary identifty $\mathcal{K} \otimes \mathbf{C}$ with $\mathbf{C} \times \mathbf{C}$ by $x \otimes y \mapsto (\iota_\infty(x)y, \overline{\iota_\infty(x)}y)$.

We let $\mathbf{Q}_\infty \subset \overline{\mathbf{Q}}$ be the unique $\mathbf{Z}_p$-extension of $\mathbf{Q}$ and let $\mathcal{K}_\infty \subset \overline{\mathbf{Q}}$ be the unique $\mathbf{Z}_p^2$-extension of $\mathcal{K}$. We let $\mathcal{K}_\infty^+$ and $\mathcal{K}_\infty^-$ be, respectively, the cyclotomic and anti-cyclomotic

$\mathbf{Z}_p$-extensions (so $\mathcal{K}_\infty^+ = \mathcal{K}\mathbf{Q}_\infty$ and $\mathcal{K}_\infty^+ \cap \mathcal{K}_\infty^- = \mathcal{K}$). In the context of these extensions we will write $\mathbf{Q}_n$, $\mathcal{K}_n$, and $\mathcal{K}_n^\pm$ to mean the maximal subfields of conductor p^{n+1}.

2.1.2. *Galois groups.* For any subfield $F \subseteq \overline{\mathbf{Q}}$ we let $G_F := \mathrm{Gal}(\overline{\mathbf{Q}}/F)$. Given a set Σ of finite places of F, we let $G_{F,\Sigma} := \mathrm{Gal}(F_\Sigma/F)$, F_Σ being the maximal extension of F unramified at all finite places not in Σ. If Σ is a set of finite places of a subfield of F we write $G_{F,\Sigma}$ for $G_{F,\Sigma'}$, where Σ' is the set of places of F over those in Σ.

A decomposition group at a place v of F will be denoted $G_{F,v}$ and its inertia subgroup will be denoted $I_{F,v}$. If the choice of decomposition group is important, the choice will be made clear in the text. When F is understood we will often drop it from our notation for decomposition and inertia groups.

If F is finite over $\mathbf{Q}$, then for a finite place v of F we will write frob_v for a *geometric* Frobenius element (well-defined only in $G_{F,v}/I_{F,v}$). The L-function of a Galois representation of G_F will always be defined with respect to geometric Frobenius elements.

When $F = \mathbf{Q}$ or $\mathcal{K}$ we fix choices of decomposition groups. When $v = \infty$ we assume that $G_{F,v}$ is the decomposition group determined by the fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. We let $c \in G_{\mathbf{Q},\infty}$ be the unique nontrivial element; this is a complex conjugation which agrees with the usual complex conjugation on $\mathbf{C}$ via the fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. The restriction of c to $\mathcal{K}$ is the nontrivial automorphism of $\mathcal{K}$, so no confusion should result from our also denoting the action of c on an element of $\mathbf{C}$ by 'bar' (i.e., writing $\bar{x}$ to mean $c(x)$). When v is the place determined by the fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ we assume $G_{F,v}$ is the decomposition group determined by this embedding. We choose $G_{\mathcal{K},\bar{v}_0} = cG_{\mathcal{K},v_0}c^{-1}$.

2.1.3. *Reciprocity maps.* For a local or global field F we normalize the reciprocity map rec_F of class field theory so that uniformizers get mapped to geometric Frobenius elements.

2.1.4. *Hodge-Tate weights.* Unless otherwise stated, whenever we discuss Hodge-Tate weights for a p-adic Galois representation of $G_\mathcal{K}$ they are for the place v_0.

2.1.5. *The groups* $\Gamma_\mathbf{Q}$, $\Gamma_\mathcal{K}$, *and* $\Gamma_\mathcal{K}^\pm$. We let $\Gamma_\mathbf{Q} := \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ and $\Gamma_\mathcal{K} := \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$ and let $\Gamma_\mathcal{K}^\pm \subset \Gamma_\mathcal{K}$ be the subgroup on which conjugation by c acts as ± 1. Then $\Gamma_\mathcal{K} = \Gamma_\mathcal{K}^+ \oplus \Gamma_\mathcal{K}^-$. Via the canonical projection $\Gamma_\mathcal{K} \to \mathrm{Gal}(\mathcal{K}_\infty^\pm/\mathcal{K})$, $\Gamma_\mathcal{K}^\pm$ is identified with the target. Via the canonical projection $\Gamma_\mathcal{K} \to \Gamma_\mathbf{Q}$, $\Gamma_\mathcal{K}^+$ is identified with $\Gamma_\mathbf{Q}$. We fix topological generators $\gamma_\pm \in \Gamma_\mathcal{K}^\pm$ and let $\gamma \in \Gamma_\mathbf{Q}$ be the topological generator identified with γ_+. To simplify matters we will assume these have been chosen so that $\mathrm{rec}_{\mathbf{Q}_p}(1+p) = \gamma$ and $\mathrm{rec}_{\mathcal{K}_p}((1+p)^{1/2}, (1+p)^{-1/2}) = \gamma_-$.

2.2. **Characters.** Let F be a number field.

2.2.1. *Idele class characters.* For an idele class character $\chi : \mathbf{A}_F^\times \to \mathbf{C}^\times$ (so trivial on $F^\times$) we write $\mathfrak{f}_\chi$ for the conductor of χ. If $F = \mathbf{Q}$ then we make no distinction between the ideal $\mathfrak{f}_\chi$ and the positive integer that generates it. For any finite set of places S of F we set

$$L^S(\chi, s) := \prod_{v \nmid \mathfrak{f}_\chi, v \notin S} (1 - \chi_v(\varpi_v) q_v^{-s})^{-1}, \quad \chi = \otimes \chi_v,$$

where ϖ_v is a uniformizer at v. If S is a finite set of places of a subfield of F we use the same notation for the product over places that do not divide a place in S. The same convention will be used for Euler products throughout this paper.

Suppose $F = \mathbf{Q}$ or $\mathcal{K}$ and $\chi_\infty(z) = \mathrm{sgn}(z)^b z^a$ if $F = \mathbf{Q}$ and $\chi_\infty(z) = z^a \bar{z}^b$ if $F = \mathcal{K}$, a and b being integers. We denote by σ_χ the (unique) p-adic Galois character

$$\sigma_\chi : G_F \to \overline{\mathbf{Q}}_p^\times$$

such that

$$\sigma_\chi(\mathrm{frob}_v) = \chi_v(\varpi_v), \quad v \nmid p\mathfrak{f}_\chi.$$

For any finite set of places $S \supseteq \{v|p\}$

$$L^S(\sigma_\chi, s) = L^S(\chi, s).$$

If $F = \mathbf{Q}$ then the motivic weight of σ_χ is $-2a$ and its Hodge-Tate weight is $-a$. If $F = \mathcal{K}$ then the motivic weight of σ_χ is $-(a+b)$ and its Hodge-Tate weight with respect to the place v_0 is $-a$. We let σ_χ^c be the composition of σ_χ with conjugation of G_F by c.

Given an idele class character χ of $\mathbf{A}_\mathcal{K}^\times$, we let $\chi^c(x) := \chi(\bar{x})$ and $\chi' := \chi|_{\mathbf{A}_\mathbf{Q}^\times}$, where $\mathbf{A}_\mathbf{Q} \hookrightarrow \mathbf{A}_\mathcal{K}$ is the canonical inclusion. Then $\chi\chi^c = \chi' \circ N_{\mathcal{K}/\mathbf{Q}}$, so

$$\sigma_\chi \sigma_\chi^c = \sigma_{\chi\chi^c} = \sigma_{\chi' \circ N_{\mathcal{K}/\mathbf{Q}}},$$

provided σ_χ exists. Note that $\sigma_\chi^c = \sigma_{\chi^c}$.

2.2.2. *Hecke characters.* If ψ is a Hecke character of F of conductor $\mathfrak{f}_\psi$, then we associate with ψ an idele class character $\otimes\psi_v$ of $\mathbf{A}_F^\times$ in the usual way. In particular, for a finite place $v \nmid \mathfrak{f}_v$, $\psi_v(\varpi_v) = \psi(\mathfrak{p}_v)$, where ϖ_v is a uniformizer at v and $\mathfrak{p}_v$ is the prime ideal corresponding to v. We will continue to denote $\otimes\psi_v$ by ψ; this should cause no confusion.

If ψ is a Dirichlet character of conductor N then we associate with it a Hecke character, which we also denote ψ, of conductor N such that $\psi((\ell)) = \psi(\ell)$ for all $\ell \nmid N$. Then $\psi_\ell(\ell) = \psi(\ell)$ for all primes $\ell \nmid N$.

2.2.3. *The cyclotomic character.* We denote by ϵ the p-adic Galois character associated with the character $|\cdot|_\mathbf{Q}$. Then $\epsilon|_{G_F}$ is the the Galois character associated with $|\cdot|_F$. The character ϵ gives the action of Galois on p-power roots of unity: if $v \nmid p$ is a place of F then $\epsilon(\mathrm{frob}_v) = N_{F/\mathbf{Q}} v^{-1}$. The Hodge-Tate weight of ϵ is -1 and its motivic weight is -2.

2.2.4. *The Teichmüller character.* Let ω be the composition $G_{\mathbf{Q}} \stackrel{\epsilon}{\to} \mathbf{Z}_p^\times \to (\mathbf{Z}_p/p\mathbf{Z}_p)^\times \to \mathbf{Z}_p^\times$, where the second arrow is reduction modulo p and the third is the Teichmüller lift. Via the reciprocity map, ω induces a character of $\mathbf{A}^\times$, which we continue to denote by ω. In this way, we can view ω as a character of $\mathbf{Z}_p^\times$ (via the inclusion $\mathbf{Q}_p{}^\times \hookrightarrow \mathbf{A}_{\mathbf{Q}}^\times$).

2.2.5. *The character associated with $\mathcal{K}$.* Let $\chi_{\mathcal{K}} : \mathbf{A}^\times \to \mathbf{C}^\times$ be the usual quadratic character associated with $\mathcal{K}$. We also write $\chi_{\mathcal{K}}$ for the p-adic Galois character associated with this idele class character; this should cause no confusion as which character is meant will always be clear from the context.

3. Selmer groups

In this section we define the Selmer groups that are the main focus of this paper. The first two subsections below contain sorites on p-adic Selmer groups; these should be well-known to experts. In the third subsection we define the Selmer groups studied in this paper. In the fourth and fifth subsections we recall the p-adic L-functions associated with modular forms, the main conjectures for the Selmer groups associated with an ordinary cuspidal eigenform and their corollaries, and the results of Kato [Ka04] about these conjectures. The last subsection is devoted to the statements of the main results of this paper.

We make no claim of originality for the results in 3.1 and 3.2 below. Most of the results therein can be found in different guises elsewhere in the literature, especially in the papers of Greenberg. In particular, we learned the proof of Proposition 3.2.8 from [Gr94].

3.1. **Sorites on Selmer groups.** The following develops the theory of Selmer groups as needed to understand the main results of this paper and their proofs.

3.1.1. *Shapiro's lemma.* Let E/F be a finite Galois extension of fields. By Shapiro's lemma, for any discrete G_E-module M there is a canonical isomorphism

$$H^i(E,M) = H^i(F, \mathrm{Ind}_{G_E}^{G_F} M)) \tag{3.1.1.a}$$

with $\mathrm{Ind}_{G_E}^{G_F} M := \{\phi : G_F \to M \ : \phi(gg') = g\phi(g') \ \forall g \in G_E\}$. If the G_E-action on M is the restriction of a G_F-action, then we have an isomorphism of G_F-modules

$$\mathrm{Ind}_{G_E}^{G_F} M \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(E/F)], M)$$

given by

$$\phi \mapsto \left(\sum_{g \in \mathrm{Gal}(E/F)} n_g g \mapsto \sum_{g \in \mathrm{Gal}(E/F)} n_g g \phi(g^{-1}) \right).$$

The action of $\sigma \in \mathrm{Gal}(E/F)$ on $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(E/F)], M)$ is $(\sigma \cdot f)(x) = \sigma f(\sigma^{-1}x)$. Therefore, if M is a G_F-module then there is a canonical isomorphism

$$H^i(E, M) = H^i(F, \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(E/F)], M)).$$

3.1.2. *Shapiro's lemma and restriction.* For analyzing Selmer groups, it is useful to know how Shapiro's lemma interacts with restrictions at finite places. To this end, let v be a finite place of F and $G_{F,v}$ a decomposition group for v. Let $w_0|v$ be a place of E fixed by $G_{F,v}$. Then $G_{E,w_0} := G_{F,v} \cap G_E$ is a decomposition group for w_0. For each place $w|v$ of E fix $g_w \in G_F$ such that $g_w w_0 = w$ and put $G_{F,w} := g_w G_{F,v} g_w^{-1}$ and $G_{E,w} := g_w G_{E,w_0} g_w^{-1}$. The latter are decomposition groups for v and w, respectively. There is a $G_{F,v}$-module isomorphism

$$\mathrm{Ind}_{G_E}^{G_F} M \xrightarrow{\sim} \prod_{w|v} \mathrm{Ind}_{G_{E,w}}^{G_{F,w}} M, \qquad (3.1.2.a)$$
$$\phi \mapsto (\phi_w)_{w|v}, \quad \phi_w(g) := \phi(g g_w).$$

The $G_{F,v}$-action on the right-hand side is given by $(g \cdot \phi_w)(g') = \phi_w(g' g_w g g_w^{-1})$. The isomorphism (3.1.2.a) induces an isomorphism

$$H^i(F_v, \mathrm{Ind}_{G_E}^{G_M} M) \xrightarrow{\sim} \prod_{w|v} H^i(F_v, \mathrm{Ind}_{G_{E,w}}^{G_{F,w}} M) \xrightarrow{\sim} \prod_{w|v} H^i(G_{F,w}, \mathrm{Ind}_{G_{E,w}}^{G_{F,w}} M) = \prod_{w|v} H^i(E_w, M),$$

with the equality denoting the canonical identification coming from Shapiro's lemma and the middle isomorphism given on cocycles by $(c_w)_{w|v} \mapsto (c'_w)_{w|v}$ with $c'_w(g) = c_w(g_w^{-1} g g_w)$. This isomorphism fits into a commutative diagram

$$\begin{array}{ccc} H^i(F, \mathrm{Ind}_{G_E}^{G_F} M) & = & H^1(E, M) \\ \downarrow{\scriptstyle res} & & \downarrow{\scriptstyle res} \\ H^i(F_v, \mathrm{Ind}_{G_E}^{G_M} M) & \xrightarrow{\sim} & \prod_{w|v} H^i(E_w, M). \end{array} \qquad (3.1.2.b)$$

Let $I_v \subset G_{F,v}$, $I_{F,w} \subset G_{F,w}$, and $I_w \subset G_{E,w}$ be the respective inertia subgroups. From (3.1.2.a) we obtain an isomorphism

$$(\mathrm{Ind}_{G_E}^{G_F} M)^{I_v} \xrightarrow{\sim} \prod_{w|v} (\mathrm{Ind}_{G_{E,w}}^{G_{F,w}} M)^{I_{F,w}}$$

and that the natural inclusion

$$\mathrm{Ind}_{G_{k_w}}^{G_{k_v}} M^{I_w} \hookrightarrow (\mathrm{Ind}_{G_{E,w}}^{G_{F,w}} M)^{I_{F,w}}$$

is an isomorphism. Here k_v and k_w are, respectively, the residue fields of F_v and E_w. It follows that the bottom isomorphism of (3.1.2.b) identifies $H^1(k_v, (\mathrm{Ind}_{G_E}^{G_F} M)^{I_v})$ with $\prod_{w|v} H^1(k_w, M^{I_w})$. In particular, a class in $H^1(F, \mathrm{Ind}_{G_E}^{G_F} M)$ is unramified at v if and only if the corresponding class in $H^1(E, M)$ is unramified at all $w|v$.

Suppose M is a G_F-module. The isomorphism (3.1.1.a) can be rewritten as

$$\mathrm{Ind}_{G_E}^{G_F} M \xrightarrow{\sim} M \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(E/F)], \mathbf{Z}).$$

If $M^+ \subseteq M$ is a $G_{F,v}$-submodule, then from (3.1.2.a) we obtain an isomophism of $G_{F,v}$-submodules

$$M^+ \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(E/F)], \mathbf{Z}) \xrightarrow{\sim} \prod_{w|v} M_w^+ \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(E_w/F_v)], \mathbf{Z}),$$

where $M_w^+ := g_w M^+ g_w^{-1}$, and hence the bottom isomorphism of (3.1.2.b) identifies the image of $H^1(F_v, M^+ \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(E/F)], \mathbf{Z}))$ in $H^1(F_v, M \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(E/F)], \mathbf{Z}))$ with the image of $\prod_{w|v} H^1(E_w, M_w^+)$ in $\prod_{w|v} H^1(E_w, M)$. In particular, the restriction of a class in $H^1(F, \mathrm{Ind}_{G_E}^{G_F} M)$ to $H^1(F_v, \mathrm{Ind}_{G_E}^{G_F} M) = H^1(F_v, M \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(E/F)], \mathbf{Z}))$ is in the image of $H^1(F_v, M^+ \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(E/F)], \mathbf{Z}))$ if and only if the restriction of the corresponding class in $H^1(E, M)$ to $\prod_{w|v} H^1(E_w, M)$ is in the image of $\prod_{w|v} H^1(E_w, M_w^+)$.

3.1.3. *Σ-primitive Selmer groups.* Let $F \subseteq \overline{\mathbf{Q}}$. Let T be a free module of finite rank over a profinite $\mathbf{Z}_p$-algebra A and assume that T is equipped with a continuous action of G_F. We assume that for each place $v|p$ of F we are given a G_v-stable free A-direct summand $T_v \subset T$. Let Σ be a set of finite places of F. We denote by $Sel_F^\Sigma(T, (T_v)_{v|p})$ the kernel of the restriction map

$$Sel_F^\Sigma(T, (T_v)_{v|p}) := \ker\{H^1(F, T \otimes_A A^*) \to \prod_{\substack{v \notin \Sigma \\ v \nmid p}} H^1(I_v, T \otimes_A A^*) \times \prod_{v|p} H^1(I_v, T/T_v \otimes_A A^*)\},$$

where $A^* := \mathrm{Hom}_{cont}(A, \mathbf{Q}_p/\mathbf{Z}_p)$ is the Pontrjagin dual of A. (We will similarly denote by M^* the Pontrjagin dual of any locally compact $\mathbf{Z}_p$-module M.) This is the Σ-primitive Selmer group. The Selmer groups are independent of the choices of the decomposition groups. If Σ contains all the places at which T is ramified and all the places over p, then

$$Sel_F^\Sigma(T, (T_v)_{v|p}) = \ker\{H^1(G_{F,\Sigma}, T \otimes_A A^*) \to \prod_{v|p} H^1(I_v, T/T_v \otimes_A A^*)\}.$$

We also put

$$X_F^\Sigma(T, (T_v)_{v|p}) := \mathrm{Hom}_A(Sel_F^\Sigma(T, (T_v)_{v|p}), A^*).$$

We will just write $X_F^\Sigma(T)$ or $Sel_F^\Sigma(T)$ when the T_v's are clear from the context or are not important.

Given T, $\{T_v\}_{v|p}$ and Σ as above for a given F, for any extension E/F (not necessarily finite) we put $Sel_E^\Sigma(T) := Sel_E^{\Sigma_E}(T, (T_w)_{w|p})$ and $X_E^\Sigma(T) := X_E^{\Sigma_E}(T, (T_w)_{w|p})$, where Σ_E is the set of places of E over those in Σ and if $w|v|p$ then $T_w = g_w T_v$ for $g_w \in G_F$ such that $g_w^{-1} G_{E,w} g_w \subseteq G_{F,v}$. This is independent of the choices of g_w's. We have

$$Sel_E^\Sigma(T) = \varinjlim_{F \subseteq F' \subseteq E} Sel_{F'}^\Sigma(T) \quad \text{and} \quad X_E^\Sigma(T) = \varprojlim_{F \subseteq F' \subseteq E} X_{F'}^\Sigma(T),$$

where F' runs over the finite extensions of F contained in E.

When Σ is empty or contains only primes over p we drop it from the notation. The corresponding Selmer group is called the primitive Selmer group.

3.1.4. *Passing from F to F^+.* Assume that F is a CM number field and let F^+ be its maximal totally real subfield (so c restricts to the nontrivial element of $\mathrm{Gal}(F/F^+)$). Let T be as above with the additional assumption that the G_F action on T is the restriction of a G_{F^+}-action. Let $v|p$ be a place of F^+. If v is inert in F then we assume that $G_{F,v} \subset G_{F^+,v}$. If v splits in F then we fix a splitting $v = ww^c$ and assume that $G_{F,w} = G_{F,v}$ and $G_{F,w^c} = cG_{F^+,v}c^{-1}$. For each place $v|p$ of F^+, we assume we are given a $G_{F^+,v}$-stable A-summand $T_v \subset T$. If v splits in F, $v = ww^c$, then we let $T_w = T_v$ and $T_{w^c} = cT_v$.

Let Σ^+ be a finite set of finite places of F^+ and let Σ be the set of places of F over those in Σ^+. We may then define $Sel_F^{\Sigma}(T)$, $Sel_{F^+}^{\Sigma^+}(T)$, and $Sel_{F^+}^{\Sigma^+}(T \otimes \chi_F)$, where χ_F is the quadratic character of G_{F^+} corresponding to the extension F/F^+. Since $p > 2$ the usual action of $\mathrm{Gal}(F/F^+)$ on $H^1(F, T \otimes_A A^*)$ yields a decomposition

$$Sel_F^{\Sigma}(T) = Sel_F^{\Sigma}(T)^+ \oplus Sel_F^{\Sigma}(T)^-,$$

where the $\pm$ superscript denotes that c acts as ± 1.

Lemma 3.1.5. *The restriction map from G_{F^+} to G_F yields isomorphisms*

$$Sel_{F^+}^{\Sigma^+}(T) \xrightarrow{\sim} Sel_F^{\Sigma}(T)^+ \qquad Sel_{F^+}^{\Sigma^+}(T \otimes \chi_F) \xrightarrow{\sim} Sel_F^{\Sigma}(T)^-.$$

Proof. This follows easily from Shapiro's lemma, the inflation-restriction sequence, and the fact that $\mathbf{Z}_p[\mathrm{Gal}(F/F^+)] \cong \mathbf{Z}_p \oplus \mathbf{Z}_p(\chi_F)$ as a G_{F^+}-module.∎

3.1.6. *Fitting ideals and characteristic ideals.* Let A be a noetherian ring. We write $Fitt_A(X)$ for the Fitting ideal in A of a finitely generated A-module X (and therefore of finite presentation). This is the ideal generated by the determinant of the $r \times r$-minors of the matrix giving the first arrow in a given presentation of X:

$$A^s \to A^r \to X \to 0.$$

In particular, if X is not a torsion A-module then $Fitt_A(X) = 0$.

Fitting ideals behave well with respect to base change. For any noetherian A-algebra B, $Fitt_B(X \otimes_A B) = Fitt_A(X)B$. In particular, if $I \subset A$ is an ideal, then

$$Fitt_{A/I}(X/IX) = Fitt_A(X) \bmod I.$$

To define the notion of characteristic ideal we need to recall a few facts about divisorial ideals. Recall first that a divisorial ideal is an ideal which is equal to the intersection of all principal ideals containing it. In particular any principal ideal is divisorial. Let us assume now that A is a noetherian normal domain. For any prime ideal $Q \subset A$ of height one, denote by ord_Q the essential valuation attached to Q. Then any divisorial ideal I is of the form

$$I = \{x \in A \ : \ \mathrm{ord}_Q(x) \geq m_Q \ \forall \ Q \text{ of height one } \},$$

where the m_Q are non-negative integers almost all equal to zero. The m_Q's are uniquely determined and we set $\mathrm{ord}_Q(I) := m_Q$ (A_Q is a DVR and $\mathrm{ord}_Q(I)$ is the valuation of any generator of IA_Q). If I and J are two divisorial ideals, then the following are equivalent:

(i) $\mathrm{ord}_Q(I) \leq \mathrm{ord}_Q(J)$ for all prime ideals Q of height one
(ii) $I \supseteq J$.

In particular, if I is divisorial and $x \in A$, we have $(x) \supseteq I$ if and only if $\mathrm{ord}_Q(I) \geq \mathrm{ord}_Q(x)$ for all Q of height one. The characteristic ideal of an A-module X is the divisorial ideal $Char_A(X)$ defined by

$$Char_A(X) := \{x \in A \ : \ \mathrm{ord}_Q(x) \geq \ell_Q(X) \ \forall \ Q \text{ of height one } \},$$

where $\ell_Q(X)$ is the A_Q-length of the Q-localization X_Q (possibly infinite). One checks easily that

$$Char_A(A/Fitt_A(X)) = Char_A(X).$$

Unlike Fitting ideals, characteristic ideals do not behave well under base change in general. This is particularly true if X contains a nontrivial pseudo-null submodule. However, since inclusion of divisorial ideals is easier to recognize, most of the time we will work with characteristic ideals. In the cases of interest to us, for the purposes of base change we are able to make do with a weaker statement (see Corollary 3.2.9).

The following easy lemma will be useful in identifying Fitting and characteristic ideals.

Lemma 3.1.7. *Let A be a ring, $\mathfrak{a} \subset A$ a proper ideal contained in the Jacobson radical of A, and assume that $A/\mathfrak{a}$ is a domain. Let $\mathcal{L} \in A$ be such that its reduction $\bar{\mathcal{L}}$ modulo $\mathfrak{a}$ is non-zero. Let $I \subseteq (\mathcal{L})$ be an ideal and let $\bar{I}$ its image in $A/\mathfrak{a}$. If $\bar{\mathcal{L}} \in \bar{I}$, then $I = (\mathcal{L})$.*

Proof. We need to show that $\mathcal{L} \in I$. As in the statement of the lemma, we denote the image of reduction modulo $\mathfrak{a}$ by a 'bar.' By assumption, there exist s $\alpha \in I$ such that $\bar{\alpha} = \bar{\mathcal{L}}$. On the other hand, since $I \subseteq (\mathcal{L})$, there exists $\beta \in A$ such that $\alpha = \beta\mathcal{L}$. Therefore $\bar{\mathcal{L}} = \bar{\beta}\bar{\mathcal{L}}$, and hence $\bar{\beta} = 1$ since $A/\mathfrak{a}$ is a domain and $\bar{\mathcal{L}}$ is non-zero. As $\mathfrak{a}$ is contained in the radical of A, it then follows that β is a unit in A, so $\mathcal{L} = \beta^{-1}\alpha \in I$. ∎

3.1.8. *Fitting and characteristic ideals of Selmer groups.*

Lemma 3.1.9. *Let F be a number field and S a finite set of finite places of F. Let A be a profinite $\mathbf{Z}_p$-algebra and let M be a finitely generated A-module equipped with a continuous action of $G_{F,S}$. Then $H^1(G_{F,S}, M \otimes_A A^*)$ is co-finitely generated over A.*

Recall that an A-module X is co-finitely generated if $\mathrm{Hom}_A(X, A^*)$ is finitely generated. A consequence of this lemma is that the dual Selmer groups $X_F^{\Sigma}(T, (T_v)_{v|p})$ defined before are finitely generated over A.

Proof. See the proposition in Section 4 of [Gr94] where it is essentially deduced from the arguments used to prove Proposition 3.2.8 below. ∎

For a number field F and T and A as before, we set

$$Ft^{\Sigma}_{F,A}(T) := Fitt_A(X^{\Sigma}_F(T)) \text{ and } Ch^{\Sigma}_{F,A}(T) := Char_A(X^{\Sigma}_F(T)).$$

Of course, we have only defined $Ch^{\Sigma}_{F,A}(T)$ if A is noetherian and normal.

3.2. **Iwasawa theory of Selmer groups.** We now develop the Iwasawa theory of Selmer group over the fields $\mathbf{Q}_\infty$, $\mathcal{K}_\infty$, and $\mathcal{K}^{\pm}_\infty$.

3.2.1. *Iwasawa algebras.* Let $\Lambda_{\mathbf{Q}} := \mathbf{Z}_p[\![\Gamma_{\mathbf{Q}}]\!]$, $\Lambda_{\mathcal{K}} := \mathbf{Z}_p[\![\Gamma_{\mathcal{K}}]\!]$, and $\Lambda^{\pm}_{\mathcal{K}} := \mathbf{Z}_p[\![\Gamma^{\pm}_{\mathcal{K}}]\!]$. The projection $\Gamma^{+}_{\mathcal{K}} \xrightarrow{\sim} \Gamma_{\mathbf{Q}}$ determines an isomorphism $\Lambda^{+}_{\mathcal{K}} \xrightarrow{\sim} \Lambda_{\mathbf{Q}}$.

Let $\boldsymbol{\varepsilon}_{\mathcal{K}} : G_{\mathcal{K}} \to \Gamma_{\mathcal{K}} \hookrightarrow \Lambda^{\times}_{\mathcal{K}}$ be the canonical character. We similarly define characters $\boldsymbol{\varepsilon}_{\mathcal{K},\pm} : G_{\mathcal{K}} \to \Lambda^{\pm,\times}_{\mathcal{K}}$ and $\boldsymbol{\varepsilon}_{\mathbf{Q}} : G_{\mathbf{Q}} \to \Lambda^{\times}$. Note that

$$\boldsymbol{\varepsilon}_{\mathbf{Q}} \bmod (\gamma - (1+p)^m) = \omega^{-m}\epsilon^m.$$

For a profinite $\mathbf{Z}_p$-algebra A we set $\Lambda_{\mathbf{Q},A} := A\hat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathbf{Q}}$, where $\hat{\otimes}_{\mathbf{Z}_p}$ denotes the tensor product in the category of profinite $\mathbf{Z}_p$-modules (and continuous morphisms); in particulary $\Lambda_{\mathbf{Q},A} = A[\![\Gamma_{\mathbf{Q}}]\!]$. We similarly define $\Lambda_{\mathcal{K},A}$ and $\Lambda^{\pm}_{\mathcal{K},A}$.

3.2.2. *Selmer groups as modules over Iwasawa algebras.* Let A be a profinite $\mathbf{Z}_p$-algebra and let T be a free A-module of finite rank equipped with a continuous A-linear action of $G_{\mathbf{Q}}$. We assume given a G_p-stable A-free direct summand T_p of T.

Shapiro's lemma provides the following.

Proposition 3.2.3. *Let $F = \mathbf{Q}$ or $\mathcal{K}$. There is a canonical isomorphism of $\Lambda_{F,A}$-modules*

$$Sel^{\Sigma}_{F_\infty}(T) \cong Sel^{\Sigma}_F(T \otimes_A \Lambda_{F,A}(\boldsymbol{\varepsilon}_F^{-1})).$$

The right-hand side is defined by viewing $T \otimes_A \Lambda_{F,A}$ as a $\Lambda_{F,A}[G_F]$-module. When $F = \mathcal{K}$ the same isomorphism holds with F_∞ replaced by $\mathcal{K}^{\pm}_\infty$, $\Lambda_{F,A}$ by $\Lambda^{\pm}_{\mathcal{K},A}$, and $\boldsymbol{\varepsilon}_F$ by $\boldsymbol{\varepsilon}_{\mathcal{K},\pm}$.

Proof. This should be well-known, but for the reader's convenience we provide a proof. For this we use Shapiro's lemma as recalled in 3.1.1, noting first that

$$\begin{aligned}\varinjlim_n \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(F_n/F)], A^*) &= \varinjlim_n \mathrm{Hom}_{cont}(\Lambda_n, \mathbf{Q}_p/\mathbf{Z}_p)\\ &= \mathrm{Hom}_{cont}(\varprojlim_n \Lambda_n, \mathbf{Q}_p/\mathbf{Z}_p) = \Lambda^*_{F,A}(\boldsymbol{\varepsilon}_F^{-1}),\end{aligned}$$

where $\Lambda_n := A[\mathrm{Gal}(F_n/F)]$ and the last identication is as $\Lambda_{F,A}[G_F]$-modules. Appealing to Shapiro's lemma, it follows that

$$\begin{aligned} H^1(F_\infty, T\otimes_A A^*) &= \varinjlim_n H^1(F_n, T\otimes_A A^*) \\ &= \varinjlim_n H^1(F, \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(F_n/F)], T\otimes_A A^*)) \\ &= \varinjlim_n H^1(F, T\otimes_A \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(F_n/F)], A^*)) \\ &= \varinjlim_n H^1(F, T\otimes_A \Lambda_n^*) = H^1(F, T\otimes_A \Lambda_{F,A}^*(\boldsymbol{\varepsilon}_F^{-1})). \end{aligned}$$

That this identifies $Sel^\Sigma_{F_\infty}(T)$ with $Sel^\Sigma_F(T\otimes_A \Lambda_{F,A}(\epsilon_F^{-1}))$ then follows from the analysis in 3.1.2. The same arguments apply to the situation where $F=\mathcal{K}$ and $F_\infty = \mathcal{K}_\infty^\pm$. ∎

Remark 3.2.4. This recovers [Gr94, Prop. 3.2].

As a direct consequence of the preceding proposition and Lemma 3.1.9, the dual group $X^\Sigma_{\mathbf{Q}_\infty,A}(T) = X^\Sigma_{\mathbf{Q},\Lambda_{\mathbf{Q}}}(T\otimes_A \Lambda_{\mathbf{Q},A}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1}))$ is finitely generated over $\Lambda_{\mathbf{Q},A}$. We then put

$$Ft^\Sigma_{\mathbf{Q}_\infty,A}(T) := Ft^\Sigma_{\mathbf{Q},\Lambda_{\mathbf{Q},A}}(T\otimes_A \Lambda_{\mathbf{Q},A}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1}))$$

and

$$Ch^\Sigma_{\mathbf{Q}_\infty,A}(T) := Ch^\Sigma_{\mathbf{Q},\Lambda_{\mathbf{Q},A}}(T\otimes_A \Lambda_{\mathbf{Q},A}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1})).$$

These belong to $\Lambda_{\mathbf{Q},A}$. We similarly define $Ft^\Sigma_{\mathcal{K}_\infty,A}(T)$, $Ch^\Sigma_{\mathcal{K}_\infty,A}(T)\in\Lambda_{\mathcal{K},A}$ and $Ft^\Sigma_{\mathcal{K}^\pm_\infty,A}(T)$, $Ch^\Sigma_{\mathcal{K}^\pm_\infty,A}(T)\in\Lambda^\pm_{\mathcal{K},A}$.

Combining Proposition 3.2.3 with Lemma 3.1.5 yields the following.

Lemma 3.2.5. *There are* $\Lambda^+_{\mathcal{K},A}$*-isomorphisms*

$$Sel^\Sigma_{\mathcal{K}^+_\infty}(T)\cong Sel^\Sigma_{\mathbf{Q}_\infty}(T)\oplus Sel^\Sigma_{\mathbf{Q}_\infty}(T\otimes\chi_{\mathcal{K}})$$

and

$$X^\Sigma_{\mathcal{K}^+_\infty}(T)\cong X^\Sigma_{\mathbf{Q}_\infty}(T)\oplus X^\Sigma_{\mathbf{Q}_\infty}(T\otimes\chi_{\mathcal{K}}).$$

3.2.6. *Dual Selmer groups as torsion modules.* Suppose that A is a domain and finite over $\mathbf{Z}_p$. Let (T,T_p) be as before and assume that T is geometric (in the sense of Fontaine-Mazur) and pure with regular Hodge-Tate weights and such that the rank of T_p is equal to the rank of the $+1$-eigenspace for the action of the complex conjugation c. Then it is conjectured (by Greenberg, Bloch-Kato, Fontaine-Perrin-Riou) that $X_{\mathbf{Q}_\infty}(T)$ (resp. $X_{\mathcal{K}_\infty}(T)$) is torsion over $\Lambda_{\mathbf{Q},A}$ (resp. $\Lambda_{\mathcal{K},A}$). When T is one dimensional this fact is a simple consequence of class field theory. In general, this seems to be a deep fact. It has been proved by K. Kato [Ka04] when T is the Galois module associated with an elliptic cuspidal eigenform f of weight $k\geq 2$ and $p\nmid N_f$ is a prime at which f is ordinary and T_p is the rank-one unramified G_p-subrepresentation; Kato's proof uses an Euler system constructed from Siegel units and the K-theory of modular curves.

3.2.7. *Control of Selmer groups.* Let A be a profinite $\mathbf{Z}_p$-algebra. By elementary properties of Pontrjagin duality, for any ideal $\mathfrak{a} \subset A$ we have a canonical isomorphism

$$A^*[\mathfrak{a}] \cong (A/\mathfrak{a})^*$$

and hence for any free A-module M a canonical identification $M/\mathfrak{a} \otimes_{A/\mathfrak{a}} (A/\mathfrak{a})^* = M \otimes_A A^*[\mathfrak{a}]$. This implies that for any (T, T_p) as in paragraph 3.2.2 we have a canonical map

$$Sel_F^{\Sigma}(T/\mathfrak{a}T) \to Sel_F^{\Sigma}(T)[\mathfrak{a}].$$

Proposition 3.2.8. *Suppose the action of I_p on T/T_p factors through the image of I_p in $\Gamma_{\mathbf{Q}}$ and that $\Sigma \cup \{p\}$ contains all primes at which T is ramified. Let $F = \mathbf{Q}_\infty$, $\mathcal{K}_\infty$, or $\mathcal{K}_\infty^+$, and suppose also that there is no nontrivial A-subquotient of T^* on which G_F acts trivially. Then the above map induces isomorphisms*

$$Sel_F^{\Sigma}(T/\mathfrak{a}T) \cong Sel_F^{\Sigma}(T)[\mathfrak{a}] \quad \textit{and} \quad X_F^{\Sigma}(T/\mathfrak{a}T) \cong X_F^{\Sigma}(T)/\mathfrak{a}X_F^{\Sigma}(T).$$

Proof. Let $S = \Sigma \cup \{p\}$. Let $x_1, \ldots, x_k$ be a system of A-generators of $\mathfrak{a}$. We prove by induction on j that $H^1(G_{F,S}, T \otimes A^*[x_1, \ldots, x_j]) \to H^1(G_{F,S}, T \otimes A^*)[x_1, \ldots, x_j]$ is an isomorphism. Assume this is known for $x_1, ...x_j$ replaced with $x_1, ..., x_{j-1}$. Consider the exact sequence

$$0 \to A^*[x_1, \ldots, x_j] \to B \overset{\times x_j}{\to} x_j B \to 0$$

with $B = A^*[x_1, \ldots, x_{j-1}]$. After tensoring with T, from the associated long exact cohomology sequence and noting that $H^0(G_{F,S}, T \otimes_A x_j B) = 0$ by the hypotheses on T^*, we find that there is an exact sequence

$$0 \to H^1(G_{F,S}, T \otimes_A A^*[x_1, \ldots, x_j]) \to H^1(G_{F,S}, T \otimes_A B) \overset{\phi}{\to} H^1(G_{F,S}, T \otimes_A x_j B).$$

From the long exact cohomology sequence associated with $0 \to x_j B \hookrightarrow B \to B/x_j B \to 0$ and the hypotheses on T^*, we get an exact sequence

$$0 \to H^1(G_{F,S}, T \otimes_A x_j B) \overset{\phi'}{\to} H^1(G_{F,S}, T \otimes_A B).$$

Since the composition $\phi' \circ \phi$ is just multiplication by x_j, it follows that $\ker \phi = H^1(G_{F,S}, T \otimes_A B)[x_j]$. By the induction hypothesis, we thus have

$$H^1(G_{F,S}, T \otimes_A A^*[x_1, \ldots, x_j]) = H^1(G_{F,S}, T \otimes_A A^*)[x_1, \ldots, x_j],$$

and therefore $H^1(G_{F,S}, T \otimes_A A^*[\mathfrak{a}]) = H^1(G_{F,S}, T \otimes_A A^*)[\mathfrak{a}]$.

Let $w|p$ be a place of F and let Let $I_w = g_w I_p g_w^{-1} \cap G_{\mathbf{Q}_\infty}$. By our hypothesis on the action of inertia at p, I_w acts trivially on $T/T_w = T/g_w T$. Therefore $H^1(I_w, T/T_w \otimes_A A^*[\mathfrak{a}]) = \mathrm{Hom}_{\mathbf{Z}}(I_w, T/T_w \otimes_A A^*[\mathfrak{a}])$ and $H^1(I_w, T/T_w \otimes_A A^*) = \mathrm{Hom}_{\mathbf{Z}}(I_w, T/T_w \otimes_A A^*)$, so $H^1(I_w, T/T_w \otimes_A A^*[\mathfrak{a}]) \hookrightarrow H^1(I_w, T/T_w \otimes_A A^*)$.

The proposition now follows from the commutativity of the diagram

$$\begin{array}{ccc}
H^1(G_{F,S}, T \otimes_A A^*[\mathfrak{a}]) & \xrightarrow{res} & \prod_{w|p} H^1(I_w, T/T_w \otimes_A A^*[\mathfrak{a}]) \\
\| & & \downarrow \\
H^1(G_{F,S}, T \otimes_A A^*)[\mathfrak{a}] & \xrightarrow{res} & \prod_{w|p} H^1(I_w, T/T_w \otimes_A A^*)
\end{array}$$

where the vertical arrows are the maps from the preceding paragraphs. ∎

Corollary 3.2.9. *Let $F = \mathbf{Q}_\infty$, $\mathcal{K}_\infty$, or $\mathcal{K}_\infty^+$. With the hypotheses and notation of the preceding proposition,*

(i) $Ft^\Sigma_{F,A/\mathfrak{a}}(T/\mathfrak{a}T) = Ft^\Sigma_{F,A}(T) \bmod \mathfrak{a}$;
(ii) *if A and $A/\mathfrak{a}$ are noetherian normal domains then $(f) \bmod \mathfrak{a}$ divides $Ch^\Sigma_{F,A/\mathfrak{a}}(T/\mathfrak{a})$ for any principal ideal $(f) \supseteq \mathrm{Ch}^\Sigma_{F,A}$; in particular, if A is a unique factorization domain then $Ch^\Sigma_{F,A}(T) \bmod \mathfrak{a}$ divides $Ch^\Sigma_{F,A/\mathfrak{a}}(T/\mathfrak{a})$.*

Proof. Part (i) follows from basic properties of Fitting ideals (cf. 3.1.6), and part (ii) follows from the fact that the characteristic ideal is the smallest divisorial ideal containing the Fitting ideal (and that principal ideals are divisorial). ∎

3.2.10. *Descent from $\mathcal{K}_\infty$ to $\mathcal{K}_\infty^+$.* Let $I^- \subset \Lambda_\mathcal{K}$ be the kernel of the surjection $\Lambda_\mathcal{K} \to \Lambda_\mathbf{Q}$ induced by the canonical projection $\Gamma_\mathcal{K} \to \Gamma_\mathbf{Q}$; we also write I^- for the kernel of the map $\Lambda_\mathcal{K}^- \to \mathbf{Z}_p$ induced by the trivial map $\Gamma_\mathcal{K}^- \to 1$. Note that the inclusion $\Lambda^+_{\mathcal{K},A} \subseteq \Lambda_{\mathcal{K},A}$ identifies $\Lambda^+_{\mathcal{K},A}$ with $\Lambda_{\mathcal{K},A}/I^-\Lambda_{\mathcal{K},A}$.

For (T, T_p) and A as in 3.2.2 there is a canonical map

$$Sel^\Sigma_{\mathcal{K}_\infty^+}(T) \to Sel^\Sigma_{\mathcal{K}_\infty}(T)[I^-]$$

of $\Lambda^+_{\mathcal{K},A}$-modules.

Proposition 3.2.11. *Under the hypotheses of Proposition 3.2.8, the above map is an isomorphism and induces an isomorphism*

$$X^\Sigma_{\mathcal{K}_\infty}(T)/I^- X^\Sigma_{\mathcal{K}_\infty}(T) \xrightarrow{\sim} X^\Sigma_{\mathcal{K}_\infty^+}(T)$$

of $\Lambda^+_{\mathcal{K},A}$-modules. Furthermore, if A is an unique factorization domain then $Ch^\Sigma_{\mathcal{K}_\infty}(T)$ mod I^- divides $Ch^\Sigma_{\mathcal{K}_\infty^+}(T)$.

Proof. The canonical map $Sel^\Sigma_{\mathcal{K}_\infty^+}(T) \to Sel^\Sigma_{\mathcal{K}_\infty}(T)[I^-]$ equals the composition map

$$Sel^\Sigma_{\mathcal{K}_\infty^+}(T) \xrightarrow{\sim} Sel^\Sigma_{\mathcal{K}_\infty^+}(T \otimes_A \Lambda^-_{\mathcal{K},A}(\varepsilon^{-1}_{\mathcal{K},-}))[I^-] = Sel^\Sigma_\mathcal{K}(T \otimes_A \Lambda_{\mathcal{K},A}(\varepsilon_\mathcal{K}^{-1}))[I^-] = Sel^\Sigma_{\mathcal{K}_\infty}(T)[I^-],$$

where the first isomorphism comes from Proposition 3.2.8 and the second and third from Proposition 3.2.3. The claim about characteristic ideals then follows from part (ii) of Corollary 3.2.9. ∎

3.2.12. *Specializing the cyclotomic variable.* Specializing the cyclotomic variable is more subtle in general; control can fail when the associated p-adic L-function has a trivial zero. The following proposition establishes a control statement for a situation in which there should be no trivial zeros.

Let (T, T_p) and A be as in 3.2.2. Let $I_{\mathbf{Q}}$ be the kernel of the surjection $\Lambda_{\mathbf{Q}} \to \mathbf{Z}_p$ induced by the trivial homomorphism $\Gamma_{\mathbf{Q}} \to 1$.

Proposition 3.2.13. *Suppose there is no nontrivial A-subquotient of T^* on which $G_{\mathbf{Q}}$ acts trivially. Assume $\Sigma \cup \{p\}$ contains all prime at which T is ramified. Then there is an exact sequence*

$$0 \to Sel^{\Sigma}_{\mathbf{Q}}(T) \to Sel^{\Sigma}_{\mathbf{Q}_\infty}(T)^{\Gamma_{\mathbf{Q}}} \to H^0(I_p, T/T_p \otimes_A \Lambda^*_{\mathbf{Q},A}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1})) \otimes_{\Lambda_{\mathbf{Q}}} \Lambda_{\mathbf{Q}}/I_{\mathbf{Q}})^{G_p}.$$

In particular, if $(H^0(I_p, T/T_p \otimes_A \Lambda^*_{\mathbf{Q},A}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1})) \otimes_{\Lambda_{\mathbf{Q}}} \Lambda_{\mathbf{Q}}/I_{\mathbf{Q}})^{G_p} = 0$, then restriction yields an isomorphism

$$Sel^{\Sigma}_{\mathbf{Q}}(T) \xrightarrow{\sim} Sel^{\Sigma}_{\mathbf{Q}_\infty}(T)^{\Gamma_{\mathbf{Q}}}$$

and even an isomorphism

$$Sel^{\Sigma}_{\mathcal{K}}(T) \xrightarrow{\sim} Sel^{\Sigma}_{\mathcal{K}^+_\infty}(T)^{\Gamma^+_{\mathcal{K}}}.$$

if there is no non-trivial A-subquotient of T^* on which $G_{\mathcal{K}}$ acts trivially.

Proof. Let $S = \Sigma \cup \{p\}$. Arguing as in the proof of Proposition 3.2.8 establishes

$$H^1(G_{\mathbf{Q}_\infty,S}, T \otimes_A A^*) \xrightarrow{\sim} H^1(G_{\mathbf{Q},S}, T \otimes_A \Lambda^*_{\mathbf{Q},A}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1}))[I_{\mathbf{Q}}].$$

On the other hand, the exact sequence

$$0 \to A^* \hookrightarrow \Lambda^*_{\mathbf{Q},A} \xrightarrow{\times(\gamma-1)} \Lambda^*_{\mathbf{Q},A} \to 0$$

yields an exact sequence

$$(H^0(I_p, T/T_p \otimes_A \Lambda^*_{\mathbf{Q},A}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1})) \otimes_{\Lambda_{\mathbf{Q}}} \Lambda_{\mathbf{Q}}/I_{\mathbf{Q}})^{G_p} \hookrightarrow H^1(I_p, T/T_p \otimes_A A^*)^{G_p} \\ \to H^1(I_p, T/T_p \otimes_A \Lambda^*_{\mathbf{Q},A}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1}))^{G_p}.$$

We deduce from this that there is an exact sequence

$$Sel^{\Sigma}_{\mathbf{Q}}(T) \hookrightarrow Sel^{\Sigma}_{\mathbf{Q}}(T \otimes_A \Lambda_{\mathbf{Q},A}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1}))[I_{\mathbf{Q}}] \to H^0(I_p, T/T_p \otimes_A \Lambda^*_{\mathbf{Q},A}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1})) \otimes_{\Lambda_{\mathbf{Q}}} \Lambda_{\mathbf{Q}}/I_{\mathbf{Q}})^{G_p},$$

where the first map is induced from the inclusion $A \hookrightarrow \Lambda_{\mathbf{Q},A}$ and, by Proposition 3.2.3, is identified with the restriction map $Sel^{\Sigma}_{\mathbf{Q}}(T) \to Sel^{\Sigma}_{\mathbf{Q}_\infty}(T)^{\Gamma_{\mathbf{Q}}}$. These arguments are easily adapted to apply to $Sel^{\Sigma}_{\mathcal{K}}(T) \to Sel^{\Sigma}_{\mathcal{K}^+_\infty}(T)^{\Gamma^+_{\mathcal{K}}}$. ∎

3.2.14. *Relaxing the ramification.* Let T and A be as in 3.2.2.

Lemma 3.2.15. *Suppose F is a number field an $v \nmid p$ a place of F at which T is unramified. Suppose also that A is noetherian and normal. Then*

$$Char_A((H^1(I_v, T\otimes_A A^*)^{G_{F,v}})^*) = (\det_A(1-q_v^{-1}\mathrm{frob}_v^{-1}|_T)).$$

Here q_v is the order of the residue field of v. The lemma is immediate from the hypotheses on T and v. As an immediate consequence of the lemma we have

Corollary 3.2.16. *Suppose F is a number field and S is a finite set of places of F not dividing p and such that T is unramified at all $v \in S$. Let Σ be any finite set of places containing S. Suppose also that A is noetherian and normal. Then*

$$Ch^{\Sigma}_{F,A}(T) \supseteq Ch^{\Sigma/S}_{F,A}(T)\cdot(\prod_{v\in S}\det_A(1-q_v^{-1}\mathrm{frob}_v^{-1}|_T)).$$

The relationship between the Selmer groups $Sel^{\Sigma}_{\mathbf{Q}_\infty}(T)$ as Σ varies has been analyzed by Greenberg and Vatsal in more detail in the second section of [GV00]. Before explaining their results, we introduce some notation. Let A be a profinite $\mathbf{Z}_p$-algebra. For any finite set of primes Σ and any compact $A[G_F]$-module M, we put

$$H^1_\Sigma(F,M) := \prod_{v\in\Sigma_F, v\nmid p,v} H^1(G_{F,v}, M),$$

where Σ_F is the set of places of F over those in Σ.

Let T and A be as in 3.2.2. Let ℓ be a prime. As explained in 3.1.2, there is an isomorphism

$$H^1(G_{\mathbf{Q},\ell}, \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q}), T\otimes_A A^*)) \xrightarrow{\sim} \prod_{w|\ell} H^1(G_{\mathbf{Q}_n,w}, T\otimes_A A^*).$$

Taking the inductive limit over n yields an isomorphism

$$H^1(G_{\mathbf{Q},\ell}, T\otimes_A \Lambda^*_{\mathbf{Q},A}(\epsilon_{\mathbf{Q}}^{-1})) \xrightarrow{\sim} \prod_{w|\ell} H^1(G_{\mathbf{Q}_\infty,w}, T\otimes_A A^*).$$

In particular, there is a $\Lambda_{\mathbf{Q},A}$-isomorphism

$$H^1_\Sigma(\mathbf{Q}, T\otimes_A \Lambda^*_{\mathbf{Q},A}(\epsilon_{\mathbf{Q}}^{-1})) \xrightarrow{\sim} H^1_\Sigma(\mathbf{Q}_\infty, T\otimes_A A^*).$$

Lemma 3.2.17. *Let Σ be a finite set of primes.*

(i) *$H^1_\Sigma(\mathbf{Q}_\infty, T\otimes_A A^*)^*$ is a finitely generated torsion $\Lambda_{\mathbf{Q},A}$-module having no non-trivial pseudo-null $\Lambda_{\mathbf{Q},A}$-submodules.*
(ii) *If A is noetherian and normal, then the characteristic ideal of $H^1_\Sigma(\mathbf{Q}_\infty, T\otimes_A A^*)$ is the ideal generated by*

$$\prod_{\ell\in\Sigma} P_{T,\ell}(\ell^{-1}\boldsymbol{\varepsilon}_{\mathbf{Q}}(\mathrm{frob}_\ell)),$$

where $P_{T,\ell}(X) := \det_{F_A}(1 - X \cdot \mathrm{frob}_\ell^{-1}; (T \otimes_A F_A)^{I_\ell})$ *with* F_A *the total ring of fractions of* A.

Proof. The first part follows from a simple computation and actually holds even if we assume only that T is finitely generated over A. The second part follows from a simple adaptation of the proof of Proposition 2.4 in [GV00]. ∎

Proposition 3.2.18. *Let* (T, T_p) *and* A *be as in 3.2.2. Suppose* A *has a decreasing sequence of ideals* $A \supseteq I_1 \supseteq I_2 \supseteq \cdots$ *such that* $\cap_{n=1}^{\infty} I_n = 0$ *and each* A/I_n *is a free* $\mathbf{Z}_p$*-module of finite rank. Suppose also that* $X_{\mathbf{Q}_\infty,A}(T,T_p)$ *is a torsion* $\Lambda_{\mathbf{Q},A}$*-module and there are no nontrivial* A*-submodules of* T^* *on which* $G_{\mathbf{Q}}$ *acts trivially. Suppose* $\Sigma \cup \{p\}$ *contains all primes at which* T *is ramified. For any finite sets of primes* $\Sigma' \subset \Sigma$ *there is an exact sequence of* $\Lambda_{\mathbf{Q},A}$*-modules*

$$0 \to Sel_{\mathbf{Q}_\infty}^{\Sigma'}(T,T_p) \hookrightarrow Sel_{\mathbf{Q}_\infty}^{\Sigma}(T,T_p) \stackrel{res}{\to} H^1_{\Sigma/\Sigma'}(\mathbf{Q}_\infty, T \otimes_A A^*) \to 0,$$

and hence a dual exact sequence of $\Lambda_{\mathbf{Q},A}$*-modules*

$$0 \to H^1_{\Sigma/\Sigma'}(\mathbf{Q}_\infty, T \otimes_A A^*)^* \to X_{\mathbf{Q}_\infty}^{\Sigma}(T,T_p) \to X_{\mathbf{Q}_\infty}^{\Sigma'}(T,T_p) \to 0.$$

Proof. When A is the ring of integers of a finite extension of $\mathbf{Q}_p$, this is just Corollary 2.3 of [GV00], proved by a Poitou-Tate duality argument. The general case follows from this one. Clearly we just need to show that $Sel_{\mathbf{Q}_\infty}^{\Sigma}(T,T_p) \stackrel{res}{\to} H^1_{\Sigma/\Sigma'}(\mathbf{Q}_\infty, T \otimes_A A^*)$ is surjective. But by Proposition 3.2.8, $Sel_{\mathbf{Q}_\infty}(T/I_n) = Sel_{\mathbf{Q}_\infty}(T)[I_n]$, and these are torsion $\Lambda_{\mathbf{Q},A/I_n}$-modules. Appealing to the case of the proposition proved in [GV00] (with $\mathbf{Z}_p$ in place of A) gives a surjection

$$Sel_{\mathbf{Q}_\infty}^{\Sigma}(T)[I_n] \stackrel{res}{\to} H^1_{\Sigma}(\mathbf{Q}_\infty, T \otimes_A A^*[I_n]) \to 0$$

from which the desired surjection follows upon taking the direct limit over n. ∎

3.3. **Selmer groups and modular forms.** In this section we introduce the Selmer groups for ordinary modular forms. We begin by recalling some of the standard definitions and results for modular forms.

3.3.1. *Elliptic modular forms.* For positive integers N and k we let $S_k(N)$ and $M_k(N)$ denote the space of cusp forms and modular forms, respectively, of weight k for the congruence subgroups $\Gamma_1(N)$. For a Dirichlet character χ modulo N we let $S_k(N,\chi)$ and $M_k(N,\chi)$ be the respective subspaces of $S_k(N)$ and $M_k(N)$ of forms with Nebentypus χ. For $f \in M_k(N)$ we write its Fourier expansion (q-expansion) at the infinite cusp as

$$f(\tau) = \sum_{n=0}^{\infty} a(n,f) q^n, \quad q = e^{2\pi i \tau},$$

where τ is an element of the upper half-plane $\mathfrak{h}$. For a subring $A \subset \mathbf{C}$ (resp. a subring $A \subset \mathbf{C}$ containing the values of χ) we let $S_k(N;A)$ and $M_k(N;A)$ (resp. $S_k(N,\chi;A)$ and $M_k(N,\chi;A)$) be the submodules consisting of forms with q-expansion coefficients in A.

Recalling that we have identified $\overline{\mathbf{Q}}_p$ with $\mathbf{C}$ this defines modules $S_k(N;A)$, $M_k(N;A)$, $S_k(N,\chi;A)$, and $M_k(N,\chi;A)$ for subrings $A\subseteq\overline{\mathbf{Q}}_p$.

For a holomorphic function $f:\mathfrak{h}\to\mathbf{C}$ and $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbf{R})$, we define $f|_k\gamma$ as usual by $(f|_k\gamma)(\tau):=(c\tau+d)^{-k}f(\frac{a\tau+b}{c\tau+d})$. (So if $f\in S_k(N,\chi)$, then $f|_k\gamma=\chi(d)f$ for all $\gamma\in\Gamma_0(N)$.)

Recall that there is an action of the Hecke algebra $h(N)$ of level N on the spaces $S_k(N;A)$, $M_k(N;A)$, $S_k(N,\chi;A)$, and $M_k(N,\chi;A)$. The ring $h(N)$ is generated over $\mathbf{Z}$ by the so-denoted $T(n)$-operators and the diamond operators $\langle c\rangle$ for $(c,N)=1$; when n is a prime $T(n)$ is the double coset $\Gamma_1(N)\left(\begin{smallmatrix}1&0\\0&n\end{smallmatrix}\right)\Gamma_1(N)$ and $\langle c\rangle=\sigma_c$ where $\sigma_c\in\mathrm{SL}_2(\mathbf{Z})$ is such that $\sigma_c\equiv\left(\begin{smallmatrix}c^{-1}&0\\0&c\end{smallmatrix}\right)$ mod N. If $f\in S_k(N,\chi;A)$ then $f|_k\langle c\rangle=f|_k\sigma_c=\chi(c)f$.

3.3.2. *Eigenforms.* Let $f\in S_k(N)$ be a normalized eigenform for the action of $h(N)$. Here 'normalized' means $a(1,f)=1$ (so $f|_kT(n)=a(n,f)f$). Let N_f be the conductor of f (i.e., the level of the associated newform) and let χ_f be the Nebentypus of f (the unique character mod N_f such that $f|_k\langle c\rangle=\chi_f(c)f$ for all $c\in\mathbf{Z}$ such that $(c,N)=1$). The coefficients $a(n,f)$ generate a finite extension of $\mathbf{Q}$ that we denote by $\mathbf{Q}(f)$. We let $\mathbf{Z}(f)$ be the ring of integers of $\mathbf{Q}(f)$. Let $\lambda_f:h(N)\to\mathbf{Z}(f)$ be the homomorphism associated with f; λ_f is characterized by $\lambda_f(T(n))=a(n,f)$.

3.3.3. *Periods of eigenforms.* Let $f\in S_k(N)$, $k\geq 2$, be a normalized eigenform. We recall the definition of the periods of f that we will use to define the p-adic L-function of f.

Recall that the Eichler-Shimura map

$$Per:S_k(N)\to H^1(\Gamma_1(N),\mathrm{Sym}^{k-2}(\mathbf{C}^2))$$

is defined by putting $Per(g)$ equal to the class of the cocycle

$$\gamma\mapsto\int_\tau^{\gamma(\tau)}g(z)(z^{k-1},z^{k-1},\ldots,1)dz,$$

where the integration is over any path between τ and $\gamma(\tau)$. This map is $h(N)$-invariant.

Let $\mathbf{Z}(f)_{(p)}:=\mathbf{Q}(f)\cap\iota_p^{-1}(\overline{\mathbf{Z}}_p)$ and let

$$M(f)_{(p)}:=H^1(\Gamma_1(N),\mathrm{Sym}^{k-2}(\mathbf{Z}(f)_{(p)}))[\lambda_f].$$

Suppose $N=N_f$. Then $M(f)_{(p)}$ is free of rank 2 over $\mathbf{Z}(f)_{(p)}$, and via the inclusion $\mathbf{Z}(f)_{(p)}\subset\mathbf{C}$ we can view $M(f)_{(p)}$ as a submodule of $H^1(\Gamma_1(N),\mathrm{Sym}^{k-2}(\mathbf{C}^2))$ that spans the two-dimensional $\mathbf{C}$-space $H^1(\Gamma_1(N),\mathrm{Sym}^{k-2}(\mathbf{C}^2))[\lambda_f]$. We fix a $\mathbf{Z}(f)_{(p)}$-basis (γ^+,γ^-) such that $(\gamma^\pm)^\iota=\pm\gamma^\pm$, where ι is the involution associated with the conjugation action of $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ on $H^1(\Gamma_1(N),\mathrm{Sym}^{k-2}(\mathbf{C}^2))$. We define the periods $\Omega_f^\pm\in\mathbf{C}^\times$ of f by

$$Per(f)=\Omega_f^+\gamma^++\Omega_f^-\gamma^-.$$

These periods are well-defined up to units in $\mathbf{Z}(f)_{(p)}$.

3.3.4. *Galois representations for eigenforms.* Let $f \in S_k(N)$, $k \geq 2$, be a normalized eigenform. We fix $L \subset \overline{\mathbf{Q}}_p$ a finite extension of $\mathbf{Q}_p$ containing $\mathbf{Q}(f)$. Let O_L be the ring of integers of L and $\mathbf{F}_L$ its residue field. As proved by Eichler, Shimura, and Deligne, there exists a continuous semisimple Galois representation (ρ_f, V_f) over L with V_f a two-dimensional L-space and

$$\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_L(V_f)$$

a continuous homomorphism, characterized by being unramified at primes $\ell \nmid pN_f$ and the property that

$$\mathrm{tr}\,\rho_f(\mathrm{frob}_\ell) = a(\ell, f), \quad \ell \nmid pN.$$

This representation is irreducible and satisfies

$$\det \rho_f = \sigma_{\chi_f} \epsilon^{1-k}.$$

In particular, $\det \rho_f(c) = -1$.

By the continuity of ρ_f there exist $G_{\mathbf{Q}}$-stable O_L-lattices in V_f. Let $T_f \subset V_f$ be such a lattice and let $\overline{T}_f := T_f \otimes_{O_L} \mathbf{F}_L$. Then $\overline{T}_f$ has an induced continuous $\mathbf{F}_L$-linear $G_{\mathbf{Q}}$-action; we denote the corresponding homomorphism $G_{\mathbf{Q}} \to \mathrm{GL}_{\mathbf{F}_L}(\overline{T}_f)$ by $\bar{\rho}_f$. We distinguish the following case:

(**irred**) the representation $\bar{\rho}_f$ is irreducible.

When this is the case for one choice of T_f then it is true for all choices of T_f and the resulting $\mathbf{F}_L$-representations are isomorphic.

3.3.5. *p-ordinary eigenforms.* Recall that an eigenform of level divisible by p is said to be ordinary at p (or just 'ordinary' since p is fixed) if

(**ord**) $a(p, f)$ is a unit in $\overline{\mathbf{Z}}_p$.

We will say that an ordinary eigenform f of level Np^r, $p \nmid N$, is a p-stabilized newform if either $N_f = Np^r$ or $N_f = N$ and $r = 1$.

When the condition (**ord**) holds, the Galois representation (ρ_f, V_f) restricted to G_p contains a G_p-stable L-line $V_f^+ \subset V_f$ such that the action of G_p on V_f^+ is given by the unramified character whose value on frob_p is $a(p, f)$. Then I_p acts on the quotient $V_f^- := V_f / V_f^+$ by $\sigma_{\chi_f} \epsilon^{1-k}$. Let $\psi_p^\pm$ denote the $O_L^\times$-valued character giving the action of G_p on $V_f^\pm$. We distinguish the following situation:

(**dist**) ψ_p^+ and ψ_p^- are distinct modulo the maximal ideal of O_L.

3.3.6. *Selmer groups attached to ordinary eigenforms.* Let $f \in S_k(N)$, $k \geq 2$, be an ordinary normalized eigenform. Let $L \subset \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$ containing $\mathbf{Q}(f)$. Let (ρ_f, V_f) be the Galois representation associated with f as above. Fix $T_f \subset V_f$ a $G_{\mathbf{Q}}$-stable O_L-lattice and let $T_f^+ := T_f \cap V_f^+$. Let $T := T_f(\det \rho_f^{-1})$ and $T^+ := T_f^+(\det \rho_f^{-1})$. Let $F = \mathbf{Q}$, $\mathcal{K}$, $\mathbf{Q}_\infty$, $\mathcal{K}_\infty$, or $\mathcal{K}_\infty^\pm$ and let ξ be a continuous $O_L^\times$-valued character of G_F.

We define the Selmer groups and dual Selmer groups associated with f and ξ as follows. For any finite set of primes Σ we set

$$Sel^{\Sigma}_{F,L}(f,\xi) := Sel^{\Sigma}_{F}(T\otimes\xi, T^+\otimes\xi) \quad \text{and} \quad X^{\Sigma}_{F,L}(f,\xi) := X^{\Sigma}_{F}(T\otimes\xi, T^+\otimes\xi).$$

A priori these groups depend on the choice of T_f. However, if (**irred**) holds, then any two such lattices are homothetic and so their corresponding Selmer groups are isomorphic as $\Lambda_{F,\mathcal{O}_L}$-modules. In general, for different choices of lattices these modules only differ in their support at primes above p.

For $F = \mathbf{Q}_\infty$, $\mathcal{K}_\infty$, or $\mathcal{K}^{\pm}_\infty$ we put

$$Ch^{\Sigma}_{F,L}(f,\xi) := Ch^{\Sigma}_{F,O_L}(T\otimes\xi, T^+\otimes\xi) \quad \text{and} \quad Ft^{\Sigma}_{F,L}(f,\xi) := Ft^{\Sigma}_{F,O_L}(T\otimes\xi, T^+\otimes\xi).$$

When $\Sigma \subseteq \{p\}$ or ξ is trivial, then we drop it from our notation.

The following is an important fact about these Selmer groups. Its proof is due to Kato (see [Ka04, Thm. 17.4(1)]).

Theorem 3.3.7 (Kato)**.** *For f as above with $p\nmid N_f$, the dual Selmer groups $X^{\Sigma}_{\mathbf{Q}_\infty}(f,\xi)$ are torsion $\Lambda_{\mathbf{Q},O_L}$-modules.*

3.3.8. *Some p-adic deformations of characters.* For $F = \mathbf{Q}$ or $\mathcal{K}$ let $\Psi_F : \mathbf{A}_F^\times/F^\times \to \Lambda_F^\times$ be $\Psi_F := \boldsymbol{\varepsilon}_F \circ \mathrm{rec}_F$. We similarly define $\Psi^{\pm}_{\mathcal{K}} : \mathbf{A}^\times_{\mathcal{K}}/\mathcal{K}^\times \to \mathbf{Z}_p[\![\mathrm{Gal}(\mathcal{K}^{\pm}_\infty/\mathcal{K})]\!]^\times = \Lambda^{\pm,\times}_{\mathcal{K}}$ via the projection $\Gamma^{\pm}_{\mathcal{K}} \xrightarrow{\sim} \mathrm{Gal}(\mathcal{K}^{\pm}_\infty/\mathcal{K})$.

For an indeterminate W we let $\Lambda_W := \mathbf{Z}_p[\![W]\!]$, and for any $\mathbf{Z}_p$-algebra A we let $\Lambda_{W,A} := A[\![W]\!]$. We can identify Λ_W with $\Lambda_{\mathbf{Q}}$ (resp. $\Lambda^+_{\mathcal{K}}$) by indentifying γ (resp. γ^+) with $1+W$. Via this identification $\Psi_{\mathbf{Q}}$ (resp. $\Psi^+_{\mathcal{K}}$) defines a homomorphism $\Psi_W : \mathbf{A}^\times/\mathbf{Q}^\times \to \Lambda_W^\times$ (resp. $\Psi^+_W : \mathbf{A}^\times_{\mathcal{K}}/\mathcal{K}^\times \to \Lambda^\times_W$) and $\boldsymbol{\varepsilon}_{\mathbf{Q}}$ defines a homomorphism $\boldsymbol{\varepsilon}_W : G_{\mathbf{Q}} \to \Lambda^\times_W$.

If $\phi : \Lambda_W \to \overline{\mathbf{Q}}_p$ is a $\mathbf{Z}_p$-homomorphism such that $\phi(1+W) = \zeta(1+p)^k$ with k an integer and ζ a primitive p^rth root of unity, then $x^k_\infty x_p^{-k}(\phi\circ\Psi_W)(x) = \chi_\zeta\omega^{-k}(x)|x|^k_{\mathbf{Q}}$, where χ_ζ is the unique idele class character of p-power order and conductor such that $\chi_{\zeta,p}(1+p) = \zeta$. Also, $\phi\circ\boldsymbol{\varepsilon}_W = \chi_\zeta\omega^{-k}\epsilon^k$.

3.3.9. *Arithmetic homomorphisms.* Let A be a finite integral extension of $\mathbf{Z}_p$ in $\overline{\mathbf{Q}}_p$. Given a topological A-algebra R we let $\mathcal{X}_{R,A} := \mathrm{Hom}_{cont\,A-\mathrm{alg}}(R,\overline{\mathbf{Q}}_p)$. If $A = \mathbf{Z}_p$ then we just write $\mathcal{X}_R$ for $\mathcal{X}_{R,A}$. Given $r\in R$ and $\phi\in\mathcal{X}_R$ we put $r(\phi) := \phi(r)$. We put $\mathcal{X}_{W,A} := \mathcal{X}_{\Lambda_{W,A},A}$ and $\mathcal{X}_{F,A} := \mathcal{X}_{\Lambda_{F,A},A}$ for $F = \mathbf{Q}$ or $\mathcal{K}$.

Recall that $\phi\in\mathcal{X}_{\Lambda_W}$ is called *arithmetic* if $\phi(1+W) = \zeta(1+p)^{k-2}$ for some p-power root of unity ζ and some integer k. The integer k is called the weight of ϕ and denoted k_ϕ. We let $t_\phi > 0$ be the integer such that ζ is a primitive $p^{t_\phi-1}$th root of unity; we sometimes call this the level of ϕ. We let $\chi_\phi := \chi_{\zeta^{-1}}$ with $\chi_{\zeta^{-1}}$ as in 3.3.8.

Let $\mathbb{I}$ be a local reduced finite integral extension of $\Lambda_{W,A}$. We let

$$\mathcal{X}^a_{\mathbb{I},A} := \{\phi\in\mathcal{X}_{\mathbb{I},A}\ :\phi|\Lambda_W \text{ is arithmetic},\ k_{\phi|\Lambda_W}\geq 2\}.$$

Given $\phi \in \mathcal{X}^a_{\mathbb{I},A}$ we write k_ϕ, t_ϕ, and χ_ϕ for the weight, level, and character of $\phi|\Lambda_W$.

The identification of $\Lambda_{\mathbf{Q}}$ and $\Lambda^+_{\mathcal{K}}$ with Λ_W defines a notion of arithmetic homomorphisms ϕ in $\mathcal{X}_{\Lambda_{\mathbf{Q}}}$ and $\mathcal{X}_{\Lambda^+_{\mathcal{K}}}$, with corresponding weights k_ϕ, levels t_ϕ, and characters χ_ϕ.

3.3.10. *Hida families.* Let χ be an even Dirichlet character modulo Np, $p \nmid N$. Let $\mathbb{I}$ be a local reduced finite integral extension of $\Lambda_{W,\mathbf{Z}_p[\chi]}$. Recall that an $\mathbb{I}$-adic elliptic modular form of tame level N and character χ is a q-expansion $\mathbf{f} = \sum_{n=0}^\infty \mathbf{a}(n)q^n \in \mathbb{I}[\![q]\!]$ such that for all $\phi \in \mathcal{X}^a_{\mathbb{I},\mathbf{Z}_p[\chi]}$,

$$\mathbf{f}_\phi = \sum_{n=0}^\infty \phi(\mathbf{a}(n))q^n \in M_{k_\phi}(Np^{t_\phi}, \chi\omega^{k_\phi-2}\chi_\phi; \phi(\mathbb{I})).$$

If $\mathbf{f}_\phi$ is always an eigenform (resp. cusp form, p-stabilized newform) then we say $\mathbf{f}$ is an $\mathbb{I}$-adic eigenform (resp. cusp form, newform). Similarly, if $\mathbf{f}_\phi$ is always ordinary then we say $\mathbf{f}$ is an ordinary $\mathbb{I}$-adic modular form.

Given an $\mathbb{I}$-adic form $\mathbf{f}$ we will write $\chi_{\mathbf{f}}$ for the associated tame character χ; this is called the Nebentypus of $\mathbf{f}$. We will also write $a(n, \mathbf{f})$ for the coefficient of q^n in the series defining $\mathbf{f}$.

3.3.11. *Selmer groups for Hida families.* Let $\mathbf{f}$ be an $\mathbb{I}$-adic cusp eigenform (so in particular, $\mathbb{I}$ is a local reduced finite integral extension of $\Lambda_{W,\mathbf{Z}_p[\chi_{\mathbf{f}}]}$). Assume that

(irred)$_{\mathbf{f}}$ $\qquad\qquad$ $\rho_{\mathbf{f}_\phi}$ satisfies **(irred)** for some (hence all) $\phi \in \mathcal{X}^a_{\mathbb{I},\mathbf{Z}_p[\chi_{\mathbf{f}}]}$.

From the theory of pseudo-representations it then follows that there exists a continuous $\mathbb{I}$-linear Galois representation $(\rho_{\mathbf{f}}, T_{\mathbf{f}})$ with $T_{\mathbf{f}}$ a free $\mathbb{I}$-module of rank two and $\rho_{\mathbf{f}} : G_{\mathbf{Q}} \to \mathrm{GL}_{\mathbb{I}}(T_{\mathbf{f}})$ a continuous representation characterized by the property that $\rho_{\mathbf{f}}$ is unramified at all primes $\ell \nmid Np$ and satisfies

$$\mathrm{trace}\rho_{\mathbf{f}}(\mathrm{frob}_\ell) = a(\ell, \mathbf{f}), \quad \ell \nmid Np,$$

and

$$\det \rho_{\mathbf{f}} = \sigma_{\chi_{\mathbf{f}}}\epsilon^{-1}\boldsymbol{\varepsilon}_W^{-1}.$$

The induced $G_{\mathbf{Q}}$-action on $T_{\mathbf{f}} \otimes_{\mathbb{I}} \phi(\mathbb{I})$ is isomorphic to $\rho_{\mathbf{f}_\phi}$ (possibly after extension of scalars).

Let $V_{\mathbf{f}} = T_{\mathbf{f}} \otimes_{\mathbb{I}} F_{\mathbb{I}}$, where $F_{\mathbb{I}}$ is the ring of fractions of $\mathbb{I}$. If $\mathbf{f}$ is ordinary, then one can use the corresponding property for each $\rho_{\mathbf{f}_\phi}$ to deduce the existence of a $F_{\mathbb{I}}$-line $V^+_{\mathbf{f}} \subset V_{\mathbf{f}}$ which is stable under the action of G_p and on which G_p acts via the unramified character $\delta_{\mathbf{f}}$ characterized by $\delta_{\mathbf{f}}(\mathrm{frob}_p) = a(p, \mathbf{f})$. The action of G_p on $V^-_{\mathbf{f}} := V_{\mathbf{f}}/V^+_{\mathbf{f}}$ is then via $\delta_{\mathbf{f}}^{-1} \det \rho_{\mathbf{f}}$. The condition **(dist)** for some (hence all) $\rho_{\mathbf{f}_\phi}$ is equivalent to

(dist)$_{\mathbf{f}}$ $\qquad\qquad$ $\delta^2_{\mathbf{f}} \not\equiv \sigma_{\chi_{\mathbf{f}}}\omega^{-1} \bmod \mathfrak{m}_{\mathbb{I}},$

where $\mathfrak{m}_{\mathbb{I}}$ is the maximal ideal of $\mathbb{I}$. If $(\mathbf{dist})_{\mathbf{f}}$ holds, then $T_{\mathbf{f}}^+ := T_{\mathbf{f}} \cap V_{\mathbf{f}}^+$ is a free $\mathbb{I}$-summand of $T_{\mathbf{f}}$ of rank one.

Assuming that $(\mathbf{dist})_{\mathbf{f}}$ holds, for any finite set of primes Σ and for $F = \mathbf{Q}_\infty$, $\mathcal{K}_\infty$, or $\mathcal{K}_\infty^\pm$ we put

$$Sel_F^\Sigma(\mathbf{f}) := Sel_F^\Sigma(T_{\mathbf{f}}(\det\rho_{\mathbf{f}}^{-1}), T_{\mathbf{f}}^+(\det\rho_{\mathbf{f}}^{-1})) \text{ and } X_F^\Sigma(\mathbf{f}) := Sel_F^\Sigma(T_{\mathbf{f}}(\det\rho_{\mathbf{f}}^{-1}), T_{\mathbf{f}}^+(\det\rho_{\mathbf{f}}^{-1})).$$

We also put

$$Ch_F^\Sigma(\mathbf{f}) := Ch_{F,\mathbb{I}}^\Sigma(T_{\mathbf{f}}(\det\rho_{\mathbf{f}}^{-1}), T_{\mathbf{f}}^+(\det\rho_{\mathbf{f}}^{-1})) \text{ and } Ft_F^\Sigma(\mathbf{f}) := Ft_{F,\mathbb{I}}^\Sigma(T_{\mathbf{f}}(\det\rho_{\mathbf{f}}^{-1}), T_{\mathbf{f}}^+(\det\rho_{\mathbf{f}}^{-1})),$$

recalling that $Ch_F^\Sigma(\mathbf{f})$ has only been defined if $\mathbb{I}$ is normal (so $\Lambda_{\mathcal{K},\mathbb{I}}$ is a noetherian normal domain).

3.3.12. *Relating $Sel_F^\Sigma(\mathbf{f})$ to $Sel_{F,L}^\Sigma(\mathbf{f}_\phi)$.* Assume $(\mathbf{irred})_{\mathbf{f}}$ and $(\mathbf{dist})_{\mathbf{f}}$ hold. Let $\phi \in \mathcal{X}_{\mathbb{I},\mathbf{Z}[\chi_{\mathbf{f}}]}^a$ be an arithmetic homomorphism. Let $L \subseteq \overline{\mathbf{Q}}_p$ be any finite extension of $\mathbf{Q}_p$ containing $\phi(\mathbb{I})$ (and hence $\mathbf{Q}(\mathbf{f}_\phi)$). There is a $G_{\mathbf{Q}}$-isomorphism $T_{\mathbf{f}_\phi} \cong T_{\mathbf{f}} \otimes_{\mathbb{I},\phi} O_L$ which when restricted to G_p determines a G_p-isomorphism $T_{\mathbf{f}_\phi}^+ \cong T_{\mathbf{f}}^+ \otimes_{\mathbb{I},\mathfrak{p}} O_L$. Here we have written $\otimes_{\mathbb{I},\phi}$ to emphasize that O_L is being considered as an $\mathbb{I}$-module via ϕ. Let $\mathfrak{p}_\phi := \ker\phi$. If $\Sigma \cup \{p\}$ contains all the primes dividing Np, then for $F = \mathbf{Q}_\infty$, $\mathcal{K}_\infty$, or $\mathcal{K}_\infty^+$ there are isomorphisms

$$\text{(3.3.12.a)} \qquad Sel_F^\Sigma(\mathbf{f})[\mathfrak{p}_\phi] \otimes_{\mathbb{I},\phi} O_L = Sel_F^\Sigma(T_{\mathbf{f}}/\mathfrak{p}_\phi T_{\mathbf{f}}(\det\rho_{\mathbf{f}}^{-1})) \otimes_{\mathbb{I},\phi} O_L \cong Sel_{F,L}^\Sigma(\mathbf{f}_\phi)$$

and

$$\text{(3.3.12.b)} \qquad X_{F,L}^\Sigma(\mathbf{f}_\phi) \cong X_F^\Sigma(T_{\mathbf{f}}/\mathfrak{p}_\phi T_{\mathbf{f}}(\det\rho_{\mathbf{f}}^{-1})) \otimes_{\mathbb{I},\phi} O_L = (X_F^\Sigma(\mathbf{f})/\mathfrak{p}_\phi X_F^\Sigma(\mathbf{f})) \otimes_{\mathbb{I},\phi} O_L.$$

If $F = \mathbf{Q}_\infty$ (resp. $F = \mathcal{K}_\infty$ or $\mathcal{K}_\infty^+$) these are isomorphisms of $\Lambda_{\mathbf{Q},\mathbb{I}}$-modules (resp $\Lambda_{\mathcal{K},\mathbb{I}}$- or $\Lambda_{\mathcal{K},\mathbb{I}}^+$-modules). In (3.3.12.a) and (3.3.12.b) respectively, the first and second identifications follow from Proposition 3.2.8.

Lemma 3.3.13. *Let $\mathbf{f}$ be an $\mathbb{I}$-adic ordinary eigenform for which $(\mathbf{dist})_{\mathbf{f}}$ and $(\mathbf{irred})_{\mathbf{f}}$ hold. Then for $F = \mathbf{Q}$ or $\mathcal{K}$, $X_{F_\infty}^\Sigma(\mathbf{f})$ is a torsion $\Lambda_{F,\mathbb{I}}$-module.*

Proof. It is sufficient to prove the lemma under the hypothesis that Σ contains all the primes dividing Np.

Let $\phi \in \mathcal{X}_{\mathbb{I},\mathbf{Z}_p[\chi_{\mathbf{f}}]}^a$ be such that $p \nmid N_{\mathbf{f}_\phi}$ (there are infinitely many such ϕ) and $L \subset \overline{\mathbf{Q}}_p$ containing $\phi(\mathbb{I})$. As $X_{\mathbf{Q}_\infty,L}^\Sigma(\mathbf{f}_\phi)$ is a torsion $\Lambda_{\mathbf{Q},O_L}$-module, it follows from (3.3.12.b) that $X_{\mathbf{Q}_\infty}^\Sigma(\mathbf{f})$ is a torsion $\Lambda_{\mathbf{Q},\mathbb{I}}$-module. The case $F = \mathcal{K}$ follows from the case $F = \mathbf{Q}$ and Proposition 3.2.11 and Lemma 3.2.5. ∎

3.3.14. *Relating* $Sel^{\Sigma}_{\mathbf{Q}_\infty,L}(f)$ *to* $Sel^{\Sigma}_{\mathbf{Q},L}(f)$. Let $f \in S_k(N)$, $k \geq 2$, be an ordinary normalized eigenform. Let $L \subset \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$ containing $\mathbf{Q}(f)$. Assume that (**irred**) and (**ord**) hold for f and let Σ be a finite set of primes containing all those dividing pN_f. Let ζ be a pth-power root of unity and $0 \leq m \leq k-2$ an integer. Let $\phi : \Lambda_{\mathbf{Q},O_L} \to O_L[\zeta]$ the homomorphism sending γ to $\zeta(1+p)^m$ and let $\mathfrak{p}_\phi$ be the kernel of ϕ. Let $T_f \subset V_f$ be a $G_{\mathbf{Q}}$-stable O_L-lattice and let $T = (T_f \otimes_{O_L} O_L[\zeta])(\det \rho_f^{-1}\chi_\phi\omega^m\epsilon^{-m})$ and $T^+ = (T_f^+ \otimes_{O_L} O_L[\zeta])(\det \rho_f^{-1}\chi_\phi\omega^m\epsilon^{-m})$. Then by Proposition 3.2.13 the cokernel of the restriction map

$$Sel^{\Sigma}_{\mathbf{Q},L[\zeta]}(f,\chi_\phi\omega^m\epsilon^{-m}) = Sel^{\Sigma}_{\mathbf{Q}}(T,T^+) \hookrightarrow (Sel^{\Sigma}_{\mathbf{Q}_\infty}(f)\otimes_{\Lambda_{O_L}}\Lambda_{O_L[\zeta]})[\mathfrak{p}_\phi] = Sel^{\Sigma}_{\mathbf{Q}_\infty}(T)^{\Gamma_{\mathbf{Q}}}$$

injects into $(H^0(I_p, T\otimes_{O_L[\zeta]}\Lambda^*_{O_L[\zeta]}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1})\otimes_{\Lambda_{O_L[\zeta]}}\mathcal{O}_L[\zeta])^{G_p}$, which vanishes unless $m=1$ and $\zeta = 1$, in which case it is isomorphic to $\frac{1}{a(p,f)-1}O_L/O_L$. This proves the following lemma.

Lemma 3.3.15. *With the notation and hypotheses as above*

(i) *If* $m \neq 1$ *or* $\zeta \neq 1$ *then*

$$Sel^{\Sigma}_{\mathbf{Q},L[\zeta]}(f,\chi_\phi\omega^m\epsilon^{-m}) = (Sel^{\Sigma}_{\mathbf{Q}_\infty,L}(f)\otimes_{\Lambda_{O_L}}\Lambda_{O_L[\zeta]})[\mathfrak{p}_\phi], \tag{3.3.15.a}$$

and therefore

$$X^{\Sigma}_{\mathbf{Q},L[\zeta]}(f,\chi_\phi\omega^m\epsilon^{-m}) = X^{\Sigma}_{\mathbf{Q}_\infty,L}(f)\otimes_{\Lambda_{O_L}}\Lambda_{O_L[\zeta]}/\mathfrak{p}_\phi(X^{\Sigma}_{\mathbf{Q}_\infty,L}(f)\otimes_{\Lambda_{O_L}}\Lambda_{O_L[\zeta]}). \tag{3.3.15.b}$$

(ii) *If* $m = 1$ *and* $\zeta = 1$ *then there are exact sequences*

$$0 \to Sel^{\Sigma}_{\mathbf{Q},L]}(f) \to (Sel^{\Sigma}_{\mathbf{Q}_\infty,L}(f)[\mathfrak{p}_\phi] \to \frac{1}{a(p,f)-1}O_L/O_L \tag{3.3.15.c}$$

and

$$\mathcal{O}_L/(a(p,f)-1)\mathcal{O}_L \to X^{\Sigma}_{\mathbf{Q}_\infty,L}(f)/\mathfrak{p}_\phi X^{\Sigma}_{\mathbf{Q}_\infty,L}(f) \to X^{\Sigma}_{\mathbf{Q},L}(f) \to 0. \tag{3.3.15.d}$$

3.3.16. *No pseudo-null submodules.* Some of the most arithmetically interesting consequences of main conjectures occur when the dual Selmer groups do not have non-zero pseudo-null submodules (non-zero submodules whose localizations at each height one prime are zero). Greenberg [GSel] has identified conditions under which this can hold for very general Selmer groups. The results in this section should be viewed as special cases of Greenberg's.

Let $f \in S_k(N)$, $k \geq 2$, be an ordinary p-stabilized eigenform. Let $L \subset \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$ containing $\mathbf{Q}(f)$ and let O_L be the ring of integers of L. Assume that (**irred**) and (**ord**) hold for f. Let Σ be a finite set of primes of $\mathbf{Q}$ containing all those that divide pN_f.

Let $T_f \subset V_f$ be a $G_{\mathbf{Q}}$-stable O_L-lattice and $T_f^+ \subset T_f$ the unramified rank one G_p-stable $\mathcal{O}_L$-summand. Let $T := T_f(\det\rho_f^{-1})$ and $T^+ := T_f^+(\det\rho_f^{-1})$. Put $M := T\otimes_{O_L}\Lambda^*_{\mathbf{Q},O_L}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1})$, $M^+ := T^+\otimes_{O_L}\Lambda^*_{\mathbf{Q},O_L}(\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1})$, and $M^- := M/M^+$. Given an element $0 \neq x \in \Lambda_{\mathbf{Q},O_L}$ we put $T_x := \mathrm{Hom}_{O_L}(L/O_L, M[x])$, $T_x^\pm := \mathrm{Hom}_{O_L}(L/O_L, M^\pm[x])$, $V_x :=$

$\mathrm{Hom}_{O_L}(L, M[x])$, and $V_x^{\pm} := \mathrm{Hom}(L, M^{\pm}[x])$. Then $V_x/T_x \xrightarrow{\sim} M[x]$ and $V_x^{\pm}/T_x^{\pm} \xrightarrow{\sim} M^{\pm}[x]$ are isomorphisms of $G_\Sigma := G_{\mathbf{Q},\Sigma}$-modules.

Lemma 3.3.17. *Let $x = \gamma - (1+p)^m \in \Lambda_{\mathbf{Q},O_L}$ with $0 \neq m \in \mathbf{Z}$. Then*

$$H^1(I_p, M^-)^{G_p}[x] = H^1(I_p, M^-[x])^{G_p} \cong L/O_L.$$

Proof. Since $m \neq 0$, $H^1(I_p, M^-[x]) = H^1(I_p, M)[x]$. Let $\mathfrak{m}$ be the maximal ideal of $\Lambda_{\mathbf{Q},O_L}$. By local duality $H^2(\mathbf{Q}_p, M^-[\mathfrak{m}])$ is dual to $H^0(\mathbf{Q}_p, M^-[\mathfrak{m}]^*(1))$, and the latter is zero since $M^-[\mathfrak{m}]^*(1)$ is ramified at p. It follows that $H^1(\mathbf{Q}_p, M^-[x])$ is divisible (since $\mathfrak{m} = (x, \varpi_L)$ with ϖ_L a uniformizer of L). By the local Euler characteristic formula $\dim_L H^1(\mathbf{Q}_p, V_x^-) = 1 + \dim_L H^0(\mathbf{Q}_p, V_x^-) + \dim_L H^0(\mathbf{Q}_p, (V_x^-)^\vee(1)) = 1$, from which it follows that $H^1(\mathbf{Q}_p, M^-[x]) \cong L/O_L$. Since $M^-[x]^{I_p}$ is finite, it then follows that $H^1(I_p, M^-[x])^{G_p} \cong L/\mathcal{O}_L$.∎

Lemma 3.3.18. *Suppose $X^\Sigma_{\mathbf{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathbf{Q},O_L}$-module. Then for all but finitely many $x = \gamma - (1+p)^m \in \Lambda_{\mathbf{Q},O_L}$, $m \in \mathbf{Z}$:*

- (i) $H^1(G_\Sigma, M[x]) \twoheadrightarrow H^1(I_p, M^-[x])^{G_p}$;
- (ii) $Ш^1(\Sigma, T_x^\vee(1)) = 0$;
- (iii) $H^2(G_\Sigma, M[x]) = Ш^2(\Sigma, M[x]) = 0$;
- (iv) $H^1(G_\Sigma, M)/xH^1(G_\Sigma, M) = 0$.

Recall that $Ш^i(\Sigma, \cdot) := \ker\{H^i(G_\Sigma, \cdot) \to \prod_{v\in\Sigma} H^1(\mathbf{Q}_v, \cdot)\}$. Also, $T_x^\vee := \mathrm{Hom}_{O_L}(T_x, O_L)$.

Proof. By Tate-Poitou duality

$$\begin{aligned}\dim_L H^1(G_\Sigma, V_x) = {}& \dim_L Ш^1(\Sigma, V_x^\vee(1)) + \dim_L H^0(G_\Sigma, V_x) - \dim_L H^0(G_\Sigma, V_x^\vee(1)) \\ & - \dim_L H^0(\mathbf{R}, V_x) + \sum_{v\in\Sigma} \dim_L H^0(\mathbf{Q}_v, V_x^\vee(1)) + \dim_L V_x.\end{aligned}$$

As $H^0(G_\Sigma, V_x) = 0 = H^0(G_\Sigma, V_x^\vee(1))$, $\dim_L H^0(\mathbf{R}, V_x) = 1$, and $H^0(\mathbf{Q}_v, V_x^\vee(1)) = 0$ for all but finitely many x, it follows that

$$\dim_L H^1(G_\Sigma, V_x) = \dim_L Ш^1(\Sigma, V_x^\vee(1)) + 1 \tag{3.3.18.a}$$

for all but finitely many x. On the other hand, letting $H^1_{\mathrm{ord}}(G_\Sigma, V_x)$ be the kernel of the restriction map $H^1(G_\Sigma, V_x) \to H^1(I_p, V_x^-)^{G_p}$ and Im_x its image, we also have that

$$\dim_L H^1(G_\Sigma, V_x) = \dim_L H^1_{\mathrm{ord}}(G_\Sigma, V_x) + \dim_L Im_x.$$

Since $X^\Sigma_{\mathbf{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathbf{Q},O_L}$-module, $Sel^\Sigma_{\mathbf{Q}_\infty,L}(f)[x] = \ker\{H^1(G_\Sigma, M[x]) \to H^1(I_p, M^-)^{G_p}[x]\}$ is finite for all but finitely many x, and hence $H^1_{\mathrm{ord}}(G_\Sigma, V_x) = 0$ for all but finitely many x. In particular, for all but finitely many x

$$\dim_L H^1(G_\Sigma, V_x) = \dim_L Im_x. \tag{3.3.18.b}$$

From Lemma 3.3.17 it follows that $\dim_L Im_x \leq 1$. Combining this with (3.3.18.a) and (3.3.18.b) yields

$$\dim_L H^1(G_\Sigma, V_x) = \dim_L Im_x = 1 \quad \text{and} \quad \dim_L Ш^1(\Sigma, V_x^\vee(1)) = 0$$

for all but finitely many x. Part (i) of the lemma then follows from Lemma 3.3.17 and the first equality. Part (ii) follows from the second equality and the fact that $Ш^1(\Sigma, T_x^\vee(1))$ is torsion-free (which follows from (**irred**)) with O_L-rank equal to $\dim_L Ш^1(\Sigma, V_x^\vee(1))$.

Since $H^2(\mathbf{Q}_v, M[x])$ is dual to $H^1(\mathbf{Q}_v, T_x^\vee(1))$ and the latter is zero for all but finitely many x, it follows that for all but finitely many x, $H^2(G_\Sigma, M[x]) = Ш^2(\Sigma, M[x])$. But by global duality $Ш^2(\Sigma, M[x])$ is dual to $Ш^1(\Sigma, T_x^\vee(1))$, which is zero for all but finitely many x by (ii). This proves (iii). Part (iv) follows easily from part (iii). ∎

Proposition 3.3.19. *Let $f \in S_k(N)$, $k \geq 2$, be an ordinary p-stabilized eigenform. Let $L \subset \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$ containing $\mathbf{Q}(f)$. Assume that* (**irred**) *and* (**ord**) *hold for f. Let Σ be a finite set of primes of $\mathbf{Q}$ containing all those that divide pN_f. If $X^\Sigma_{\mathbf{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathbf{Q},O_L}$-module, then $X^\Sigma_{\mathbf{Q}_\infty,L}(f)$ has no non-zero pseudo-null $\Lambda_{\mathbf{Q},O_L}$-submodules.*

Proof. The dual Selmer group $X^\Sigma_{\mathbf{Q}_\infty,L}(f)$ has no non-zero pseudo-null $\Lambda_{\mathbf{Q},O_L}$-submodule if and only if the Selmer group $Sel^\Sigma_{\mathbf{Q}_\infty,L}(f)$ has no non-zero finite $\Lambda_{\mathbf{Q},O_L}$-quotient. We will prove the latter under the hypotheses of the proposition.

Let Im denote the image of the restriction map $H^1(G_\Sigma, M) \to H^1(I_p, M^-)^{G_p}$, the kernel of which is $Sel^\Sigma_{\mathbf{Q}_\infty,L}(f)$. Let $x \in \Lambda_{\mathbf{Q},O_L}$ be as in Lemmas 3.3.17 and 3.3.18 (so in particular such that (i)-(iv) of Lemma 3.3.18 hold for x). Then the image of $H^1(G_\Sigma, M[x]) = H^1(G_\Sigma, M)[x]$ in $H^1(I_p, M^-[x])^{G_p} = H^1(I_p, M^-)^{G_p}[x]$ is contained in $Im[x]$ but is also everything by part (i) of Lemma 3.3.18, so the restriction map $H^1(I_p, M)[x] \to Im[x]$ is surjective. Multiplying the short exact sequence

$$0 \to Sel^\Sigma_{\mathbf{Q}_\infty,L}(f) \to H^1(G_\Sigma, M) \to Im \to 0$$

by x, we deduce from the preceding surjection and the snake lemma that there is an injection

$$Sel^\Sigma_{\mathbf{Q}_\infty,L}(f)/xSel^\Sigma_{\mathbf{Q}_\infty,L}(f) \hookrightarrow H^1(G_\Sigma, M)/xH^1(\mathbf{G}_\Sigma, M).$$

The right-hand side is zero by part (iv) of Lemma 3.3.18, hence the left-hand side is zero. If N is a finite $\Lambda_{\mathbf{Q},O_L}$-quotient of $Sel^\Sigma_{\mathbf{Q}_\infty,L}(f)$, then N/xN is a quotient of $Sel^\Sigma_{\mathbf{Q}_\infty,L}(f)/xSel^\Sigma(\mathbf{Q}_\infty, L)(f)$ and hence zero. It then follows from Nakayama's Lemma that N, being finite, is zero. ∎

When combined with Lemma 3.3.15 this proposition yields the following.

Corollary 3.3.20. *In the notation of 3.3.14 and under the hypotheses of Proposition 3.3.19:*

(i) *If $m \neq 1$ of $\zeta \neq 1$, then*

$$\#X^\Sigma_{\mathbf{Q},L[\zeta]}(f, \chi_\phi\omega^m\epsilon^{-m}) = \#\Lambda_{\mathbf{Q},O_L[\zeta]}/(\mathfrak{p}_\phi, Ch^\Sigma_{\mathbf{Q}_\infty,L}(f)).$$

(ii) *If $m = 1$ and $\zeta = 1$, then*

$$\#X^\Sigma_{\mathbf{Q},L}(f)|\#\Lambda_{\mathbf{Q},O_L}/(\mathfrak{p}_\phi, Ch^\Sigma_{\mathbf{Q}_\infty,L}(f))|\#X^\Sigma_{\mathbf{Q},L}(f) \cdot \#\mathcal{O}_L/(a(p,f)-1)\mathcal{O}_L.$$

In particular, if $a(p,f)-1$ *is a unit then* $\#X^{\Sigma}_{\mathbf{Q},L}(f) = \#\Lambda_{\mathbf{Q},O_L}/(\gamma-1, Ch^{\Sigma}_{\mathbf{Q}_\infty,L}(f))$.

(iii) $Ft^{\Sigma}_{\mathbf{Q}_\infty,L}(f) = Ch^{\Sigma}_{\mathbf{Q}_\infty,L}(f)$.

Here an equality $\#A = \#B$ is to be understood to mean that if one side is infinite, then so is the other. Similary, a division $\#A|\#B$ always holds if B is infinite and if A is infinite then so is B.

Proof. It is a standard result in Iwasawa theory that if a finite $\Lambda_{\mathbf{Q},\mathcal{O}[\zeta]}$-module $\mathcal{M}$ has no non-zero psuedo-null submodule, then

$$\#\mathcal{M}/x\mathcal{M} = \#\Lambda_{\mathbf{Q},\mathcal{O}[\zeta]}/(x, Char_{\Lambda_{\mathbf{Q},O_L[\zeta]}}(\mathcal{M})).$$

Since $\mathfrak{p}_\phi\Lambda_{\mathbf{Q},\mathcal{O}_L[\zeta]} = (\gamma-\zeta(1+p)^m)$, parts (i) and (ii) then follow from the proposition and Lemma 3.3.15. To prove part (iii), we begin with the natural surjection

$$\Lambda_{\mathbf{Q},O_L}/Ft^{\Sigma}_{\mathbf{Q}_\infty,L}(f) \twoheadrightarrow \Lambda_{\mathbf{Q},O_L}/Ch^{\Sigma}_{\mathbf{Q}_\infty,L}(f). \tag{3.3.20.a}$$

For a ϕ as in part (i), tensoring (3.3.20.a) with $\otimes_{\Lambda_{\mathbf{Q},O_L}}\Lambda_{\mathbf{Q},O_L[\zeta]}$ and reducing modulo $\mathfrak{p}_\phi$ yields an isomorphism. Since ϕ can be chosen such that $\Lambda_{\mathbf{Q},O_L[\zeta]}/(Ch^{\Sigma}_{\mathbf{Q}_\infty,L}(f))$ has no $\mathfrak{p}_\phi$-torsion, we conclude easily from this that (3.3.20.a) is an isomorphism (after tensoring with $\Lambda_{\mathbf{Q},O_L[\zeta]}$ and so before tensoring by faithful flatness). ∎

3.4. ***p*-adic *L*-functions.** We now introduce the p-adic L-functions that appear in the statements of the Iwasawa-Greenberg main conjectures for modular forms.

3.4.1. *L-functions of Galois representations.* Let F be a number field, $L \subseteq \overline{\mathbf{Q}}_p$ a finite extension of $\mathbf{Q}_p$, and $\rho : G_F \to \mathrm{GL}_L(V)$ a continuous homomorphism with V a finite-dimensional L-space. Recall that for a finite place $v \nmid p$ of F one defines the local L-function ρ by

$$L_v(\rho,s) := P_{\rho,v}(q_v^{-s})^{-1}$$

with $P_{\rho,v}(X) := det(1 - X\cdot \mathrm{frob}_v; V^{I_v})$ and q_v the size of the residue field at v. Assuming that $P_{\rho,v}(X) \in \overline{\mathbf{Q}}[X]$ for all v and that V has weight w for almost all v (i.e., for almost all v, the inverses of the roots of $P_{\rho,v}(X)$, viewed as elements of $\mathbf{C}$ via ι_p', all have absolute value $\leq q_v^{w/2}$) then for any finite set of primes S containing p the product

$$L_F^S(\rho,s) := \prod_{v\notin S} L_v(\rho,s)$$

converges absolutely for $\mathrm{Re}(s) > 1 + w/2$. When $F = \mathbf{Q}$ we drop it from the notation.

3.4.2. *L-functions for modular forms.* For an eigenform $f \in S_k(N)$ we let $L(f,s)$ denote its usual L-function. Let (ρ_f, V_f) be the Galois representation associated with f. Then ρ_f satisfies the hypotheses of the preceding section with $w = \frac{k-1}{2}$, and if S contains all the primes dividing Np then $L^S(\rho_f,s) = L^S(f,s)$. If f is a newform (resp. a p-stabilized newform) then the same equality holds for any set S (resp. any set S containing p).

Let ξ be a finite order Hecke character of $\mathcal{K}$. We let

$$L^S_{\mathcal{K}}(f,\xi,s) := L^S_{\mathcal{K}}(\rho_f \otimes \sigma_\xi, s).$$

If S contains all the primes that divide $Np\mathrm{Nm}(\mathfrak{f}_\xi)$ then

$$L^S_{\mathcal{K}}(f,\xi,s) = \sum_{\mathfrak{a}} \xi(\mathfrak{a}) a(\mathrm{Nm}(\mathfrak{a}), f)\mathrm{Nm}(\mathfrak{a})^{-s},$$

where the sum is over the integral ideals $\mathfrak{a}$ of $\mathcal{K}$ prime to $\mathfrak{f}_\xi$ and the primes in S. If π is the usual unitary automorphic representation of representation of $\mathrm{GL}_2(\mathbf{A})$ associated with f, then

$$L^S(f,s) = L^S(\pi, s - \frac{k-1}{2})$$

if S contains all primes dividing Np. If $BC_{\mathcal{K}}(\pi)$ denotes the base change of π to a representation of $\mathrm{GL}_2(\mathbf{A}_{\mathcal{K}})$, then, provided S contains all the primes dividing $Np\mathrm{Nm}(\mathfrak{f}_\xi)$,

$$L^S(BC_{\mathcal{K}}(\pi) \otimes \xi, s - \frac{k-1}{2}) = L^S_{\mathcal{K}}(f,\xi,s).$$

3.4.3. *p-adic L-functions for Dirichlet characters.* Let L be a finite extension of $\mathbf{Q}_p$ and let O_L be its ring of integers. Let χ be an even L-valued Dirichlet character. Let S be any finite set of primes. It is well known from the work of Kubota-Leopoldt that there exists a p-adic meromorphic function $L^S_p(s,\chi)$ on $\mathbf{Z}_p$, such that for any integer $m \geq 1$, we have

$$L^S_p(1-m,\chi) = L^{\{p\}}(1-m,\chi\omega^m) \prod_{\ell \in S, \ell \neq p} (1 - \chi^{-1}\omega^{-m}(\ell)\ell^{-m}).$$

When $\chi \neq 1$ or S contains a prime other than p this function is analytic on $\mathbf{Z}_p$, and when χ is the trivial character and $S \subseteq \{p\}$ it has a pole of order one at $s = 1$.

We define $H^S_\chi \in \Lambda_{\mathbf{Q}}$ to be $\gamma - 1$ if χ is trivial and $S \subseteq \{p\}$ and otherwise we set $H^S_\chi = 1$. We denote by G^S_χ the unique element in $\Lambda_{\mathbf{Q},O_L}$ such that for any $\phi \in \mathcal{X}_{\mathbf{Q},O_L}$ satisfying $\phi(\gamma) = \zeta(1+p)^m$ with $m \geq 1$ an integer and ζ a p-power root of unity,

$$L^S_p(1-m, \chi\chi_\phi) = \frac{G^S_\chi(\phi)}{H^S_\chi(\phi)}.$$

(Recall that χ_ϕ is the Dirichlet character corresponding via the reciprocity map to the finite order character of $\Gamma_{\mathbf{Q}}$ sending γ to ζ^{-1}.) If $S \subseteq \{p\}$ then we omit it from our notation. Recall that $G_1 \in \Lambda_{\mathbf{Q}}^\times$ and that if $S \not\subseteq \{p\}$ then

$$(\gamma - 1)G^S_1 \sim \prod_{\ell \in S, \ell \neq p} (1 - \Psi_{\mathbf{Q},\ell}(\ell)),$$

where $\sim$ denotes equality up to a unit in $\Lambda_{\mathbf{Q}}$.

3.4.4. *The Manin-Vishik p-adic L-function.* Let $f \in S_k(Np^r, \chi; L)$ be an ordinary p-stabilized newform, $L \subseteq \overline{\mathbf{Q}}_p$ being a finite extension of $\mathbf{Q}_p$. Given a primitive O_L-valued Dirichlet character ψ of conductor C prime to p, Amice-Vélu [AV75] and Vishik [Vi76] (but see also [MTT86]) have constructed a p-adic L-function $\mathcal{L}_{f,\psi} \in \Lambda_{\mathbf{Q},O_L}$ such that if $\phi \in \mathcal{X}_{\mathbf{Q},O_L}$ satisfies $\phi(\gamma) = \zeta(1+p)^m$, $0 \le m \le k-2$, with ζ a primitive $p^{t_\phi - 1}$th root of unity, then

$$\mathcal{L}_{f,\psi}(\phi) = e_p(\phi)\frac{(p^{t'_\phi}C)^{m+1} m! L(f, \psi\omega^{-m}\chi_\phi^{-1}, m+1)}{(-2\pi i)^m G(\psi\omega^{-m}\chi_\phi^{-1})\Omega_f^{\mathrm{sgn}((-1)^m\psi(-1))}},$$

$$e_p(\phi) = a(p,f)^{-t_\phi}\left(1 - \frac{\omega^{-m}\chi_\phi^{-1}\psi(p)\chi(p)p^{\kappa-2-m}}{a(p,f)}\right)\left(1 - \frac{\omega^m\chi_\phi\bar{\psi}(p)p^m}{a(p,f)}\right),$$

where $t'_\phi = 0$ if $t_\phi = 1$ and $p-1|m$ and otherwise $t'_\phi = t_\phi$. Here $G(\psi\omega^{-m}\chi_\phi^{-1})$ is the usual Gauss sum.

For any finite set S of primes we set

$$\mathcal{L}^S_{f,\psi} := \mathcal{L}_{f,\psi}\prod_{\ell\in S, \ell\neq p}\Big(1 - a(\ell,f)\ell^{-1}\psi(\ell)\Psi_{\mathbf{Q},\ell}(\ell) + \chi(\ell)\ell^{k-3}\psi(\ell^2))\Phi_{\mathbf{Q},\ell}(\ell^2)\Big).$$

Then for ϕ as above

$$\mathcal{L}^S_{f,\psi}(\phi) = e_p(\phi)\frac{(p^{t'_\phi}C)^{m+1} m! L^{S/\{p\}}(f, \psi\omega^{-m}\chi_\phi^{-1}, m+1)}{(-2\pi i)^m G(\psi\omega^{-m}\chi_\phi^{-1})\Omega_f^{\mathrm{sgn}((-1)^m\psi(-1))}}.$$

We omit the subscript ψ when ψ is trivial and, of course, drop the superscript S when $S \subseteq \{p\}$. If $f \otimes \psi$ is the ordinary p-stabilized newform associated with $\sum_{n=1}^\infty \psi(n)a(n,f)q^n$ and S contains all the primes dividing NC, then

$$\mathcal{L}^S_{f,\psi} \sim \mathcal{L}^S_{f\otimes\psi}.$$

3.4.5. *A three-variable p-adic L-function.* Let $L \subset \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$ with integer ring O_L, and let $\mathbb{I}$ be a domain that is a finite integral local reduced extension of Λ_{W,O_L}. Let $\mathbf{f}$ be an $\mathbb{I}$-adic ordinary eigenform of tame level N as above. Without loss of generality we may suppose that L contains the values of $\chi_{\mathbf{f}}$. Suppose also that L contains $\mathbf{Q}[\boldsymbol{\mu}_{Np}, i, D_\mathcal{K}^{1/2}]$.

Put $\mathbb{I}_\mathcal{K} := \Lambda_{\mathcal{K},\mathbb{I}}$ (so $\mathbb{I}_\mathcal{K} = \mathbb{I}[\![\Gamma_\mathcal{K}]\!]$) and let

$$\mathcal{X}^a_{\mathbb{I}_\mathcal{K},O_L} := \{\phi \in \mathcal{X}_{\mathbb{I}_\mathcal{K},O_L} \ : \ \phi|\mathbb{I} \in \mathcal{X}^a_{\mathbb{I},\mathbf{Z}_p[\chi_\mathbf{f}]}, \phi(\gamma_+) = \zeta_+(1+p)^{k_{\phi|\mathbb{I}}-2}, \phi(\gamma_-) = \zeta_-\}$$

with $\zeta_\pm$ p-power roots of unity. For $\phi \in \mathcal{X}^a_{\mathbb{I}_\mathcal{K},O_L}$ we let k_ϕ, t_ϕ, and χ_ϕ denote the corresponding objects for $\phi|\mathbb{I}$. For $\phi \in \mathcal{X}^a_{\mathbb{I}_\mathcal{K},O_L}$ put

$$\xi_\phi := \phi \circ (\Psi_\mathcal{K}/\Psi_W^+) \quad \text{and} \quad \theta_\phi := \omega^{2-k_\phi}\chi_\phi^{-1}\xi_\phi.$$

These are finite-order idele class characters of $\mathbf{A}_\mathcal{K}^\times$. Let

$$\mathcal{X}'_{\mathbb{I}_\mathcal{K},O_L} := \{\phi \in \mathcal{X}^a_{\mathbb{I}_\mathcal{K},O_L} \ : \ p|\mathfrak{f}_{\chi_\mathbf{f}\xi_\phi}, p^{t_\phi}|\mathrm{Nm}(\mathfrak{f}_{\chi_\mathbf{f}\xi_\phi}), p|\mathfrak{f}_{\theta_\phi}\}.$$

Let S be a finite set of primes containing all those dividing $NpD_{\mathcal{K}}$. Assume there exists a finite idele class character ψ of $\mathbf{A}_{\mathcal{K}}^{\times}$ unramified outside S and satisfying $\psi|_{\mathbf{A}_{\mathbf{Q}}^{\times}} = \chi_{\mathbf{f}}$ and $\mathrm{cond}(\psi_p)|p$. If $(\mathbf{irred})_{\mathbf{f}}$ and $(\mathbf{dist})_{\mathbf{f}}$ hold, then there exists $\mathcal{L}_{\mathbf{f},\mathcal{K}}^S \in \mathbb{I}_{\mathcal{K}}$ such that for any $\phi \in \mathcal{X}'_{\mathbb{I}_{\mathcal{K}},O_L}$ we have
(3.4.5.a)

$$\mathcal{L}_{\mathbf{f},\mathcal{K}}^S(\phi) = u_{\mathbf{f}_\phi} a(p,\mathbf{f}_\phi)^{-\mathrm{ord}_p(\mathrm{Nm}(\mathfrak{f}_{\theta_\phi}))} \frac{((k_\phi-2)!)^2 \mathfrak{g}(\theta_\phi)\mathrm{Nm}(\mathfrak{f}_{\theta_\phi}\mathfrak{d})^{k_\phi-2} L_{\mathcal{K}}^S(\mathbf{f}_\phi,\theta_\phi,k_\phi-1)}{(-2\pi i)^{2k_\phi-2}\Omega_{\mathbf{f}_\phi}^+\Omega_{\mathbf{f}_\phi}^-},$$

where $u_{\mathbf{f}_\phi}$ is a p-adic unit depending only on $\mathbf{f}_\phi$. This is part (ii) of Theorem 12.3.1: take $\mathcal{L}_{\mathbf{f},\mathcal{K}}^S := \mathcal{L}_{\mathbf{f},\mathcal{K},\chi_{\mathbf{f}}}^S$ with the latter as in the theorem. Here $\mathfrak{g}(\theta_\phi)$ is a Gauss sum; see 8.1.3 for a definition.

The kernels of the homomorphisms $\phi \in \mathcal{X}'_{\mathbb{I}_K,O_L}$ are Zariski dense in $\mathrm{Spec}\,\mathbb{I}_{\mathcal{K}}$, so the specialization property (3.4.5.a) characterizes $\mathcal{L}_{\mathbf{f},\mathcal{K}}^S$.

If $S' \supseteq S$, then it is easy to see that

$$\mathcal{L}_{\mathbf{f},\mathcal{K}}^{S'} = \mathcal{L}_{\mathbf{f},\mathcal{K}}^S \cdot \prod_{v|\ell\in S'\setminus S} \det_{\mathbb{I}_{\mathcal{K}}}(1 - q_v^{-1}\varepsilon_{\mathcal{K}}\rho_{\mathbf{f}}(\mathrm{frob}_v)), \tag{3.4.5.b}$$

where v is a place of $\mathcal{K}$ and q_v is the order of its residue field.

Remark. This must be a multiple of the p-adic L-function constructed by Hida [Hi88a, Thm 5.1d] for λ' (in the notation of *loc. cit.*) a particular CM Hida family, but we have not checked this.

3.4.6. *One and two variable specializations of* $\mathcal{L}_{\mathbf{f},\mathcal{K}}^S$. Let $f \in S_k(Np^r,\chi;L)$, $p \nmid N$, be an ordinary eigenform. Write $\chi = \omega^{k-2}\chi_1\chi_2$ with χ_1 a character modulo Np and χ_2 a character modulo p^r of p-power order; this decomposition is unique. Suppose (**irred**) and (**dist**) hold for f. Let

$$\mathcal{X}_{\mathcal{K},O_L}^k := \{\phi \in \mathcal{X}_{\mathcal{K},O_L} \ : \ \phi(\gamma^+) = \zeta_+(1+p)^{k-2}, \phi(\gamma^-) = \zeta_-\}$$

with $\zeta_\pm$ p-power roots of unity. Given $\phi \in \mathcal{X}_{\mathcal{K},O_L}^k$ let $\xi_\phi : \mathbf{A}_{\mathcal{K}}^\times \to \overline{\mathbf{Q}}_p^\times$ be defined by

$$\xi_\phi(x) = \chi_2(x)|x/x_\infty|_{\mathcal{K}}^{2-k}(x_{v_0}x_{\bar{v}_0})^{2-k}\phi(\Psi_{\mathcal{K}}(x)).$$

This is a finite order idele class character. Put $\theta_\phi := \omega^{2-k}\chi_2^{-1}(x)\xi_\phi$. Let

$$\mathcal{X}'_{\mathcal{K},O_L} := \{\phi \in \mathcal{X}_{\mathcal{K},O_L}^k \ : \ p|\mathfrak{f}_{\chi_0\xi_\phi}, p^{t_\phi}|\mathrm{Nm}(\mathfrak{f}_{\chi_0\xi_\phi}), p|\mathfrak{f}_{\theta_\phi}\}.$$

Under the same hypotheses as in 3.4.5 there exists $\mathcal{L}_{f,\mathcal{K}}^S \in \Lambda_{\mathcal{K},O_L}$ such that for $\phi \in \mathcal{X}'_{\mathcal{K},O_L}$ we have

$$\mathcal{L}_{f,\mathcal{K}}^S(\phi) = u_f a(p,f)^{-\mathrm{ord}_p(\mathrm{Nm}(\mathfrak{f}_{\theta_\phi}))} \frac{((k-2)!)^2 \mathfrak{g}(\theta_\phi)\mathrm{Nm}(\mathfrak{f}_{\theta_\phi}\mathfrak{d})^{k-2} L_{\mathcal{K}}^S(f,\theta_\phi,k-1)}{(-2\pi i)^{2k-2}\Omega_f^+\Omega_f^-},$$

with u_f a p-adic unit depending only on f. This is just part (ii) of Theorem 12.3.2, where $\mathcal{L}_{f,\mathcal{K}}^S$ is constructed as a specialization of $\mathcal{L}_{\mathbf{f},\mathcal{K}}^S$ for some $\mathbf{f}$; since $f = \mathbf{f}_{\phi_0}$ for some

$\mathbb{I}$-adic ordinary eigenform $\mathbf{f}$ as in 3.4.5 with $\chi_{\mathbf{f}} = \chi_1$ and some $\phi_0 \in \mathcal{X}_{\mathbb{I},O_L}$ with $\chi_{\phi_0} = \chi_2$, we have

$$\mathcal{L}^S_{f,\mathcal{K}} = (\phi_0 \otimes id)(\mathcal{L}^S_{\mathbf{f},\mathcal{K}}),$$

where

$$\phi_0 \otimes id : \mathbb{I}_K = \mathbb{I}\hat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathcal{K}} \to \phi(\mathbb{I}) \otimes_{\mathbf{Z}_p} \Lambda_{\mathcal{K}} = \Lambda_{\mathcal{K},\phi(\mathbb{I})}$$

is the obvious extension of ϕ_0.

Let $\mathcal{L}^{S,+}_{f,\mathcal{K}}$ be the image of $\mathcal{L}^S_{f,\mathcal{K}}$ under the projection $\Lambda_{\mathcal{K},O_L} \to \Lambda_{\mathbf{Q},O_L}$. It follows easily that

$$\mathcal{L}^{S,+}_{f,\mathcal{K}} = a_f \mathcal{L}^S_f \mathcal{L}^S_{f\otimes\chi_{\mathcal{K}}},$$

where $\mathcal{L}^S_f$ and $\mathcal{L}^S_{f\otimes\chi_{\mathcal{K}}}$ are the p-adic L-functions from 3.4.4 and a_f is a p-adic unit (see Proposition 12.3.4).

3.5. **The Main Conjectures.** Before recalling the main conjectures whose proofs are the primary goals of this paper, we recall the theorem of Mazur and Wiles [MW84] which was the original Main Conjecture of Iwasawa. For the history of this conjecture and its proof we refer the reader to the original paper of Mazur and Wiles.

Let χ be an even Dirichlet character and L a finite extension of $\mathbf{Q}_p$ containing $\mathbf{Z}_p[\chi]$. Let S be a finite set of primes. Let $X^S_{\mathbf{Q}_\infty,L}(\chi\epsilon) := X^S_{\mathbf{Q}_\infty}(T_\chi, T^+_\chi)$ with $T_\chi := O_L(\chi\epsilon)$ and $T^+_\chi = 0$ and let $Ch^S_{\mathbf{Q}_\infty,L}(\chi\epsilon) := Ch^S_{\mathbf{Q}_\infty}(T_\chi, T^+_\chi)$.

Theorem 3.5.1 (Mazur-Wiles)**.** $Ch^S_{\mathbf{Q}_\infty,L}(\chi\epsilon) = (G^S_\chi)$.

We recall now the Iwasawa-Greenberg main conjectures for the Selmer groups associated with ordinary modular forms (see [Gr94]). In the case of an ordinary elliptic curve, the cyclotomic case of this conjecture is due to Mazur [Ma72] but the formulation below is due to Greenberg.

3.5.2. *The three-variable main conjecture.* Let $L \subseteq \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$ and let $\mathbb{I}$ be a local reduced finite integral extension of Λ_{W,O_L}. Let $\mathbf{f}$ be an $\mathbb{I}$-adic ordinary eigenform of tame level N as in 3.3.10. From Lemma 3.3.13, we know that the dual Selmer group $X^\Sigma_{\mathcal{K}_\infty}(\mathbf{f})$ is torsion over $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$.

Conjecture 3.5.3. *Suppose $\mathbb{I}$ is a normal domain and S is a finite set of primes. The ideal $Ch^S_{\mathcal{K}_\infty}(\mathbf{f})$ is principal and generated by an element $\mathcal{L}^S_{\mathbf{f},\mathcal{K}} \in \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ satisfying* (3.4.5.a) *for all $\phi \in \mathcal{X}'_{\Lambda_{\mathcal{K},\mathbb{I}},O_L}$.*

Of course, when L contains $\mathbf{Q}[\boldsymbol{\mu}_{Np}, \chi_{\mathbf{f}}, i, D^{1/2}_{\mathcal{K}}]$ and S is a finite set of primes containing all those dividing $ND_{\mathcal{K}}$ then the element $\mathcal{L}^S_{\mathbf{f},\mathcal{K}}$ would be the three-variable p-adic L-function so-denoted in 3.4.5.

Recall that in defining the Selmer groups for $\mathbf{f}$ and their characteristic ideals, we assumed that $(\mathbf{irred})_{\mathbf{f}}$ held, so the choice of the lattice $T_{\mathbf{f}}$ does not introduce any ambiguity into the statement of this conjecture.

3.5.4. *The main conjecture for p-ordinary eigenforms.* Let $f \in S_k(Np^r, \chi; L)$, $p \nmid N$, be an ordinary eigenform; here we take $L \subset \overline{\mathbf{Q}}_p$ a finite extension of $\mathbf{Q}_p$. By Theorem 3.3.7 (due to Kato), the dual Selmer group $X^{\Sigma}_{\mathbf{Q}_\infty,L}(f)$ is torsion over $\Lambda_{\mathbf{Q},O_L}$.

Conjecture 3.5.5. *For any finite set of primes Σ, the ideal $Ch^{\Sigma}_{\mathbf{Q}_\infty,L}(f)$ is principal and $Ch^{\Sigma}_{\mathbf{Q}_\infty,L}(f) = (\mathcal{L}^{\Sigma}_f)$ in $\Lambda_{\mathbf{Q},O_L} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, with the equality holding in $\Lambda_{\mathbf{Q},O_L}$ if* (**irred**) *holds.*

The following result, due to Kato (Theorem 17.14 of [Ka04]), was the first major breakthrough towards this conjecture.

Theorem 3.5.6 (Kato)**.** *Suppose $p \nmid N_f$. The characteristic ideal $Ch^{\Sigma}_{\mathbf{Q}_\infty,L}(f)$ divides $\mathcal{L}^{\Sigma}_f$ in $L \otimes_{O_L} \Lambda_{\mathbf{Q},O_L}$. Furthermore, if there exists an O_L-basis of T_f with respect to which the image of ρ_f contains $\mathrm{SL}_2(\mathbf{Z}_p)$, then the divisibility also holds in $\Lambda_{\mathbf{Q},O_L}$.*

3.6. **Main results.** We deduce the main results of this paper from the following theorem. This theorem will be proved at the end of §7, assuming the existence of a certain p-adic Eisenstein family. The existence of this family is then established in the remainder of this paper (§§8-13).

Theorem 3.6.1. *Let $L \subseteq \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$ and $\mathbb{I}$ a normal domain and a finite integral extension of Λ_{W,O_L}. Let $\mathbf{f}$ be an $\mathbb{I}$-adic ordinary eigenform of tame level N with $\chi_{\mathbf{f}} = 1$. Assume that L contains $\mathbf{Q}[\boldsymbol{\mu}_{Np}, i, D_{\mathcal{K}}^{1/2}]$. Suppose $N = N^+N^-$ with N^+ divisible only by primes that split in $\mathcal{K}$ and N^- divisible only by primes inert in $\mathcal{K}$. Suppose also*

- (**irred**)$_{\mathbf{f}}$ *and* (**dist**)$_{\mathbf{f}}$ *hold;*
- *N^- is square-free and has an odd number of prime factors;*
- *the reduction $\bar{\rho}_{\mathbf{f}}$ of $\rho_{\mathbf{f}}$ modulo the maximal ideal of $\mathbb{I}$ is ramified at all $\ell | N^-$.*

Let Σ be a finite set of primes containing all those that divide $ND_{\mathcal{K}}$. Then

$$Ch^{\Sigma}_{\mathcal{K}_\infty}(\mathbf{f}) \subseteq (\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}).$$

Remarks on the hypotheses. We comment on the roles played by the various hypotheses of this theorem.

(i) The hypotheses on L are present as it is only under such conditions that we prove the existence of $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$.
(ii) The hypotheses on N and N^- and $\bar{\rho}_{\mathbf{f}}|I_\ell$ for $\ell | N^-$ are needed because of an appeal to the work of Vatsal [Va03] (see Proposition 12.3.6).
(iii) The hypothesis that $\chi_{\mathbf{f}} = 1$ is also needed to appeal to the results of Vatsal but is also used in the Galois arguments in §7.
(iv) The hypotheses (**irred**)$_{\mathbf{f}}$ and (**dist**)$_{\mathbf{f}}$ are used to conclude the existence of the $\mathbb{I}$-free representations $T_{\mathbf{f}}$ and $T^-_{\mathbf{f}}$ and to construct $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$ and the Eisenstein family in 12.4 used in the analysis of the Eisenstein ideal.

(v) The condition that Σ contains all primes dividing $ND_{\mathcal{K}}$ is used in the construction of $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$ and the Eisenstein family; it is also made to avoid any need to appeal to compatibilities of global Galois representations with the local Langlands correspondence in the Galois arguments in §7.

Corollary 3.6.2. *Let $f \in S_k(Np^r, \chi; L)$, $k \geq 2$ and $L \subseteq \mathbf{Q}_p$ a finite extension of $\mathbf{Q}_p$, be a p-ordinary cuspidal eigenform. Assume that L contains $\mathbf{Q}[\boldsymbol{\mu}_{Np}, i, D_{\mathcal{K}}^{1/2}]$. Suppose $N = N^+N^-$ with N^+ divisible only by primes that split in $\mathcal{K}$ and N^- divisible only by primes inert in $\mathcal{K}$. Suppose also*

- (**irred**) *and* (**dist**) *hold;*
- *$\chi = \omega^{k-2}\chi_1$ with χ_1 of p-power order and conductor;*
- *N^- is square-free and has an odd number of prime factors;*
- *$\bar{\rho}_f$ is ramified at all $\ell | N^-$.*

Let Σ be a finite set of primes containing all those that divide $ND_{\mathcal{K}}$. Then

$$Ch^{\Sigma}_{\mathcal{K}_\infty,L}(f) \subseteq (\mathcal{L}^{\Sigma}_{f,\mathcal{K}}).$$

Proof. Note first that the corollary is true if and only if it is true for some finite extension of L. Since f can be taken to be a specialization of some $\mathbb{I}$-adic ordinary form $\mathbf{f}$ as in Theorem 3.6.1 after possibly replacing L with a finite extension, this corollary follows from combining Theorem 3.6.1 with (3.3.12.b) and part (ii) of Corollary 3.2.9 and the relation between $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}$ and $\mathcal{L}^{\Sigma}_{f,\mathcal{K}}$ explained in 3.4.6. ∎

Corollary 3.6.3. *Let $f \in S_k(Np^r, \chi; L)$, $k \geq 2$ and $L \subseteq \mathbf{Q}_p$ a finite extension of $\mathbf{Q}_p$, be a p-ordinary cuspidal eigenform. Suppose*

- (**irred**) *and* (**dist**) *hold for ρ_f;*
- *$\chi = \omega^{k-2}\chi_1$ with χ_1 of p-power order and conductor;*
- *N^- is square-free and has an odd number of prime factors;*
- *there exists a prime $q \neq p$ such that $q || N$ and $\bar{\rho}_f$ is ramified at q.*

Then for any finite set of primes Σ

$$Ch^{\Sigma}_{\mathbf{Q}_\infty,L}(f)Ch^{\Sigma}_{\mathbf{Q}_\infty,L}(f \otimes \chi_{\mathcal{K}}) \subseteq (\mathcal{L}^{\Sigma}_{f}\mathcal{L}^{\Sigma}_{f\otimes\chi_{\mathcal{K}}}).$$

Proof. From the definition of $\mathcal{L}^{\Sigma}_{f}$ and $\mathcal{L}^{\Sigma}_{f\otimes\chi_{\mathcal{K}}}$ and Lemma 3.2.17 it follows that the corollary is true for Σ if it is true for some $\Sigma' \supset \Sigma$ with $\Sigma' \cup p$ containing all primes dividing $ND_{\mathcal{K}}$. We may therefore assume this for Σ. The corollary then follows from combining Corollary 3.6.2, Proposition 3.2.11, Lemma 3.2.5, Corollary 3.2.9 and the relation between $\mathcal{L}^{\Sigma}_{f,\mathcal{K}}$ and $\mathcal{L}^{\Sigma}_{f}\mathcal{L}^{\Sigma}_{f\otimes\chi_{\mathcal{K}}}$ explained in 3.4.6.∎

Combining these results with Kato's theorem (Theorem 3.5.6) we deduce some cases of the Iwasawa-Greenberg main conjectures.

Theorem 3.6.4. *Let $f \in S_k(Np^r, \chi; L)$, $k \geq 2$ and $L \subseteq \overline{\mathbf{Q}}_p$ a finite extension of $\mathbf{Q}_p$, be a p-ordinary cuspidal eigenform. Suppose*

- $\chi = 1$ *and* $k \equiv 2 \bmod p-1$*;*
- (**irred**) *and* (**dist**) *hold for* ρ_f*;*
- *there exists a prime* $q \neq p$ *such that* $q || N_f$ *and* $\bar{\rho}_f$ *is ramified at* q*;*
- $p \nmid N_f$.

Then for any set of primes Σ,

$$Ch^{\Sigma}_{\mathbf{Q}_\infty, L}(f) = (\mathcal{L}^{\Sigma}_f)$$

in $\Lambda_{\mathbf{Q}, O_L} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$*. If furthermore*

- *there exists an* O_L*-basis of* T_f *with respect to which the image of* ρ_f *contains* $\mathrm{SL}_2(\mathbf{Z}_p)$,

then the equality holds in $\Lambda_{\mathbf{Q}, O_L}$*; that is, the Iwasawa-Greenberg Main Conjecture (Conjecture 3.5.5) is true for* f*.*

For the proof of this corollary we allow the field $\mathcal{K}$ to depend on f.

Proof. Choose an imaginary quadratic field $\mathcal{K}$ in which p splits, all prime divisors of N_f/q split, and q is inert (so $N^- = q$). Suppose first that Σ contains all primes dividing $pN_f D_{\mathcal{K}}$. By Kato's Theorem 3.5.6

$$(\mathcal{L}^{\Sigma}_f) \subseteq Ch^{\Sigma}_{\mathbf{Q}_\infty, L}(f) \quad \text{and} \quad (\mathcal{L}^{\Sigma}_{f \otimes \chi_{\mathcal{K}}}) \subseteq Ch^{\Sigma}_{\mathbf{Q}_\infty, L}(f \otimes \chi_{\mathcal{K}})$$

in $\Lambda_{\mathbf{Q}, O_L} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ and even in $\Lambda_{\mathbf{Q}, O_L}$ if the image of ρ_f contains $\mathrm{SL}_2(\mathbf{Z}_p)$. If the first of these is not an equality then we have

$$(\mathcal{L}^{\Sigma}_f \mathcal{L}^{\Sigma}_{f \otimes \chi}) \subsetneq Ch^{\Sigma}_{\mathbf{Q}_\infty, L}(f) Ch^{\Sigma}_{\mathbf{Q}_\infty, L}(f \otimes \chi_{\mathcal{K}}),$$

which contradicts Corollary 3.6.3. Therefore it must be that

$$(\mathcal{L}^{\Sigma}_f) = Ch^{\Sigma}_{\mathbf{Q}_\infty, L}(f).$$

That this holds for all choices of Σ then follows from Proposition 3.2.18 and Lemma 3.2.17 and the relation between $\mathcal{L}^{\Sigma}_f$ and $\mathcal{L}^{\Sigma'}_f$ for $\Sigma' \subset \Sigma$. ∎

Remarks on the hypotheses.

(i) The hypothesis that $p \nmid N_f$ comes from Kato's theorem (Theorem 3.5.6). When combined with the condition $\chi = \omega^{k-2}\chi_1$ from Corollary 3.6.3, this forces the first listed condition of the theorem.

(ii) The condition that $T_{\mathbf{f}}$ have an $\mathcal{O}_L$-basis with respect to which the image of ρ_f contains $\mathrm{SL}_2(\mathbf{Z}_p)$ also comes from Kato's theorem.

Theorem 3.6.5. *Let $f \in S_k(Np^r, \chi; L)$, $k \geq 2$ and $L \subseteq \overline{\mathbf{Q}}_p$ a finite extension of $\mathbf{Q}_p$, be a p-ordinary cuspidal eigenform. Assume that L contains $\mathbf{Q}[\boldsymbol{\mu}_{Np}, i, D_{\mathcal{K}}^{1/2}]$. Suppose $N = N^+N^-$ with N^+ divisible only by primes that split in $\mathcal{K}$ and N^- divisible only by primes inert in $\mathcal{K}$. Suppose also*

- *$\chi = 1$ and $k \equiv 2 \bmod p-1$;*
- *(**irred**) and (**dist**) hold;*
- *N^- is square-free and has an odd number of prime factors;*
- *$\bar{\rho}_f$ is ramified at all $\ell | N^-$;*
- *$p \nmid N_f$;*
- *there exists an O_L-basis of T_f with respect to which the image of ρ_f contains $\mathrm{SL}_2(\mathbf{Z}_p)$.*

Then for Σ a finite set of primes containing all those that divide $ND_{\mathcal{K}}$,

$$Ft^{\Sigma}_{\mathcal{K}_\infty,L}(f) = Ch^{\Sigma}_{\mathcal{K}_\infty,L}(f) = (\mathcal{L}^{\Sigma}_{f,\mathcal{K}}).$$

Proof. Let $A := O_L[\![\Gamma_{\mathcal{K}}]\!]$ and let $\mathfrak{a} \subset A$ be the kernel of the the homomorphism $O_L[\![\Gamma_{\mathcal{K}}]\!] \to O_L[\![\Gamma_{\mathbf{Q}}]\!]$ induced by the canonical projection $\Gamma_{\mathcal{K}} \to \Gamma_{\mathbf{Q}}$. Put $I := Ch^{\Sigma}_{\mathcal{K}_\infty,L}(f)$, $J := Ft^{\Sigma}_{\mathcal{K}_\infty,L}(f)$, and $\mathcal{L} := \mathcal{L}^{\Sigma}_{f,\mathcal{K}}$. By Corollary 3.6.2, $J \subseteq I \subseteq (\mathcal{L})$. We also have $J \bmod \mathfrak{a} = Ft^{\Sigma}_{\mathbf{Q}_\infty,L}(f)Ft^{\Sigma}_{\mathbf{Q}_\infty,L}(f\otimes\chi)$ by Proposition 3.2.11, and the latter ideal equals $\mathrm{C}h^{\Sigma}_{\mathbf{Q}_\infty,L}(f)\mathrm{C}h^{\Sigma}_{\mathbf{Q}_\infty,L}(f\otimes\chi)$ by Corollary 3.3.20(iii), and this ideal equals $(\mathcal{L}^{\Sigma}_{f}\mathcal{L}^{\Sigma}_{f\otimes\chi_{\mathcal{K}}}) = (\mathcal{L}) \bmod \mathfrak{a}$ by Theorem 3.6.4. The equalities $J = I = (\mathcal{L})$ then follows from Lemma 3.1.7. ∎

Theorem 3.6.6. *Let $L \subseteq \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$ and $\mathbb{I}$ a normal domain and a finite integral extension of Λ_{W,O_L}. Let $\mathbf{f}$ be an $\mathbb{I}$-adic ordinary eigenform of tame level N with $\chi_{\mathbf{f}} = 1$. Assume that L contains $\mathbf{Q}[\boldsymbol{\mu}_{Np}, i, D_{\mathcal{K}}^{1/2}]$. Suppose $N = N^+N^-$ with N^+ divisible only by primes that split in $\mathcal{K}$ and N^- divisible only by primes inert in $\mathcal{K}$. Suppose also*

- *(**irred**)$_{\mathbf{f}}$ and (**dist**)$_{\mathbf{f}}$ hold;*
- *N^- is square-free and has an odd number of prime factors;*
- *the reduction of $\rho_{\mathbf{f}}$ modulo the maximal ideal of $\mathbb{I}$ is ramified at all $\ell | N^-$;*
- *there exists an arithmetic homomorphism $\phi \in \mathcal{X}^a_{\mathbb{I},O_L}$ with $\phi(1+W) = (1+p)^{k_\phi}$ for some $k_\phi > 2$ and $p-1 | k_\phi - 2$ and such that $T_{\mathbf{f}_\phi}$ has an O_{L_ϕ} (where L_ϕ is the field of fractions of $\phi(\mathbb{I})$) with respect to which the image of $\rho_{\mathbf{f}_\phi}$ contains $\mathrm{SL}_2(\mathbf{Z}_p)$.*

Let Σ be a finite set of primes containing all those that divide $ND_{\mathcal{K}}$. Then

$$Ft^{\Sigma}_{\mathcal{K}_\infty,L}(\mathbf{f}) = Ch^{\Sigma}_{\mathcal{K}_\infty,L}(\mathbf{f}) = (\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}).$$

That is, the three-variable Iwasawa-Greenberg Main Conjecture (Conjecture 3.5.3) holds.

Proof. It is easy to see that the theorem is true if it is true in the case where $\phi(\mathbb{I}) = O_L$ (by extending scalars so that this holds). So we assume $\phi(\mathbb{I}) = O_L$. We then let $A := \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$

and $\mathfrak{a} \subset A$ be the kernel of the homomorphism $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] \to O_L[\Gamma_{\mathcal{K}}]$ induced by ϕ. Put $J := Ft^{\Sigma}_{\mathcal{K}_\infty}(\mathbf{f})$, $I := Ch^{\Sigma}_{K_\infty}(\mathbf{f})$, and $\mathcal{L} := \mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}} \in A$. Then $J \subseteq I \subseteq (\mathcal{L})$ by Theorem 3.6.1. We also have

$$J \bmod \mathfrak{a} = Ft^{\Sigma}_{\mathcal{K}_\infty}(\mathbf{f}_\phi) = (\mathcal{L}^{\Sigma}_{f_\phi,\mathcal{K}}) = (\mathcal{L}) \bmod \mathfrak{a}$$

by (3.3.12.b) and Theorem 3.6.5. That $J = I = (\mathcal{L})$ then follows from Lemma 3.1.7. ∎

3.6.7. *Elliptic curves.* The Iwasawa-Greenberg main conjectures have many consequences for relations between special values of L-functions and orders of Selmer groups. Here we record a few such consequences for elliptic curves.

Let $E_{/\mathbf{Q}}$ be an elliptic curve over $\mathbf{Q}$ having good ordinary reduction at p. Let T_pE be the p-adic Tate-module of E and let $\rho_{E,p} : G_{\mathbf{Q}} \to \mathrm{GL}_{\mathbf{Z}_p}(T_pE)$ give the action of $G_{\mathbf{Q}}$ on T_pE. The usual p-adic Selmer group of E over $\mathbf{Q}_\infty$ is just $Sel_{\mathbf{Q}_\infty}(T_pE, T_p^+E)$, where $T_pE^+ \subset T_pE$ is the rank-one $\mathbf{Z}_p$-summand on which I_p acts via ϵ. Let $\mathcal{F}_E := Ch_{\mathbf{Q}_\infty}(T_pE, T_pE^+)$.

Let N_E be the conductor of E and let $f \in S_2(N_E; \mathbf{Z}_p)$ be the weight 2 cuspidal eigenform of trivial character associated with E (so $L(E,s) = L(f,s)$); this exists by the modularity of elliptic curves. Then $Sel_{\mathbf{Q}_\infty}(E) = Sel_{\mathbf{Q}_\infty,\mathbf{Q}_p}(f)$ (as $T_f(\det \rho_f^{-1}) = T_pE$), so $\mathcal{F}_E = Ch_{\mathbf{Q}_\infty,\mathbf{Q}_p}(f)$. Let $\mathcal{L}_E \in \Lambda_{\mathbf{Q}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ be the usual p-adic L-function for E. *A prioiri* this is a $\mathbf{Q}_p^\times$-multiple of $\mathcal{L}_f$, and if $E[p]$ is an irreducible $G_{\mathbf{Q}}$-representation, then $\mathcal{L}_E$ is a p-adic unit multiple of $\mathcal{L}_f$. The main conjecture for E is the following.

Conjecture 3.6.8. *$\mathcal{F}_E$ is principal and generated by $\mathcal{L}_E$.*

As a special case of the results of the preceding section, we have the following case of this conjecture.

Theorem 3.6.9. *Assume that*

- *E has good ordinary reduction at p;*
- *$\bar\rho_{E,p} := \rho_{E,p} \bmod p$ is irreducible;*
- *there exists a prime $q||N$, $q \neq p$ such that $\bar\rho_{E,p}$ is ramified at q.*

Then $\mathcal{F}_E = (\mathcal{L}_E)$ in $\Lambda_{\mathbf{Q}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. If the image of $\rho_{E,p}$ is surjective, then this equality holds in $\Lambda_{\mathbf{Q}}$; that is, the main conjecture holds for E.

We note that the conditions on E in this theorem are always satisfied if E has semistable reduction and $p \geq 11$; in this case $\rho_{E,p}$ is even surjective. The surjectivity follows from a celebrated result of Mazur [Ma78] while the second condition follows easily from a result of Ribet [Ri91].

Corollary 3.6.10. *If E is semistable and $p \geq 11$ is a prime of good ordinary reduction, then the main conjecture holds for E.*

For each prime $\ell | N_E$ we write $c_\ell(E)$ for the order of the p-primary part of $E(\mathbf{Q}_\ell)/E_0(\mathbf{Q}_\ell)$, where $E_0(\mathbf{Q}_\ell) \subseteq E(\mathbf{Q}_\ell)$ is the subgroup of points having nonsingular reduction modulo

ℓ; this is the mazimal power of p that divides the Tamagawa number of E at the prime ℓ. We denote by $Ш(E/\mathbf{Q})_p$ the p-primary part of the Tate-Shafarevich group of E over $\mathbf{Q}$ (by a theorem of Kolyvagin, this is finite if $L(E,1) \neq 0$). Let Ω_E be the canonical period of E.

Theorem 3.6.11. *Let E be an elliptic curve over $\mathbf{Q}$ with conductor N_E. Suppose*

- *E has good ordinary reduction at p;*
- *there exists a prime $q||N_E$, $q \neq p$, such that $\bar{\rho}_{E,p}$ is ramified at q;*
- *$\bar{\rho}_{E,p}$ is irreducible.*

(a) If $L(E,1) \neq 0$ and $\bar{\rho}_{E,p}$ is surjective then

(3.6.11.a)
$$\left|\frac{L(E,1)}{\Omega_E}\right|_p^{-1} = \#Ш(E/\mathbf{Q})_p \cdot \prod_{\ell|N_E} c_\ell(E).$$

(b) If $L(E,1) = 0$ then the corank of the Selmer group $Sel_{p^\infty}(E/\mathbf{Q})$ is at least one.

Proof. This follows from the equality $(\mathcal{L}_f) = (\mathcal{L}_E) = \mathcal{F}_E$, the interpolation properties of $\mathcal{L}_f$ and $\mathcal{L}_E$, and Theorem 4.1 of [Gr99]. ∎

Remark. (a) In particular, (3.6.11.a) holds if E is semistable and $p \geq 11$ is a prime of good ordinary reduction.

(b) If the sign of the functional equation of $L(E,s)$ is -1 (i.e., if the order of vanishing at $s = 1$ is odd) then the positivity of the corank of $Sel_{p^\infty}(E/\mathbf{Q})$ has been established without assuming the existence of the prime q or the irreducibility of $\bar{\rho}_{E,p}$ (see [Ne01] and [SU06]). However, the conclusion of (b) is new for the case when the sign is $+1$, in which case it can be strengthened in combination with [Ne01, Thm. A] to conclude that the corank is at least two.

3.6.12. *Orders of other Selmer groups.* As a consequence of the main conjectures for modular forms we can deduce that the orders of various Selmer groups are given by special values of L-functions. We give some examples here.

Theorem 3.6.13. *Let $f \in S_k(Np^r, \chi; L)$, $k \geq 2$ and $L \subseteq \overline{\mathbf{Q}}_p$ a finite extension of $\mathbf{Q}_p$, be a p-ordinary cuspidal eigenform. Suppose*

- *$\chi = 1$ and $k \equiv 2 \bmod p-1$;*
- *(**irred**) and (**dist**) hold for ρ_f;*
- *there exists a prime $q \neq p$ such that $q||N_f$ and $\bar{\rho}_f$ is ramified at q;*
- *$p \nmid N_f$.*
- *there exists an O_L-basis of T_f with respect to which the image of ρ_f contains $\mathrm{SL}_2(\mathbf{Z}_p)$.*

Let Σ be a finite set of primes containing all those that divide pN_f, and $0 \leq m \leq k-1$ be an integer, and let ζ be a primitive p^{t-1}th-power root of unity ζ. If $m \neq 0$ or $\zeta \neq 1$

or $a(p,f)-1\in O_L^\times$ *then*

$$\#Sel^{\Sigma}_{\mathbf{Q},L}(f,\omega^m\chi_\zeta^{-1}\epsilon^{-m}) = \#O_L/L_p^{\Sigma}(f,\omega^{-m}\chi_\zeta,m+1),$$

where

$$L_p^{\Sigma}(f,\omega^{-m}\chi_\zeta,m+1) = a(p,f)^{-t}\left(1-\frac{\omega^{-m}\chi_\zeta\chi(p)p^{\kappa-2-m}}{a(p,f)}\right)\left(1-\frac{\omega^m\chi_\zeta^{-1}(p)p^m}{a(p,f)}\right) \times \frac{p^{t'(m+1)}m!L^{\Sigma/\{p\}}(f,\omega^{-m}\chi_\zeta,m+1)}{(-2\pi i)^m G(\omega^{-m}\chi_\zeta)\Omega_f^{\mathrm{sgn}((-1)^m}}$$

with $t'=0$ *if* $t=1$ *and* $p-1|m$ *and otherwise* $t'=t$.

Proof. This follows from Theorem 3.6.4 and Corollary 3.3.20. ∎

3.6.14. *Other main conjectures.* It is clearly possible to consider other one- and two-variable specializations of $\mathcal{L}^S_{\mathbf{f},\mathcal{K}}$.

Anticyclotomic Main Conjectures. Specializing the Hida family to some $f=\mathbf{f}_\phi$ of weight 2 and mapping γ_+ to 1 defines an anticyclotomic p-adic L-function in $\Lambda^-_{\mathcal{K},O_L}$ and an anticyclotomic Selmer group. When f corresponds to an elliptic curve, under the hypotheses on f in Theorem 3.6.5 this anticyclotomic L-function is easily seen to agree (up to a p-adic unit) with the p-adic L-function considered by Bertolini and Darmon in [BD05], and it is relatively straightforward to see that the results of this paper complete the work of Bertolini-Darmon.

Theorem 3.6.15. *In the notation of Theorem 1 of* [BD05]*, the characteristic power series* $\mathcal{C}$ *is generated by the p-adic L-function* $L_p(E,\mathcal{K})$*.*

We leave the details of these anticyclotomic results to the industrious reader.

Cyclotomic Main Conjectures for Hida families. Setting $\gamma_-=1$ yields a cyclotomic p-adic L-function for a Hida family $\mathbf{f}$. In some cases Ochiai [Och06] has extended Kato's theorem (Theorem 3.5.6) to this setting. In combination with the preceding results of this paper, it is possible to show that in many cases the divisibility in Ochai's main theorem is actually an equality (i.e., the cyclotomic main conjecture for the Hida family is true). As the hypotheses become even more cumbersome, we leave the formulation of such a result to the interested reader.

4. Constructing coycles

In this section we give an abstract framework which gives rise to groups of extensions of representations. This is later used to construct subgroups of the Selmer groups defined in §3.

4.1. **Some notations and conventions.** Unless otherwise clear, all rings herein are assumed commutative and to a have a unit element.

4.1.1. *Representations.* Let G be a group and C a ring. A C-representation of G is a pair (V, r) consisting of a finite C-module V and a homomorphism $r : G \to \mathrm{Aut}_C(V)$. This extends by C-linearity to a homomorphism $r : C[G] \to \mathrm{End}_C(V)$ of the group algebra $C[G]$ of G to the C-algebra of C-endomorphisms of V.

If $V = C^n$ then we will just write $r : G \to \mathrm{GL}_n(C)$ to mean such a representation. If $V \cong C^n$ then we can define $\mathrm{tr}\, r(x), \det r(x) \in C$ for any $x \in C[G]$. In particular we can define the characteristic polynomial of x

$$\mathrm{C}h(r, x, T) := \det(id - T \cdot r(x)) \in C[T].$$

We say that r is defined over a subring B of C if $\mathrm{C}h(r, x, T) \in B[T]$ for all $x \in G$. In general, however, this does not mean that r can be viewed as taking values in $\mathrm{GL}_n(B)$.

4.1.2. *Residually disjoint representations.* Let (V_1, σ_1) and (V_2, σ_2) be C-representations of a group G with each V_i free over C. Assume both are defined over a local henselian subring $B \subseteq C$. We say that σ_1 and σ_2 are residually disjoint modulo the maximal ideal $\mathfrak{m}_B$ of B (or just residually disjoint if B is clear) if there exists $x \in B[G]$ such that $Ch(\sigma_1, x, T) \bmod \mathfrak{m}_B$ and $Ch(\sigma_2, x, T) \bmod \mathfrak{m}_B$ are relatively prime in $\kappa_B[T]$, where $\kappa_B := B/\mathfrak{m}_B$.

Lemma 4.1.3. *Suppose (V_1, σ_1) and (V_2, σ_2) as above are residually disjoint. Then there exists $x_1, x_2 \in B[G]$ such that for $i, j \in \{1, 2\}$, x_i acts as the identity on V_j if $i = j$ and annihilates V_j if $i \neq j$.*

Proof. Let $x \in B[G]$ be such that $Ch(\sigma_1, x, T) \bmod \mathfrak{m}_B$ and $Ch(\sigma_2, x, T) \bmod \mathfrak{m}_B$ are relatively prime in $\kappa_B[T]$. By Hensel's lemma there exist $Q_1, Q_2 \in B[T]$ such that

$$Q_1(T)Ch(\sigma_1, x, T) + Q_2(T)Ch(\sigma_2, x, T) = 1.$$

Put $x_1 := Q_2(x)Ch(\sigma_2, x, x)$ and $x_2 := Q_1(x)Ch(\sigma_1, x, x)$. We have $x_1 + x_2 = 1$, and for $i \neq j$, x_i acts trivially on V_j by the Cayley identity. ∎

4.2. **The set-up.** We now explain the set-up in which we construct extensions.

4.2.1. *The rings A and R.* Let A be a complete local DVR with fraction field F, maximal ideal $\mathfrak{m}$, and residue field $\kappa := A/\mathfrak{m}$. Let R be a local reduced finite flat A-algebra (so R is also complete). Then $R \otimes_A F$ is a product of s finite field extensions of F for some positive integer s. We denote the maximal ideal of R by $\mathcal{M}$. We assume that $R/\mathcal{M} = \kappa$.

Only in the following abstract discussion will we use a 'bar' to denote reduction modulo $\mathfrak{m}$ or $\mathcal{M}$. Given an A-representation (resp. R-representation) (M, r) we let $(M/\mathfrak{m}M, \bar{r})$ (resp. $(M/\mathcal{M}M, \bar{r})$) to mean the reduction of the representation modulo $\mathfrak{m}$ (resp. $\mathcal{M}$).

4.2.2. *The groups and their representations.* Let H be a group with a decomposition $H = G \rtimes \{1, c\}$ with $c \in H$ an element of order two normalizing G. For any C-representation (V, r) of G, we write r^c for the representation defined by $r^c(g) = r(cgc)$ for all $g \in G$. For a character ξ of G we write ξ^{-c} to mean $(\xi^c)^{-1}$. We let $r^\vee$ be the dual representation of r; if V is free over C and we fix an identification $V = C^n$ then the canonical dual basis of the C-dual of C^n identifies $r^\vee(g)$ with ${}^t r(g^{-1})$ for all $g \in G$.

Polarizations. Let $\theta : G \to \mathrm{GL}_L(V)$ be a representation of G on a vector space V over a finite extension L of F and let $\psi : H \to L^\times$ be a character. We assume that θ satisfies the ψ-polarization condition

$$\theta^c \cong \psi \otimes \theta^\vee.$$

By a ψ-polarization of θ we mean a non-degenerate L-bilinear pairing $\Phi_\theta : V \times V \to L$ such that

$$\Phi_\theta(\theta(g)v, v') = \psi(g)\Phi_\theta(v, \theta^c(g^{-1})v')$$

for all $g \in G$ and $v, v' \in V$. We let $\Phi_\theta^t(v, v') := \Phi_\theta(v', v)$. This defines another ψ-polarization. We will say that ψ is compatible with the polarization Φ_θ if

$$\Phi_\theta^t = -\psi(c)\Phi_\theta. \tag{4.2.2.a}$$

When θ is irreducible a ψ-polarization is unique up to non-zero scalar and it is necessarily true that $\Phi_\theta^t = \pm\Phi_\theta$. In this case compatibility of ψ just pins down the sign of $\psi(c)$.

We extend the notions of a ψ-polarization and its compatibility with ψ to the situation of a free $R \otimes_A F$-representation $\theta : G \to \mathrm{Aut}_{R\otimes_A F}(V)$, $V \cong (R \otimes_A F)^n$, and a character $\psi : H \to F^\times$ in the obvious way. Clearly, ψ is compatible with the polarization Φ_θ if and only if for each F-algebra homomorphism $\lambda : R \otimes_A F \to L$, L a finite extension of F, ψ is compatible with the induced polarization of $\theta_\lambda : G \to \mathrm{GL}_n(L)$, the representation obtained from θ via λ.

4.2.3. *The data.* The set-up consists of the following data:

(1) a character $\nu : H \to A^\times$;
(2) a character $\chi : G \to A^\times$ such that $\bar\chi \neq \bar\nu\bar\chi^{-c}$; let $\chi' := \nu\chi^{-c}$;
(3) a representation $\rho : G \to \mathrm{Aut}_A(V)$, $V \cong A^n$, such that
   a. $\rho^c \cong \rho^\vee \otimes \nu$,
   b. $\bar\rho$ is absolutely irreducible,
   c. ρ is residually disjoint from χ and χ';
(4) a representation $\sigma : G \to \mathrm{Aut}_{R\otimes_A F}(M)$, $M \cong (R \otimes_A F)^m$, with $m = n + 2$, such that
   a. $\sigma^c \cong \sigma^\vee \otimes \nu$,
   b. $\mathrm{tr}\,\sigma(g) \in R$ for all $g \in G$,
   c. for any $v \in M$, $\sigma(R[G])v$ is a finitely-generated R-module;
(5) a proper ideal $I \subset R$ such that $J := A \cap I \neq 0$, the natural map $A/J \to R/I$ is an isomorphism, and

$$\mathrm{tr}\,\sigma(g) \equiv \chi'(g) + \mathrm{tr}\,\rho(g) + \chi(g) \bmod I$$

   for all $g \in G$.

Noting that ρ is irreducible, we also assume that

(4.2.3.a) ν is compatible with ρ

in the sense of 4.2.2.

4.2.4. *The pairing* Φ. It follows from the assumption 4.2.3.(4a) that there exists a non-degenerate $R \otimes_A F$-linear pairing $\Phi(-,-) : M \times M \to R \otimes_A F$ such that

$$\Phi(\sigma(g)v, v') = \nu(g)\Phi(v, \sigma^c(g^{-1})v')$$

for all $g \in G$ and $v, v' \in M$.

We consider the R-linear involution $x \mapsto x^*$ of $R[G]$ defined by $g^* := cg^{-1}c$ for $g \in G$ and extended R-linearly to $R[G]$. We consider also the action σ^* of $R[G]$ on M defined by

$$\sigma^*(g) = \nu(g)^{-1}\sigma(g)$$

for $g \in G$ and extended R-linearly to an action of $R[G]$. Then

$$\Phi(\sigma(x)v, v') = \Phi(v, \sigma^*(x^*)v').$$

4.3. **The canonical lattice.** Here on we assume that $\kappa = A/\mathfrak{m} = R/\mathcal{M}$ contains at least m non-zero elements. In our applications κ will be infinite.

4.3.1. *Some projectors.* From 4.2.3.(4b) and 4.2.3.(5) it follows that for all $x \in R[G]$, $Ch(\sigma, x, T) \in R[T]$ and

(4.3.1.a) $$Ch(\sigma, x, T) \equiv (1 - T\chi'(x))Ch(\rho, x, T)(1 - T\chi(x)) \bmod I.$$

Let $\overline{Ch}(\sigma, x, T) \in \kappa[T]$ be the reduction of $\mathrm{C}h(\sigma, x, T)$ modulo $\mathcal{M}$. As $\bar{\rho}$ is irreducible and ρ, χ, and χ' are pair-wise residually disjoint, it follows from Lemma 4.1.3 that there exists $x_0 \in R[G]$ such that $\overline{Ch}(\sigma, x_0, T)$ splits over κ and has distinct non-zero roots. Since R is Henselian, $Ch(\sigma, x_0, T)$ splits in $R[T]$. We fix such an x_0 and label the roots of $Ch(\sigma, x_0, T)$ as $\alpha_1, \cdots, \alpha_m \in R$, ordered so that $\alpha_1 \equiv \chi'(x_0) \bmod I$ and $\alpha_m \equiv \chi(x_0) \bmod I$. It follows that M has an $R \otimes_A F$-basis $\epsilon_1, ..., \epsilon_m$ such that $\sigma(x_0)\epsilon_i = \alpha_i\epsilon_i$. We use this basis to identify M with $(R \otimes_A F)^m$ and σ with a homomorphism $G \to \mathrm{GL}_m(R \otimes_A F)$.

Put

$$x_i := \prod_{j \neq i}(x_0 - \alpha_j)(\alpha_i - \alpha_j)^{-1} \in R[G]$$

and

$$\pi_i := \sigma(x_i).$$

Then $\pi_i\epsilon_i = \epsilon_i$ and $\pi_i\epsilon_j = 0$ if $j \neq i$; that is, π_i projects M onto its ϵ_i-component (with respect to the basis $\epsilon_1, ..., \epsilon_m$). Note that $\pi_1 + \cdots + \pi_m = 1$ and $\pi_i\pi_j = \pi_i$ if $i = j$ and otherwise $\pi_i\pi_j = 0$. Put also

$$x_\chi := x_m, \quad x_{\chi'} := x_1, \quad x_\rho := x_2 + \cdots + x_{m-1},$$

and

$$\pi_\mu := \sigma(x_\mu), \quad \mu \in \{\chi, \chi', \rho\}.$$

Let $\beta_1 := \chi'(x_0)$ and $\beta_m := \chi(x_0)$. Let $\beta_2, ..., \beta_{m-1}$ be the eigenvalues of $\rho(x_0) \in M_n(R)$, ordered so that $\beta_i \equiv \alpha_i \bmod I$. Define $y_i, y_\mu \in R[G]$ just as x_i and x_μ but with the α_j's replaced by the β_j's. Then $\mu(y_\lambda) = 0$ if $\mu \neq \lambda$ and $\mu(y_\mu) = 1$. Also, $x_i \equiv y_i \bmod I$ and $x_\mu \equiv y_\mu \bmod I$.

Let $N := V \otimes_A R$ with G-action by ρ. Let $\delta_2, ..., \delta_{m-1}$ be an R-basis such that $\rho(x_0)\delta_i = \beta_i\delta_i$. This is possible as the β_i's are distinct modulo $\mathcal{M}$. Then $R\delta_i = \rho(y_i)N$. For $2 \leq i, j \leq m-1$ let $t_{ij} \in R[G]$ be such that the i', j'-entry of $\rho(t_{ij})$ with respect to the basis $\delta_2, ..., \delta_{m-1}$ is 0 if $(i'+1, j'+1) \neq (i, j)$ and 1 if $(i'+1, j'+1) = (i, j)$. This is possible as $\bar{\rho}$ is irreducible and so $\rho(R[G]) = \mathrm{End}_R(N)$. Note that $y_i t_{ij} = t_{ij} y_j = y_i t_{ij} y_j = t_{ij}$.

4.3.2. *The parity condition.* Recall that we have assumed that ν is compatible with ρ (see (4.2.3.a)). As $\bar{\rho}$ and hence ρ is absolutely irreducible, any ν-polarization of ρ satisfies (4.2.2.a) with ψ replaced by ν. We have not made the same assumption of σ, but as the next lemma shows this is almost immediate.

Recall that $R \otimes_A F \cong L_1 \times \cdots \times L_s$ with each L_i a finite field extension of F. Let σ_i denote the representation of G on $M_i := M \otimes_{R \otimes_A F} L_i \cong L_i^m$ induced by σ.

Lemma 4.3.3. *Assume the characteristic of κ is not 2. Let $1 \leq i \leq s$. If σ_i is absolutely irreducible, then ν is compatible with σ_i. In particular, if each σ_j, $1 \leq j \leq s$, is absolutely irreducible, then ν is compatible with σ.*

Proof. Let $L := L_i$ and let $B \subset L$ be the integral closure of A in L (so B is also a complete local DVR). Let κ_B be the residue field of B. Let v_2 be the image of ϵ_2 in M_i, and let $D := B[G]v_2 \subset M_i$, the sub-$B[G]$-module generated by v_2. The absolute irreducibility of σ_i means that D is a free B-module of rank m and $D \otimes_B L = M_i$. Let $J \subset B$ be the (proper) ideal generated by the image of I. From 4.3.1.a it follows that

$$Ch(\sigma_i, x, T) \equiv (1 - T\chi'(x))Ch(\rho, x, T)(1 - T\chi(x)) \bmod J$$

for all $x \in B[G]$. From this and the Brauer-Nesbitt theorem it follows that the semisimplification of $\overline{D} := D \otimes_B \kappa_B$ is isomorphic to $\bar{\chi}' \oplus \bar{\rho} \oplus \bar{\chi}$. As $\overline{D}$ is generated by $\bar{v}_2 := v_2 \otimes 1$, which is an eigenvector for $\bar{\rho}(x_0)$, it is easily seen that $\bar{\rho}$ is the unique irreducible $\kappa_B[G]$-quotient of $\overline{D}$. Let $\bar{\sigma}_i$ denote the G-action on $\overline{D}$.

Let Φ_i be the polarization of M_i induced by Φ. Scaling Φ by an appropriate element of $L^\times$, we may assume that $\Phi_i(D, D)$ generates B. Then the reduction $\overline{\Phi}_i$ of Φ_i modulo the maximal ideal of B defines a κ_B-linear pairing on $\overline{D} \times \overline{D}$ satisfying

$$\overline{\Phi}_i(\bar{\sigma}_i(x)d, d') = \overline{\Phi}_i(d, \bar{\sigma}_i^*(x^*)d')$$

for all $d, d' \in \overline{D}^*$ and $x \in B[G]$ (i.e., a ν-polarization of $\bar{\sigma}_i$). It is easily seen that $\overline{\Phi}_i$ induces a non-trivial polarization on the unique $\kappa_B[G]$-quotient of $\overline{D}$, namely $\bar{\rho}$. The compatibility of $\bar{\nu}$ with this polarization (following from our hypothesis that ν is compatible with ρ) then implies the compatibility of $\bar{\nu}$ with $\overline{\Phi}_i$ and hence of ν with Φ_i as κ_B has characteristic different from two.■

4.3.4. *The lattice.* Put $\epsilon := \epsilon_m$ and let $L := \sigma(R[G])\epsilon$. By hypothesis, L is a finitely-generated R-submodule of M. It has direct sum decompositions

$$L = L(\alpha_1) \oplus L(\alpha_2) \oplus \cdots \oplus L(\alpha_m) = L(\chi') \oplus L(\rho) \oplus L(\chi)$$

with

$$L(\alpha_i) := \pi_i L = \{l \in L \mid \sigma(x_0)l = \alpha_i l\}$$

and

$$L(\mu) := \pi_\mu L.$$

Here L is not necessarily a lattice in the usual terminology; it can be that $L \otimes_A F \neq M \otimes_A F$.

Lemma 4.3.5. *As an R-module, $L(\alpha_m) = L(\chi)$ is free of rank one.*

Proof. Let $l \in L$. Then $l = \sigma(x)\epsilon$ for some $x \in R[G]$ and $\pi_m(l) = \sigma(x_m x x_m)\epsilon = \mathrm{tr}\,\sigma(x_m x x_m)\epsilon$. Since $\mathrm{tr}\,\sigma(y) \in R$ for any $y \in R[G]$, it follows that $L(\alpha_m) \subseteq R\epsilon$. Since L is the $R[G]$-module generated by ϵ, the opposite inclusion is clear. ■

For $x \in R[G]$ and $\mu, \lambda \in \{\chi, \chi', \rho\}$ put

$$A_{\mu,\lambda}(x) := \sigma(x_\mu x x_\lambda) \in M_m(R \otimes_A F).$$

Note that we can also view $A_{\mu,\lambda}(x)$ as an element of $\mathrm{End}_R(L(\lambda), L(\mu))$.

Lemma 4.3.6. *Let $x, y \in R[G]$.*

(i) *If $\mu \neq \lambda$, then* $\mathrm{tr}\,A_{\mu,\lambda}(x)A_{\lambda,\mu}(y) \in I$.
(ii) $\mathrm{tr}\,A_{\mu,\mu}(x) \equiv \mathrm{tr}\,\mu(x) \bmod I$.

Proof. If $\mu \neq \lambda$ then

$$\begin{aligned}\mathrm{tr}\,A_{\mu,\lambda}(x)A_{\lambda,\mu}(y) &= \mathrm{tr}\,\sigma(x_\mu x x_\lambda y x_\mu)\\ &\equiv (\chi \oplus \rho \oplus \chi')(x_\mu x x_\lambda y x_\mu) \bmod I\\ &\equiv (\chi \oplus \rho \oplus \chi')(y_\mu x y_\lambda y y_\mu) \bmod I\\ &\equiv 0 \bmod I.\end{aligned}$$

The next to last line vanishes as either $\theta(y_\mu) = 0$ or $\theta(y_\lambda) = 0$ for $\theta \in \{\chi, \chi', \rho\}$ if $\mu \neq \lambda$ (so one of μ and λ is not equal to θ). This proves part (i). Similarly,

$$\mathrm{tr}\,A_{\mu,\mu}(x) = \mathrm{tr}\,\sigma(x_\mu x x_\mu) \equiv \mathrm{tr}\,(\chi \oplus \rho \oplus \chi')(y_\mu x y_\mu) = \mathrm{tr}\,\mu(x) \bmod I,$$

proving part (ii).■

For $2 \leq i, j \leq m-1$, let

$$\phi_{ij} := \sigma(x_i t_{ij} x_j) \in \mathrm{Hom}_R(L(\alpha_j), L(\alpha_i)).$$

Lemma 4.3.7. *Each ϕ_{ij} is an isomorphism.*

Proof. It suffices to prove that $\phi := \phi_{ij} \circ \phi_{ji} \in \mathrm{End}_R(L(\alpha_i))$ is an isomorphism. But ϕ acts via multiplication by the i,i-entry of $\sigma(x_i t_{ij} x_j t_{ji} x_i)$. This entry equals tr $\sigma(x_i t_{ij} x_j t_{ji} x_i)$ and we have

$$\begin{aligned} \mathrm{tr}\ \sigma(x_i t_{ij} x_j t_{ji} x_i) &\equiv \mathrm{tr}\ \rho(x_i t_{ij} x_j t_{ji} x_i) \bmod I \\ &\equiv \mathrm{tr}\ \rho(y_i t_{ij} y_j t_{ji} y_i) \bmod I \\ &\equiv \mathrm{tr}\ \rho(t_{ij} t_{ji}) \equiv 1 \bmod I. \end{aligned}$$

Therefore ϕ acts by multiplication by a unit of R and so is an isomorphism.■

Lemma 4.3.8. *Let $x, y \in R[G]$.*

(i) *If $\mu \neq \lambda$ the image of $A_{\mu,\lambda}(x) A_{\lambda,\mu}(y) \in \mathrm{End}_R(L(\mu), L(\mu))$ is contained in $IL(\mu)$.*
(ii) *If $\mu \neq \chi$ the image of $A_{\chi,\mu}(x) \in \mathrm{End}_R(L(\mu), L(\chi))$ is contained in $IL(\chi) = I\epsilon$.*

Proof. We first prove (i). Suppose that $\mu = \chi'$. As $L(\chi') = L(\alpha_1)$ it follows that $A_{\mu,\lambda}(x) A_{\lambda,\mu}(y)$ acts as multiplication by the $1,1$-entry of $\sigma(x_1 x x_\lambda y x_1)$. This entry is just tr $\sigma(x_1 x x_\lambda y x_1) = $ tr $A_{\mu,\lambda}(x) A_{\lambda,\mu}(y)$, which belongs to I by Lemma 4.3.6. The case $\mu = \chi$ is proved the same way, but using the m,m-entry instead.

Suppose then that $\mu = \rho$. We need to show that for all $2 \leq i, j \leq m-1$, $l \in L(\alpha_j)$, $\pi_i A_{\rho,\lambda}(x) A_{\lambda,\rho}(y) l \in IL(\alpha_i)$. As ϕ_{ji} is an isomorphism, it suffices then to show that $\phi_{ji} \pi_i A_{\rho,\lambda}(x) A_{\lambda,\rho}(y) l \in IL(\alpha_j)$. But $\phi_{ji} \pi_i A_{\rho,\lambda}(x) A_{\lambda,\rho}(y)$ acts on $L(\alpha_j)$ by multiplication by the j,j-entry of $\sigma(x_j t_{ji} x_i x_\rho x x_\lambda y x_\rho)$. This is tr $\sigma(x_j t_{ji} x_i x_\rho x x_\lambda y x_\rho x_j)$, which equals tr $\sigma(x_\rho x_j t_{ji} x_i x_\rho x x_\lambda) \sigma(x_\lambda y x_j x_\rho)$ since $x_\rho x_j = x_j = x_j x_\rho$, and the latter trace belongs to I by Lemma 4.3.6. This completes the proof of part (i).

As L is generated by ϵ over $R[G]$, for any element $l \in L(\mu)$ we have $l = \sigma(y)\epsilon = \sigma(x_\mu y x_\chi)\epsilon = A_{\mu,\chi}(y)\epsilon$ for some $y \in R[G]$. By part (i) we then have

$$A_{\chi,\mu}(x) l = A_{\chi,\mu}(x) A_{\mu,\chi}(y) \epsilon \in I\epsilon,$$

proving part (ii). ■

Put

$$\mathcal{L} := L/IL \quad \text{and} \quad \mathcal{L}(\mu) := L(\mu)/IL(\mu).$$

Let $\mathcal{A}_{\mu,\lambda}(x)$ be the image of $A_{\mu,\lambda}(x)$ in $\mathrm{End}_R(\mathcal{L}(\lambda), \mathcal{L}(\mu))$.

Corollary 4.3.9. *The sub-R-module $Sub(\mathcal{L}) := \mathcal{L}(\chi') \oplus \mathcal{L}(\rho) \subset \mathcal{L}$ is a sub-$R[G]$-module. The quotient $Quot(\mathcal{L}) := \mathcal{L}/Sub(\mathcal{L})$ is isomorphic as an $R[G]$-module to $(R/I)(\chi)$.*

Proof. Let $x \in R[G]$ and $l \in L(\chi') \oplus L(\rho)$. Then $\sigma(x) l = \sigma(x x_{\chi'}) l + \sigma(x x_\rho) l$. As the projection of $\sigma(x x_\mu) l$ to $L(\chi)$ is $\sigma(x_\chi x x_\mu) l = A_{\chi,\mu}(x) l$ and so belongs to $IL(\chi)$ by part

(ii) of the preceding lemma if $\mu \neq \chi$, the image of $\sigma(x)l$ in $\mathcal{L}$ belongs to $Sub(\mathcal{L})$. Thus $Sub(\mathcal{L})$ is a sub-$R[G]$-module.

The sub-R-module $\mathcal{L}(\chi)$ maps isomorphically onto $Quot(\mathcal{L})$. So to prove the second claim it suffices to show that for all $g \in G$, $\sigma(g)\epsilon - \chi(g)\epsilon \in L(\chi') \oplus L(\rho) \oplus IL(\chi)$; in other words, that $\pi_\chi(\sigma(g)\epsilon - \chi(g)\epsilon) \in IL(\chi) = I\epsilon$. But for any $x \in R[G]$, $\pi_\chi(\sigma(x)\epsilon) = a_m\epsilon$, where a_m is the m,m-entry of $\sigma(x)$ and so is equal to tr $\sigma(x_m x x_m)$. By part (ii) of Lemma 4.3.6, tr $\sigma(x_m g x_m) = \text{tr } A_{\chi,\chi}(g) \equiv \chi(g) \bmod I$.■

Lemma 4.3.10.

(i) *The map $x \mapsto \mathcal{A}_{\rho,\rho}(x) \in \mathrm{End}_R(\mathcal{L}(\rho))$, $x \in R[G]$, defines an R-representation.*
(ii) *There exists a finite torsion R-module $\mathcal{N}$ and an isomorphism $(\mathcal{L}(\rho), \mathcal{A}_{\rho,\rho}) \cong (\mathcal{N} \otimes_R N, 1 \otimes \rho)$ of R-representations.*
(iii) *If $L \otimes_A F = M$ then $Fitt_R(\mathcal{N}) \subseteq I$.*

Proof. Put $\mathcal{A}(x) := \mathcal{A}_{\rho,\rho}(x)$. We have

$$\mathcal{A}(xy) = \sum_{\mu \in \{\chi,\chi',\rho\}} \mathcal{A}_{\rho,\mu}(x)\mathcal{A}_{\mu,\rho}(y) = \mathcal{A}(x)\mathcal{A}(y),$$

the last equality by part (i) of Lemma 4.3.8. This proves part (i).

Let $\mathcal{L}(\alpha_i) := L(\alpha_i)/IL(\alpha_i)$. For part (ii) we take $\mathcal{N} := \mathcal{L}(\alpha_2)$. For $3 \leq j \leq m-1$ the isomorphism ϕ_{j2} (see Lemma 4.3.7) identifies $\mathcal{N}$ with $\mathcal{L}(\alpha_j)$. As $\mathcal{L}(\rho) = \mathcal{L}(\alpha_2) \oplus \cdots \oplus \mathcal{L}(\alpha_{m-1})$, $\mathcal{L}(\rho)$ is identified with $\mathcal{N}^n$. Let $\mathcal{B} := \mathrm{End}_R(\mathcal{N})$. Via this identification, the action of G on $\mathcal{L}(\rho)$ through $\mathcal{A}_{\rho,\rho}$ defines a homomorphism $\rho' : R[G] \to M_n(\mathcal{B})$. We claim that the a,b-entry of any $\rho'(x)$ is just the a,b-entry of $\rho \bmod I$. The isomorphism of part (ii) follows.

We prove the claim. For $x \in R[G]$, let $x(a,b) \in \mathcal{B}$ be the a,b-entry of $\rho'(x)$; $x(a,b)$ is just the reduction modulo I of $\phi_{a+1,2}^{-1} \circ \sigma(x_{a+1} x x_{b+1}) \circ \phi_{2,b+1}$, where we view $\sigma(x_{a+1} x x_{b+1})$ as defining an element of $\mathrm{Hom}_R(L(\alpha_{b+1}), L(\alpha_{a+1}))$. Then $x(a,a)$ is just multiplication by tr $\sigma(x_{a+1} x x_{a+1}) \bmod I$ which is just tr $\rho(y_{a+1} x y_{a+1}) \bmod I$. It follows that for any $x' \in R[G]$ we have

$$\begin{aligned} x'(a,b)x(b,a) &= \sigma(x_{a+1}x'x_{b+1}xx_{a+1})(a,a) \\ &= \text{tr } \sigma(x_{a+1}x'x_{b+1}xx_{a+1}) \bmod I \\ &= \text{tr } \rho(y_{a+1}x'y_{b+1}xy_{a+1}) \bmod I. \end{aligned}$$

Taking $x' = t_{a+1,b+1}$ we have $x'(a,b) = 1$ and so $x(b,a) = \text{tr } \rho(y_{a+1}x'y_{b+1}xy_{a+1}) \bmod I = \text{tr } \rho(t_{a+1,b+1}xy_{a+1}) \bmod I$, which is just the b,a-entry of $\rho(x) \bmod I$. This proves (ii).

Finally, if $L \otimes_A F = M$ then $L(\alpha_i) \otimes_A F \cong R \otimes_A F$. In particular, $L(\alpha_2)$ is a faithful R-module. Hence $Fitt_R(\mathcal{N}) = Fitt_R(L(\alpha_2)/IL(\alpha_2)) \subseteq I$. ■

Lemma 4.3.11. *If for each F-algebra homomorphism $\lambda : R\otimes_A F \to K$, K a finite field extension of F, the representation $\sigma_\lambda : G \to \mathrm{GL}_m(M\otimes_{R\otimes_A F,\lambda} K)$ obtained from σ via λ is either absolutely irreducible or contains an absolutely irreducible two-dimensional sub-K-representation σ'_λ such that $\mathrm{tr}\,\sigma'_\lambda(g) \equiv \chi(g)+\chi'(g) \bmod I$, then $\mathcal{L}(\chi')$ is a faithful R/I-module.*

Proof. It suffices to show that $M(\chi') := L(\chi')\otimes_A F$ is a faithful $R\otimes_A F$-module. Since $R\otimes_A F$ is a product of finite extensions of F, it therefore suffices to show that $M(\chi')\otimes_{R\otimes_A F,\lambda} K \neq 0$ for each λ as in the lemma.

Let $M_\lambda := M\otimes_{R\otimes_A F,\lambda} K \cong K^m$ be the representation space of σ_λ and let ϵ_λ be the image of ϵ in M. Then $L_\lambda := \sigma_\lambda(K[G])\epsilon_\lambda \subset M_\lambda$ is the K-span of the image of L in M_λ, and $M(\chi')\otimes_{R\otimes_A F,\lambda} K\cap L_\lambda$ is the subspace of L_λ on which x_0 acts as $\lambda(\alpha_1)$. The hypotheses on each σ_λ ensure that this last subspace is always non-zero. ∎

4.3.12. *L^* and $\mathcal{L}^*$.* Let L^* be the R-module L with $R[G]$-action by σ^*. Then

$$\mathrm{C}h(\sigma, x_0, T) = \mathrm{C}h(\sigma^*, x_0^*, T),$$

so the eigenvalues of $\sigma^*(x_0^*)$ are just α_1,...,α_m. We also have

$$\chi\nu^{-1}(x_0^*) = \chi^{-c}\nu(x_0) = \alpha_1 \quad \text{and} \quad \chi^{-c}(x_0^*) = \chi(x_0) = \alpha_m.$$

Therefore there is a decomposition

$$L^* = L^*(\chi\nu^{-1}) \oplus L^*(\rho\otimes\nu^{-1}) \oplus L^*(\chi^{-c})$$

where we have set $L^*(\chi\nu^{-1}) := L^*(\alpha_1)$, $L^*(\chi^{-c}) := L^*(\alpha_m)$, and $L^*(\rho\otimes\nu^{-1}) := L^*(\alpha_2)\oplus \cdots \oplus L^*(\alpha_{m-1})$ with $L^*(\alpha_i) = \{l \in L |\ \sigma^*(x_0^*)l = \alpha_i l\}$ for $i \in \{1,\dots m\}$. In analogy with the $A_{\mu,\lambda}(x)$'s, we define $A^*_{\mu,\lambda}(x) \in \mathrm{Hom}_R(L^*(\lambda), L^*(\mu))$ for all $x \in R[G]$ and $\mu,\lambda \in \{\chi\nu^{-1}, \rho\otimes\nu^{-1}, \chi^{-c}\}$.

Both $\sigma^*(x_0^*)\epsilon$ and $\alpha_1\epsilon$ have the same image in $Quot(\mathcal{L})$. As $\alpha_1 \in R^\times$ this image is non-zero, and it follows that $L^*(\chi\nu^{-1})$ must project surjectively onto $Quot(\mathcal{L})$ and hence that $L^* = \sigma^*(R[G])\epsilon^*$ for some $\epsilon^* \in L^*(\chi\nu^{-1})$ such that $\epsilon^* \equiv \epsilon \bmod I\mathcal{L}$. Just as for $L(\chi)$, one easily sees that $L^*(\chi\nu^{-1})$ is free of rank one over R, generated by ϵ^*.

Let $\mathcal{L}^* := L^*/IL^*$. This is just $\mathcal{L}$ with G-action twisted by ν^{-1}. There is then a decomposition

$$\mathcal{L}^* = \mathcal{L}^*(\chi\nu^{-1}) \oplus \mathcal{L}^*(\rho\otimes\nu^{-1}) \oplus \mathcal{L}^*(\chi^{-c}),$$

where $\mathcal{L}^*(\mu) := L^*(\mu)/IL^*(\mu)$. Clearly $\mathcal{L}^*(\mu)$ is just $\mathcal{L}(\mu\nu)$ with G-action twisted by ν^{-1}. For $x \in R[G]$ we denote by $\mathcal{A}^*_{\mu,\lambda} \in \mathrm{Hom}_R(\mathcal{L}^*(\lambda), \mathcal{L}^*(\mu))$ the homomorphism induced by the action of x on $\mathcal{L}^*$; this is just the homomorphism induced by $A^*_{\mu,\lambda}$. Setting

$$Sub(\mathcal{L}^*) := \mathcal{L}^*(\rho\otimes\nu^{-1}) \oplus \mathcal{L}^*(\chi^{-c}) \quad \text{and} \quad Quot(\mathcal{L}^*) := \mathcal{L}^*/Sub(\mathcal{L}^*),$$

the arguments used to prove Corollary 4.3.9 can be used to prove that

(4.3.12.a) $\qquad Sub(\mathcal{L}^*)$ is a sub-$R[G]$-module and $Quot(\mathcal{L}^*) \cong (R/I)(\chi\nu^{-1})$.

4.3.13. *The reduction of* Φ *modulo* I. We consider the restriction of the pairing Φ to $L \times L^*$.

Lemma 4.3.14.

(i) $L^*(\chi\nu^{-1}) \oplus L^*(\rho \otimes \nu^{-1})$ *is orthogonal to* $L(\chi)$.
(ii) $L(\chi) \oplus L(\rho)$ *is orthogonal to* $L^*(\chi\nu^{-1})$.
(iii) *Let* $P := \Phi(L, L^*)$. *Then*
$$P = \Phi(L(\chi), L^*(\chi^{-c})) = \Phi(L(\chi^{-c}\nu), L^*(\chi\nu^{-1})).$$
In particular, there are R*-isomorphisms* $P \cong L(\chi^{-c}\nu) \cong L^*(\chi^{-c})$.

Proof. If $l \in L(\alpha_i)$ and $l' \in L^*(\alpha_j)$, then
$$\alpha_i \Phi(l, l') = \Phi(\sigma(x_0)l, l') = \Phi(l, \sigma^*(x_0^*)l') = \alpha_j \Phi(l, l').$$
Therefore $\Phi(l, l') = 0$ if $i \neq j$. Parts (i) and (ii) follow.

For any $l \in L$ there exists $x \in R[G]$ such that $l = \sigma(x)\epsilon$. Therefore $\Phi(l, l') = \Phi(\sigma(x)\epsilon, l') = \Phi(\epsilon, \sigma^*(x^*)l')$, hence $P = \Phi(\epsilon, L^*) = \Phi(\epsilon, L^*(\chi^{-c})) = \Phi(L(\chi), L^*) = \Phi(L(\chi), L^*(\chi^{-c}))$ by part (i). Then $l' \mapsto \Phi(\epsilon, l')$ is a surjective R-homomorphism from $L^*(\chi^{-c})$ onto P. As Φ is non-degenerate, this map must also be an isomorphism. Similarly, we have $P = \Phi(L, \epsilon^*) = \Phi(L(\chi^{-c}\nu), \epsilon^*) = \Phi(L(\chi^{-c}\nu), L^*(\chi\nu^{-1}))$ and $l \mapsto \Phi(l, \epsilon^*)$ defines an isomorphism of $L(\chi^{-c}\nu)$ with P. ■

4.4. **The cocycle construction.** We now explain how the previously defined objects give rise to various extensions of representations and hence to subgroups of cocycle classes.

4.4.1. *The extensions.* Let
$$\mathcal{J}(\chi') := \sum_{x \in R[G]} \mathrm{Im}(\mathcal{A}_{\chi',\rho}(x)) \subset \mathcal{L}(\chi') \quad \text{and} \quad \mathcal{J}(\rho) := \sum_{x \in R[G]} \mathrm{Im}(\mathcal{A}_{\rho,\chi'}(x)) \subset \mathcal{L}(\rho).$$
The R-submodules $\mathcal{L}(\rho) \oplus \mathcal{J}(\chi')$ and $\mathcal{L}(\chi') \oplus \mathcal{J}(\rho)$ are both sub-$R[G]$-modules of $Sub(\mathcal{L})$. Also for any $x \in R[G]$, $\mathcal{A}_{\chi',\chi'}(x)\mathcal{J}(\chi') \subseteq \mathcal{J}(\chi')$ and $\mathcal{A}_{\rho,\rho}(x)\mathcal{J}(\rho) \subseteq \mathcal{J}(\rho)$. As explained in Lemma 4.3.10(i), $x \mapsto \mathcal{A}_{\rho,\rho}(x)$ defines an $R[G]$-action on $\mathcal{L}(\rho)$ (and hence on $\mathcal{L}(\rho)/\mathcal{J}(\rho)$). Similary, $x \mapsto \mathcal{A}_{\chi',\chi'}(x)$ defines an $R[G]$-action on $\mathcal{L}(\chi')$ (and hence on $\mathcal{L}(\chi')/\mathcal{J}(\chi')$); this action is just given by $\chi'(x)$. Therefore there are exact sequences of $R[G]$-modules

(4.4.1.a) $$0 \to \mathcal{L}(\chi')/\mathcal{J}(\chi') \to \mathcal{L}/(\mathcal{L}(\rho) \oplus \mathcal{J}(\chi')) \to Quot(\mathcal{L}) \cong R/I(\chi) \to 0$$

and

(4.4.1.b) $$0 \to \mathcal{L}(\rho)/\mathcal{J}(\rho) \to \mathcal{L}/(\mathcal{L}(\chi') \oplus \mathcal{J}(\rho)) \to Quot(\mathcal{L}) \cong R/I(\chi) \to 0.$$

The proof of Lemma 4.3.10(ii) can be adapted to prove that there is a sub-R-module $\mathcal{N}' \subseteq \mathcal{N}$ such that an isomorphism $(\mathcal{L}(\rho), \mathcal{A}_{\rho,\rho}) \cong (\mathcal{N} \otimes_R N, 1 \otimes \rho)$ induces an isomorphism $(\mathcal{J}(\rho), \mathcal{A}_{\rho,\rho}) \cong (\mathcal{N}' \otimes_R N, 1 \otimes \rho)$. Hence there is an $R[G]$-isomorphism

(4.4.1.c) $$\mathcal{L}(\rho)/\mathcal{J}(\rho) \cong (\mathcal{N}/\mathcal{N}') \otimes_R N.$$

Lemma 4.4.2. *There are no proper sub-$R[G]$-modules of $\mathcal{L}$ that project surjectively to $Quot(\mathcal{L})$. The same is then true of $\mathcal{L}/(\mathcal{L}(\rho) \oplus \mathcal{J}(\chi'))$ and $\mathcal{L}/(\mathcal{L}(\chi') \oplus \mathcal{J}(\rho))$.*

Proof. Let $\mathcal{L}' \subseteq \mathcal{L}$ be a sub-$R[G]$-module projecting onto $Quot(\mathcal{L})$. The image $\overline{\mathcal{L}}'$ of $\mathcal{L}'$ in $\overline{\mathcal{L}} := \mathcal{L} \otimes_R \kappa$ contains an eigenvector for $\sigma(x_0)$ with eigenvalue $\bar{\alpha}_m$. But in $\overline{\mathcal{L}}$ any such eigenvector belongs to $\kappa\bar{\epsilon}$ and $\bar{\epsilon}$ generates $\overline{\mathcal{L}}$ over $R[G]$. Thus $\overline{\mathcal{L}}' = \overline{\mathcal{L}}$ and hence $\mathcal{L}' = \mathcal{L}$. ∎

4.4.3. *Relations with cohomology groups.* Suppose now that

(1) A_0 is a pro-finite $\mathbf{Z}_p$-algebra and a noetherian normal domain;
(2) $P \subset A_0$ is a height one prime and $A = \hat{A}_{0,P}$ is the completion of the localization of A_0 at P;
(3) R_0 is a local reduced finite A_0-algebra;
(4) $Q \subset R_0$ is prime such that $Q \cap A_0 = P$ and $R = \hat{R}_{0,Q}$;
(5) there exist ideals $J_0 \subset A_0$ and $I_0 \subset R_0$ such that $I_0 \cap A_0 = J_0$, $A_0/J_0 = R_0/I_0$, $J = J_0A$, $I = I_0R$, $J_0 = J \cap A_0$, and $I_0 = I \cap R_0$;
(6) G and H are pro-finite groups
(7) ν and χ are continuous $A_0^\times$-valued characters;
(8) ρ is the scalar extension from A_0 to A of some continuous representation
$$\rho_0 : G_{\mathcal{K},\Sigma} \to \mathrm{Aut}_{A_0}(V_0), \quad V_0 \cong A_0^n;$$
(9) σ is defined over the image of R_0 in R.

Let $\mathfrak{L} \subset \mathcal{L}/(\mathcal{L}(\rho) \oplus \mathcal{J}(\chi'))$ be the sub-$R_0[G]$-module generated by the image $\bar{\epsilon}^*$ of ϵ^*. Let $\mathfrak{L}_1 := \mathfrak{L} \cap \mathcal{L}(\chi')/\mathcal{J}(\chi')$. These are finite R_0-modules with continuous $R_0[G]$-actions. The G-action on $\mathfrak{L}_1$ is via χ' and we have
$$\mathfrak{L}/\mathfrak{L}_1 \cong (R_0/I_0)\bar{\epsilon}^* = (A_0/J_0)\bar{\epsilon}^*$$
with G-action via χ.

For each $\phi \in \mathrm{Hom}_{A_0}(\mathfrak{L}_1, A_0^*)$ we define an $A_0^*(\chi'/\chi)$-valued 1-cocycle c_ϕ of G by
$$c_\phi(g) = \phi(\chi^{-1}(g)\sigma(g)\bar{\epsilon}^* - \bar{\epsilon}^*),$$
which equals $\phi(\chi^{-1}(g)\pi_{\chi'}\sigma(g)\bar{\epsilon}^*)$. Mapping ϕ to the class of c_ϕ defines an A_0-homomorphism

$$\mathrm{Hom}_{A_0}(\mathfrak{L}_1, A_0^*) \to H^1(G, A_0^*(\chi'/\chi)), \quad \phi \mapsto [c_\phi]. \tag{4.4.3.a}$$

Dualizing we get a map

$$H^1(G, A_0^*(\chi'/\chi))^* \to \mathrm{Hom}_{A_0}(\mathfrak{L}_1, A_0^*)^*. \tag{4.4.3.b}$$

The character χ^{-c-1} extends uniquely to a character ψ of H such that $\psi(c) = 1$. As p is odd the restriction map
$$H^1(H, A_0^*(\psi\nu)) \to H^1(G, A_0^*(\chi'/\chi))^H$$
is an isomorphism.

Proposition 4.4.4.

(i) *The map* (4.4.3.b) *is a surjection after localizing at* P.
(ii) *If* ν *is compatible with* Φ, *then the image of* (4.4.3.a) *is contained in* $H^1(H, A_0^*(\psi\nu))$.

Proof. Let K be the kernel of (4.4.3.a) and $\mathfrak{L}_1' := \cap_{\phi\in K}\ker\phi$. By Pontryagin duality $K_P^* \neq 0$ if and only if $(\mathfrak{L}_1/\mathfrak{L}_1')_P \neq 0$. The $\mathfrak{L}_1$-valued cocycle

$$c(g) = \chi^{-1}(g)\sigma(g)\bar{\epsilon}^* - \bar{\epsilon}^*$$

defines a class $[c]$ in $H^1(G, \mathfrak{L}_1/\mathfrak{L}_1'(\chi^{-1}))$ (continuous cohomology). We will show that there exists $a \in A_0$, $a \notin P$, such that $a[c]$ is zero. From this it follows that

$$0 \to (\mathfrak{L}_1/\mathfrak{L}_1')_P \to \mathcal{L}/(\mathcal{L}(\rho) \oplus \mathcal{J}(\chi')) \to Quot(\mathcal{L}) \to 0$$

is split. If $(\mathfrak{L}_1/\mathfrak{L}_1')_P$ were non-zero this would contradict Lemma 4.4.2.

For $S \subseteq K$ a finite set, let $\mathfrak{L}_S = \cap_{\phi\in S}\ker\phi$; this has finite index in $\mathfrak{L}_1$. Then $H^1(G, \mathfrak{L}_1/\mathfrak{L}_1'(\chi^{-1})) = \operatorname{proj\,lim}_S H^1(G, \mathfrak{L}_1/\mathfrak{L}_S(\chi^{-1}))$, so $a[c]$, $a \in A_0$, is zero if its image $a[c]_S$ in each $H^1(G, \mathfrak{L}_1/\mathfrak{L}_S(\chi^{-1}))$ is zero.

The map $\mathfrak{L}_1/\mathfrak{L}_S \hookrightarrow \oplus_{\phi\in S}A_0^*$ defined by $l \mapsto \oplus\phi(l)$ induces a homomorphism

$$H^1(G, (\mathfrak{L}_1/\mathfrak{L}_S)(\chi^{-1})) \to \oplus_{\phi\in S}H^1(G, A_0^*(\chi'/\chi)).$$

The image of $[c]_S$ in the right-hand side is $\oplus_{\phi\in S}[c_\phi]$ and so is zero, hence $[c]_S$ belongs to the kernel. But the kernel of this map is a quotient of some $H^0(G, C)$ with C a quotient of $\oplus_{\phi\in S}A_0^*(\chi'/\chi)$. In particular the kernel is a union of submodules isomorphic to subquotients of $A_0^*(\chi'/\chi)$ that are fixed by G. As χ'/χ is non-trivial modulo P (see 4.2.3(2)), it is easily seen that there exists $a \in A_0$, $a \notin P$, such that a annihilates all such subquotients (so a can be chosen independent of S). Thus $a[c]_S = 0$. This completes the proof of part (i).

To prove part (ii) we note that

$$\begin{aligned}\chi^{-1}(g)\Phi(\pi_{\chi'}\sigma(g)\epsilon^*, \epsilon^*) &= \chi^{-1}(g)\Phi(\sigma(g)\epsilon^*, \epsilon^*)\\ &= \psi\nu(g)\chi^c(g)\Phi(\epsilon^*, \sigma(cg^{-1}c)\epsilon^*)\\ &= -\nu(c)\psi\nu(g)\chi^c(g)\Phi(\sigma(cg^{-1}c)\epsilon^*, \epsilon^*)\\ &= -\nu(c)\psi\nu(g)\chi^c(g)\Phi(\pi_{\chi'}\sigma(cg^{-1}c)\epsilon^*, \epsilon^*),\end{aligned}$$

the third equality following from the compatibility of ν with Φ. It then follows from Lemma 4.3.14(iii) that

$$\chi^{-1}(g)\pi_{\chi'}\sigma(g)\bar{\epsilon}^* = -\nu(c)\psi\nu(g)\chi^c(g)\pi_{\chi'}\sigma(cg^{-1}c)\bar{\epsilon}^*.$$

Since $\epsilon^* \equiv \epsilon \bmod I\mathcal{L}$, the value of ϕ on the left-hand side is $c_\phi(g)$ while the value on the right-hand side is $-\nu(c)\psi\nu(g)c_\phi(cg^{-1}c)$, which equals $\psi\nu(c)c_\phi(cgc)$. Therefore $[c_\phi] \in H^1(G, A_0^*(\chi'\chi))^H$. ∎

Now let $\mathfrak{L} \subseteq \mathcal{L}/(\mathcal{L}(\chi') \oplus \mathcal{J}(\rho))$ be the sub-$R_0[G]$-module generated by the image $\bar{\epsilon}^*$ of ϵ^*, and let $\mathfrak{L}_1 := \mathfrak{L} \cap \mathcal{L}(\rho)/\mathcal{J}(\rho)$. As before $\mathfrak{L}/\mathfrak{L}_1 \cong (R_0/I_0)\bar{\epsilon}^* = (A_0/J_0)\bar{\epsilon}^*$ with G-action by χ.

Let $\mathfrak{N} \subseteq \mathcal{N}/\mathcal{N}'$ be a finitely generated sub-R_0-module such that via some fixed isomorphism $\mathcal{L}(\rho)/\mathcal{J}(\rho) \cong (\mathcal{N}/\mathcal{N}') \otimes_{A_0} V_0$, $\mathfrak{L}_1 \subseteq \mathfrak{N} \otimes_{A_0} V_0$. (For example, fixing an A_0-basis of V_0, let $\mathfrak{N}$ be the sub-R_0-module generated by the coordinate entries of each element of $\mathfrak{L}_1 \subseteq (\mathcal{N}/\mathcal{N}')^n$.) For each $\phi \in \mathrm{Hom}_{A_0}(\mathfrak{N}, A_0^*)$ we define a $V_0 \otimes_{A_0} A_0^*$-valued 1-cocycle c_ϕ of G by

$$c_\phi(g) = (\phi \otimes id)(\chi^{-1}(g)\sigma(g)\bar{\epsilon}^* - \bar{\epsilon}^*).$$

This defines an A_0-homomorphism

$$\mathrm{Hom}_{A_0}(\mathfrak{N}, A_0^*) \to H^1(G, V_0 \otimes_{A_0} A_0^*(\chi^{-1})), \quad \phi \mapsto [c_\phi], \tag{4.4.4.a}$$

and by duality an A_0-homomorphism

$$H^1(G, V_0 \otimes_{A_0} A_0^*(\chi^{-1}))^* \to \mathrm{Hom}_{A_0}(\mathfrak{N}, A_0^*)^*. \tag{4.4.4.b}$$

Proposition 4.4.5. *The map* (4.4.4.b) *is a surjection after localization at* P.

Proof. The proof is essentially the same as for Proposition 4.4.4(i).■

4.5. **Local conditions.** We fix a subgroup $D \subset G$ for which we make the following hypotheses. In applications, G will be $G_{\mathcal{K},\Sigma}$ for some finite set of primes Σ containing p and D will be a decomposition group at a prime above p.

4.5.1. *Local condition for* σ. There is a D-stable sub-$R \otimes_A F$-module $M^+ \subseteq M$ such that M^+ and $M^- := M/M^+$ are free $R \otimes_A F$-modules.

4.5.2. *Local condition for* ρ. There is a D-stable A_0-submodule $V_0^+ \subseteq V_0$ such that V_0^+ and $V_0^- := V_0/V_0^+$ are free A_0-modules. Let $V^\pm := V_0^\pm \otimes_{A_0} A$.

4.5.3. *Compatibility with the congruence condition.* We further assume that for all $x \in R[D]$, we have the congruence relation

$$Ch(M^+, x, T) \equiv Ch(V^+, x, T)(1 - T\chi(x)) \bmod I.$$

It then follows from (4.3.1.a) that for all $x \in R[D]$

$$Ch(M^-, x, T) \equiv Ch(V^-, x, T)(1 - T\chi'(x)) \bmod I.$$

Proposition 4.5.4. *Assume that hypotheses (4.5.1), (4.5.2), and (4.5.3) hold. Assume also that the D-representations $V^+ \oplus A(\chi)$ and $V^- \oplus A(\chi')$ are residually disjoint modulo P.*

(i) *There exists $t \in A_0$, $t \notin P$, such that $t \cdot$ (image of* (4.4.3.a)*) lies in the kernel of the restriction map*

$$H^1(G, A_0^*(\chi'/\chi)) \overset{res}{\to} H^1(D, A_0^*(\chi'/\chi)).$$

(ii) *There exists* $t \in A_0$, $t \notin P$, *such that* $t \cdot$ (*image of* (4.4.4.a)) *lies in the kernel of the map*

$$H^1(G, V_0 \otimes_{A_0} A_0^*(\chi^{-1})) \to H^1(D, V_0^- \otimes_{A_0} A_0^*(\chi^{-1})). \tag{4.5.4.a}$$

induced by the restriction map.

Proof. By the assumptions of the proposition there exists $x \in A[D]$ such that the characteristic polynomials $Ch(M^+, x, T)$ and $Ch(M^-, x, T)$ are relatively prime modulo the maximal ideal of R. As R is local and complete, arguing as in the poof of Lemma 4.1.3 we can construct an element $x^+ \in R[D]$ such that x^+ acts as the identity on M^+ and annihilates M^-. We note that by 4.5.3 and the construction of x^+, x^+ acts as the identity on $V^+ \otimes_A R/I$ and $(R/I)(\chi)$ and annihilates $V^- \otimes_A R/I$ and $(R/I)(\chi')$.

We now prove (ii); part (i) can be proved similarly.

Let $\mathcal{L}_1 := \mathcal{L}(\rho)/\mathcal{J}(\rho)$. Let $\mathcal{L}^+ := x^+\mathcal{L}$ and $\mathcal{L}_1^+ := x^+\mathcal{L}_1 = \mathcal{L}_1 \cap \mathcal{L}^+$. With respect to a fixed isomorphism $\mathcal{L}_1 \cong \mathcal{N}/\mathcal{N}' \otimes_A V$, $\mathcal{L}_1^+ \cong \mathcal{N}/\mathcal{N}' \otimes_A V^+$. Let $\mathfrak{L}_1$ and $\mathfrak{N}$ be as in the definition of (4.4.4.a) (in particular, $\mathfrak{L}_1 := \mathfrak{L} \cap \mathcal{L}_1$). Let $\mathfrak{L}^+ := \mathfrak{L} \cap \mathcal{L}^+$ and $\mathfrak{L}_1^+ := \mathfrak{L}_1 \cap \mathcal{L}_1^+$. Since $\mathfrak{N} \otimes_{A_0} V = \mathcal{N}/\mathcal{N}' \otimes_A V$ it follows that $\mathfrak{L}_1^+ = \mathfrak{L}_1 \cap \mathfrak{N} \otimes_{A_0} V_0^+$. Therefore $\mathfrak{L}_1/\mathfrak{L}_1^+ \hookrightarrow \mathfrak{N} \otimes_{A_0} V_0^-$.

We claim that the $R[D]$-extension

$$0 \to \mathcal{L}_1/\mathcal{L}_1^+ \to \mathcal{L}/\mathcal{L}_1^+ \to \mathcal{L}/\mathcal{L}_1 \cong (R/I)(\chi) \to 0$$

is split. This follows by applying the projector x^+ to this sequence. Consequently, the class in $H^1(G, \mathfrak{L}_1/\mathfrak{L}_1^+(\chi^{-1}))$ of the 1-cocycle of G defined by $c_1(g) = \chi^{-1}\sigma(g)\bar{\epsilon}^* - \bar{\epsilon}^*$ has zero image in $H^1(G, \mathcal{L}_1/\mathcal{L}_1^+(\chi^{-1}))$ (the usual group cohomology). It follows that there exists $t \in A_0$, $t \notin P$, such that tc_1 has zero image in $H^1(G, \mathfrak{L}_1/\mathfrak{L}_1^+(\chi^{-1}))$. Given $\phi \in \mathrm{Hom}_{A_0}(\mathfrak{N}, A_0^*)$, the image of $t[c_1]$ in $H^1(G, V_0^- \otimes_{A_0} A_0^*(\chi^{-1}))$ under the map induced by the composition homomorphism

$$\mathfrak{L}_1/\mathfrak{L}_1^+ \hookrightarrow V_0^- \otimes_{A_0} \mathfrak{N} \stackrel{id\otimes\phi}{\longrightarrow} V_0^- \otimes_{A_0} A_0^*$$

equals the image of $t[c_\phi]$ under (4.5.4.a), so the latter is zero. ∎

4.5.5. *Consequences for Selmer groups.* Keeping the notation introduced so far, suppose now that $H = G_{\mathbf{Q},\Sigma}$ for some finite set of primes Σ containing p and $G = G_{\mathcal{K},\Sigma}$ and that c is the usual complex conjugation. Assume that

$$\nu \text{ is compatible with } \Phi. \tag{4.5.5.a}$$

Suppose (4.5.1), (4.5.2), and (4.5.3) and the hypotheses of Proposition 4.5.4 hold for $D = G_{\mathcal{K},\mathfrak{p}}$ with $V_0^+ \subseteq V_0$ an A_0-free direct summand stable under the action of the decomposition group $G_{\mathbf{Q},p}$. By the polarization condition (4.5.5.a), the corresponding hypotheses are then also satisfied for $D = G_{\mathcal{K},\bar{\mathfrak{p}}}$. For $F = \mathbf{Q}$ or $\mathcal{K}$ let $Sel_F^{\Sigma}(\psi\nu)$ and $Sel_F^{\Sigma}(\rho_0 \otimes \chi^{-1})$ be the Selmer groups associated with the pairs $(A_0(\psi\nu), 0)$ and $(V_0(\chi^{-1}), V_0^+(\chi^{-1}))$ as in §3. Suppose further that A_0 is a noetherian normal domain and let $Ch_F^\Sigma(\psi\nu)$ and $Ch_F^\Sigma(\rho_0\otimes\chi^{-1})$ be the respective characteristic ideals of the Pontryagin duals of these Selmer groups.

Proposition 4.5.6. *With the above assumptions,*

(i) $\mathrm{ord}_P(Ch_{\mathbf{Q}}^\Sigma(\psi\nu))\geq \ell_P(\mathcal{L}(\chi')/\mathcal{J}(\chi'))$*;*
(ii) $\mathrm{ord}_P(Ch_{\mathcal{K}}^\Sigma(\rho_0\otimes\chi^{-1}))\geq \ell_P(\mathcal{N}/\mathcal{N}')$*; in particular, if* $\mathcal{N}'=0$ *then*

$$\mathrm{ord}_P(Ch_{\mathcal{K}}^\Sigma(\rho_0\otimes\chi^{-1}))\geq \mathrm{ord}_P(J).$$

Proof. Proposition 4.5.4 shows that the maps (4.4.3.b) and (4.4.4.b) factor through the duals of the Selmer groups of interest. The proposition then follows easily from Propositions 4.4.4 and 4.4.5.■

Corollary 4.5.7. *With the assumptions of Proposition 4.5.6, if* $\mathrm{ord}_P(Ch_{\mathbf{Q}}^\Sigma(\psi\nu))=0$ *then* $\mathrm{ord}_P(Ch_{\mathcal{K}}^\Sigma(\rho_0\otimes\chi^{-1}))\geq \mathrm{ord}_P(J)$*.*

Proof. From part (i) of the preceding proposition it follows that $\mathcal{J}(\chi')=\mathcal{L}(\chi')$. It then follows that from Lemma 4.3.8(i) that $\mathcal{J}(\rho)=0$, and so $\mathcal{N}'=0$. The corollary then follows from part (ii) of the proposition. ■

Proposition 4.5.8. *Suppose the hypotheses of Lemma 4.3.11 hold. With the assumptions of Proposition 4.5.6, if* $\mathrm{ord}_P(J)>\mathrm{ord}_P(Ch_{\mathbf{Q}}^\Sigma(\psi\nu))$*, then* $\mathrm{ord}_P(Ch_{\mathcal{K}}^\Sigma(\rho_0\otimes\chi^{-1}))\geq 1$*.*

Proof. By Lemma 4.3.11, $\mathcal{L}(\chi')$ is a faithful $R/I=A/J$-module. In combination with part (i) of Proposition 4.5.6 and the hypothesis on $\ell_P(A/J)$, it follows that $\mathcal{J}(\chi')\neq 0$ and hence that $\mathcal{L}(\rho)/\mathcal{J}(\rho)\neq 0$. As $\mathcal{L}(\rho)/\mathcal{J}(\rho)\cong(\mathcal{N}/\mathcal{N}')^n$ it also follows that $\mathcal{N}/\mathcal{N}'\neq 0$. The conclusion of the proposition now follows from part (ii) of Proposition 4.5.6. ■

5. Shimura varieties for some unitary groups

5.1. **The groups G_n.** For any group scheme $\mathcal{G}_{/\mathbf{Z}}$ there is a canonical action of the nontrivial automorphism of $\mathcal{K}$ on $\mathcal{G}(\mathcal{O}\otimes A)$, A a $\mathbf{Z}$-algebra, arising from its action on the first factor of $\mathcal{O}\otimes A$. For $g\in\mathcal{G}(\mathcal{O}\otimes A)$ we write $g\mapsto\bar g$ for this action. If ℓ splits in $\mathcal{K}$ and A is any $\mathbf{Z}_\ell$-algebra, then $\mathcal{G}(\mathcal{O}_\ell\otimes_{\mathbf{Z}_\ell}A)$ is identified with $\mathcal{G}(A)\times\mathcal{G}(A)$ via our identification $\mathcal{O}_\ell=\mathbf{Z}_\ell\times\mathbf{Z}_\ell$, and if $g=(a,b)\in\mathcal{G}(\mathcal{O}_\ell\otimes_{\mathbf{Z}_\ell}A)$ then $\bar g=(b,a)$. If v is a place of $\mathbf{Q}$ and $g\in\mathcal{G}(\mathbf{A})$ then we write g_v for the v-component of g. (When $\mathcal{G}=\mathbf{G}_a$ or $\mathbf{G}_m$ this is all consistent with our previously introduced notation.)

5.1.1. *The unitary similitude groups G_n.* For an integer $n\geq 1$ we let

$$w_n:=\begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix},$$

and let G_n be the group scheme over $\mathbf{Z}$ such that for any $\mathbf{Z}$-algebra A

$$G_n(A)=\{g\in\mathrm{GL}_{2n}(\mathcal{O}\otimes A)\ :\ gw_n{}^t\bar g=\lambda_g w_n,\lambda_g\in A^\times\}.$$

We define $\mu_n : G_n \to \mathbf{G}_m$ by $\mu_n(g) = \lambda_g$ and let $U_n \subset G_n$ be the kernel of μ_n.

The matrix w_n defines a skew-Hermitian pairing ψ_n on the $2n$-dimensional $\mathcal{K}$-space $V_n := \mathcal{K}^{2n}$. Our convention is to have elements of $\mathrm{End}_{\mathcal{K}}(V_n)$ act on the right. The unitary similitude group of this pairing (an algebraic group over $\mathbf{Q}$) is then $G_{n/\mathbf{Q}}$ and μ_n is its similitude character; the unitary group of the pairing is $U_{n/\mathbf{Q}}$.

For each prime ℓ we let $K^0_{n,\ell} := G_n(\mathbf{Z}_\ell)$. If $\ell \nmid D_{\mathcal{K}}$, then this is a hyperspecial compact subgroup of $G_n(\mathbf{Q}_\ell)$.

If ℓ splits in $\mathcal{K}$ then for any $\mathbf{Z}_\ell$-algebra A, $G_n(A) \subseteq \mathrm{GL}_{2n}(\mathcal{O} \otimes_{\mathbf{Z}_\ell} A) = \mathrm{GL}_{2n}(A) \times \mathrm{GL}_{2n}(A)$. Using this identification to write $g \in G_n(A)$ as $g = (g_1, g_2)$ with $g_1 \in \mathrm{GL}_{2n}(A)$, the map $g \mapsto (g_1, \mu_n(g))$ defines an isomorphism $G_n(A) \xrightarrow{\sim} \mathrm{GL}_{2n}(A) \times A^\times$ and hence an identification of $G_{n/\mathbf{Z}_\ell}$ with $\mathrm{GL}_{2n} \times \mathbf{G}_m$, which identifies $U_{n/\mathbf{Z}_\ell}$ with the GL_{2n}-factor. We will frequently use this identification. Note that $K^0_{n,\ell}$ is identified with $\mathrm{GL}_{2n}(\mathbf{Z}_\ell) \times \mathbf{Z}_\ell^\times$.

Similarly, for any $\mathcal{O}[1/D_{\mathcal{K}}]$-algebra A the embedding $\mathcal{O} \otimes_{\mathbf{Z}} \mathcal{O} \hookrightarrow \mathcal{O} \times \mathcal{O}$, $x \otimes y \mapsto (xy, \bar{x}y)$ identifies$\mathrm{GL}_{2n}(\mathcal{O} \otimes_{\mathbf{Z}} A)$ with $\mathrm{GL}_{2n}(A) \times \mathrm{GL}_{2n}(A)$. Using this identification to write $g \in G_n(A)$ as $g = (g_1, g_2)$, the map $G_n(A) \to \mathrm{GL}_{2n}(A) \times A^\times$, $g \mapsto (g_1, \mu_n(g))$, is an isomorphism and so defines an identification of $G_{n/\mathcal{O}}$ with $\mathrm{GL}_{2n} \times \mathbf{G}_m$. If ℓ splits in $\mathcal{K}$ then projection onto the first factor of $\mathcal{O}_\ell = \mathbf{Z}_\ell \times \mathbf{Z}_\ell$ defines an $\mathcal{O}[1/D_{\mathcal{K}}]$-algebra structure on $\mathbf{Z}_\ell$. The associated base change to $\mathbf{Z}_\ell$ of the identification $G_{n/\mathcal{O}[1/D_{\mathcal{K}}]} = \mathrm{GL}_{2n} \times \mathbf{G}_m$ is the identification of the preceding paragraph.

For $g \in G_n(R)$ let $A_g, B_g, C_g, D_g \in \mathrm{M}_n(\mathcal{O} \otimes R)$ be defined by

$$g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}.$$

If g is understood, then we may drop the subscript.

We let $T_{n/\mathcal{O}}$ be the diagonal torus of $\mathrm{GL}_{2n} = U_{n/\mathcal{O}}$ and let $B_{n/\mathcal{O}}$ be the Borel of $\mathrm{GL}_{2n} = U_{n/\mathcal{O}}$ defined by requiring $C_g = 0$ and A_g and ${}^tD_g$ to be upper-triangular. One could also take the upper-triangular Borel in place of B_n; the choice here reflects the conventions adopted in calculations in the rest of this paper[2].

When n is understood we drop it from our notation. In later sections we will be primarily interested in the case $n = 2$, and then we will drop it only in that case.

5.1.2. *Connection with* $\mathrm{GL}_{2/\mathbf{Z}}$. The canonical inclusion of GL_2 into G_1 extends to an exact sequence of group schemes over $\mathbf{Z}$:

$$1 \to \mathbf{G}_m \to \mathit{Res}_{\mathcal{O}/\mathbf{Z}}\mathbf{G}_m \times \mathrm{GL}_2 \to G_1 \to 1,$$

where the second arrow is $a \mapsto (a, a^{-1})$ and the third is $(a, g) \mapsto ag$.

[2]This makes no real difference: the modules of (ordinary) p-adic modular forms defined using either choice of Borel are isomorphic; one passes from one setting to the other by conjugation.

5.1.3. *Hermitian half-spaces.* Let

$$\mathbf{H}_n := \{Z \in M_n(\mathbf{C}) \ : \ -i(Z - {}^t\bar{Z}) > 0\}.$$

This is the Hermitian upper half-space of degree n. Note that $\mathbf{H}_1 = \mathfrak{h}$, the latter being the usual Poincaré upper half-plane. The group $G_n(\mathbf{R})^+ := \{g \in G_n(\mathbf{R}) \ : \ \mu_n(g) > 0\}$ acts on $\mathbf{H}_n$ as

$$g(Z) := (A_g Z + B_g)(C_g Z + D_g)^{-1}, \quad g \in G_n^+(\mathbf{R}).$$

Putting $\mathbf{i} := i1_n \in \mathbf{H}_n$ we let $K_{n,\infty}^+ := \{g \in U_n(\mathbf{R}) \ : \ g(\mathbf{i}) = \mathbf{i}\}$ and let $K_{n,\infty}$ be the group generated by $K_{n,\infty}^+$ and $\mathrm{diag}(1_n, -1_n)$. Then $K_{n,\infty}^+$ is a maximal compact of $U_n(\mathbf{R})$ and so also of $G_n^+(\mathbf{R})$, and $K_{n,\infty}$ is a maximal compact of $G_n(\mathbf{R})$. The center of $G_n(\mathbf{R})$ is $Z_{n,\infty} := \mathbf{C}^\times 1_{2n}$; it is contained in $G_n^+(\mathbf{R})$.

Let $X_n^\pm := G_n(\mathbf{R})/K_{n,\infty}^+ Z_{n,\infty}$ and $X_n^+ := G_n^+(\mathbf{R})/K_{n,\infty}^+ Z_{n,\infty}$. There is a real-analytic isomorphism

$$X_n^+ \xrightarrow{\sim} \mathbf{H}_n, \quad g \mapsto g(\mathbf{i}).$$

5.2. **The Shimura varieties over C.**

5.2.1. *Real Hodge structures.* Let W be a vector space over $\mathbf{Q}$. Recall that a real Hodge structure on $W_\mathbf{R} := W \otimes_\mathbf{Q} \mathbf{R}$ is an algebraic $\mathbf{R}$-linear action of $\mathbf{S} := Res_{\mathbf{C}/\mathbf{R}}\mathbf{G}_m$ on $W_\mathbf{R}$ (i.e., a homomorphism of $\mathbf{R}$-algebraic groups $\mathbf{S} \to \mathrm{GL}(W_\mathbf{R})$). Given a real Hodge structure on $W_\mathbf{R}$, there is a decomposition

$$W_\mathbf{C} := W_\mathbf{R} \otimes_\mathbf{R} \mathbf{C} = \bigoplus_{p,q} W_\mathbf{C}^{p,q}$$

with $W_\mathbf{C}^{p,q}$ the subspace on which $z \in \mathbf{S}(\mathbf{R}) = \mathbf{C}^\times$ acts by multiplication by $z^p \bar{z}^q$. The Hodge structure is pure of weight w if $p + q = w$ whenever $W_\mathbf{C}^{p,q} \neq 0$.

5.2.2. *Variation of Hodge structures of G_n-type.* Let $h : \mathbf{S} \to G_{/\mathbf{R}}$ be the $\mathbf{R}$-homomorphism defined by

$$h(x + iy) = \begin{pmatrix} x1_n & y1_n \\ -y1_n & x1_n \end{pmatrix}.$$

Then $X^\pm$ is identified with the set of $G(\mathbf{R})$-conjugacy classes of h. Viewing $V = V_n$ as a $4n$-dimensional $\mathbf{Q}$ space, h defines a real Hodge structure on $V_\mathbf{R}$. The restriction of h to $\mathbf{C}^\times$ defines a complex structure on $V_\mathbf{R}$ that differs from the canonical complex structure (the canonical complex structure is defined by the action of $\mathcal{K} \otimes \mathbf{R} = \mathbf{C}$, the identification with $\mathbf{C}$ being via the inclusion $\mathcal{K} \subset \overline{\mathbf{Q}} \subseteq \mathbf{C}$). There is a decomposition $V_\mathbf{R} = V_\mathbf{R}^+ \oplus V_\mathbf{R}^-$ where $V_\mathbf{R}^+$ (resp. $V_\mathbf{R}^-$) is the subspace on which the two complex structures coincide (resp. are conjugate). In particular, the Hodge decomposition of $V_\mathbf{C}$ defined by h is of the form $(1,0), (0,1)$ with:

$$V_\mathbf{C}^{1,0} = V_\mathbf{R}^+ \oplus \overline{V}_\mathbf{R}^- \text{ and } V_\mathbf{C}^{0,1} = \overline{V}_\mathbf{R}^+ \oplus V_\mathbf{R}^-,$$

where we have set $\overline{V}_\mathbf{R}^\pm = V_\mathbf{R}^\pm \otimes_{\mathbf{C},c} \mathbf{C}$ with $c : \mathbf{C} \to \mathbf{C}$ being complex conjugation.

Let $K = \prod_{v\nmid\infty} K_v$ be an open compact subgroup of $G(\mathbf{A}_f)$ (so $K_v = K_v^0$ for almost all v). We consider the Shimura variety over $\mathbf{C}$ defined as

$$Sh_G(K)(\mathbf{C}) := G(\mathbf{Q})\backslash X^{\pm} \times G(\mathbf{A}_f)/K = G(\mathbf{Q})\backslash G(\mathbf{A})/KK_\infty^+ Z_\infty.$$

For any point $[g] \in Sh_G(K)(\mathbf{C})$ where $[g]$ designates the class of $g = g_\infty g_f \in G(\mathbf{A})$, $z \mapsto h_{[g]}(z) := g_\infty h(z) g_\infty^{-1}$ defines a Hodge structure on $V_{\mathbf{R}}$. The space $Sh_G(K)(\mathbf{C})$ can therefore be seen as a variation of Hodge structures on $V_{\mathbf{R}}$. Let $A := V_{\mathbf{R}}/L$ where L is the lattice $L := \mathcal{O}^{2n} \subset V$. The complex structure defined by $h_{[g]}$ and the real skew-symmetric form $\mathrm{trace}_{\mathbf{C}/\mathbf{R}}(xw_n{}^t\bar{y})$ give A the structure of an abelian variety over $\mathbf{C}$ of dimension $2n$ with complex multiplication by $\mathcal{O}$. In this way $Sh_G(K)(\mathbf{C})$ can be viewed as a complex analytic family of abelian varieties.

5.3. **Moduli of abelian schemes with CM.** Let $D := D_{\mathcal{K}}$ be the absolute discriminant of $\mathcal{K}$.

5.3.1. *Modules with complex multiplication.* Let R be an $\mathcal{O}[1/2D]$-algebra and M an R-module endowed with an R-linear $\mathcal{O}$-action (i.e., a homomorphism $\iota : \mathcal{O} \to \mathrm{End}_R(M)$). Then M decomposes uniquely as

$$M = M^+ \oplus M^-$$

with $\iota(z)$ acting by multiplication by z on M^+ and by multiplication by $\bar{z}$ on M^-. If M is free over R, then M^+ and M^- are also free over R, and we will say that M is of type (r, s) if M^+ (resp. M^-) is of rank r (resp. of rank s) over R. This definition can be extended to a locally free coherent module $\mathcal{M}$ over a $\mathcal{O}[1/2D]$-scheme S by requiring that over an open affine sub-scheme $\mathrm{Spec}\, R$ of S, $\mathcal{M} \times_S \mathrm{Spec}\, R$ is associated with a free R-module of type (r, s).

5.3.2. *Abelian schemes with complex multiplication.* Let R be a $\mathcal{O}[1/2D]$-algebra and let S be an R-scheme. Let $f : A \to S$ be a semi-abelian scheme over S. We say that A has complex multiplication (CM) by $\mathcal{O}$ of type (r, s) if there is a homomorphism $\iota : \mathcal{O} \to \mathrm{End}_S(A)$ such that with the induced $\mathcal{O}$-action on the coherent sheaf $\mathrm{Lie}_S A$ is of type (r, s). So $r + s$ must equal the relative dimension of A over S. We denote by $\omega_{A/S}$ the $\mathcal{O}_S$-dual of $\mathrm{Lie}_S A$. If A has CM by $\mathcal{O}$, then there is a decomposition

$$\omega_{A/S} = \omega_{A/S}^+ \oplus \omega_{A/S}^-$$

with $\omega_{A/S}^+$ (resp. $\omega_{A/S}^-$) locally free of rank r (resp. of rank s) over $\mathcal{O}_S$.

If $S = \mathrm{Spec}\, \mathbf{C}$ and $A = V_{\mathbf{R}}/L$, then $\omega_{A/S} \cong V_{\mathbf{C}}^{1,0}$, $\omega_{A/S}^+ \cong V_{\mathbf{R}}^+$, $\omega_{A/S}^- \cong \overline{V}_{\mathbf{R}}^-$ and $H^1(A, \mathcal{O}_A) \cong V^{0,1}$, $H^1(A, \mathcal{O}_A)^+ \cong \overline{V}_{\mathbf{R}}^+$, $H^1(A, \mathcal{O}_A)^- \cong V_{\mathbf{R}}^-$.

5.3.3. *K-level structures.* Suppose now that S is locally noetherian and A is an abelian scheme with CM by $\mathcal{O}$ of type (n,n). Let $\lambda : A \to A^\vee$ be a polarization of A. Let s be a geometric point of S. For any prime ℓ, consider the Tate module(s)

$$T_\ell A_s := \varprojlim_r A_s[\ell^r] \quad \text{and} \quad V_\ell A_s := T_\ell A_s \otimes \mathbf{Q},$$

which are $\pi_1(S,s)$-modules. Then $T_\ell A_s$ is equipped with a natural $\mathbf{Z}_\ell(1)$-valued symplectic pairing $(-,-)_\ell$ induced by λ and the Weil-pairing. This extends to a $\mathbf{Q}_\ell(1)$-valued pairing on $V_\ell A_s$. These pairings are preserved by the action of the fundamental group $\pi_1(S,s)$. We equip $V_\ell A_s$ with the $\mathcal{K}_\ell$-Hermitian form defined by

$$< x, y >_\ell := (\iota(\delta_\mathcal{K})x, y)_\ell + \delta_\mathcal{K} \cdot (x,y)_\ell$$

for all $x, y \in V_\ell A_s$. The choice of a $\mathcal{K}_\ell$-basis of $V_\ell A_s$ defines an isomorphism $\eta_{\ell,s}$ of $V_{n,\ell} := \mathcal{K}_\ell^{2n}$ with $V_\ell A_s$ and endows $V_{n,\ell}$ with a Hermitian form. We consider such isomorphisms where this Hermitian form is a scalar multiple of that defined by the Hermitian matrix $\delta_\mathcal{K} w_n$ (that is, $\delta_\mathcal{K}\psi_n$) (then $\eta_{\ell,s}$ induces a homomorphism $j_{\eta_{\ell,s}} : \pi_1(S,s) \to G_n(\mathbf{Q}_\ell)$).

Let $K \subseteq G(\mathbf{A}_f)$ be as in §5.2.2 and let Σ_K be the finite set of primes ℓ such that K_ℓ is not hyperspecial. We assume that all $\ell \in \Sigma_K$ are invertible in R. Let $\eta_s = \prod_{\ell \neq p} \eta_{\ell,s}$ with each $\eta_{\ell,s}$ a $\mathcal{K}_\ell$-linear isomorphism $V_{n,\ell} \xrightarrow{\sim} V_\ell A_s$ as above. Then $G(\mathbf{A}_f^p)$ acts on the right on the set of such η_s, and $\pi_1(S,s)$ acts on the left. A K-level structure is a $K^p := \prod_{\ell \neq p} K_\ell$-orbit $\bar\eta_s$ that is stable under the action of $\pi_1(S,s)$. (These are just the K^p-orbits of those η_s such that the image of $j_{\eta_{\ell,s}}$ is contained in K_ℓ for all $\ell \neq p$.) If s and s' are geometric points on the same connected component of S then there is a canonical bijection between the set of K-level structures $\bar\eta_s$ and the set of K-level structures $\bar\eta_{s'}$ (cf. [Lan08, 1.3.7.11]). We say that $\bar\eta = (\eta_s)$, s running over the geometric points of S, defines a K-level structure on A if each $\bar\eta_s$ does and if for any two geometric points s and s' on the same connected component of S the K-level structures $\bar\eta_s$ and $\bar\eta_{s'}$ are canonically identified.

5.3.4. *S-quadruples.* For simplicity, let $R = \mathcal{O}_{(p)}$ be the localization of $\mathcal{O}$ at (p). Suppose $K = K_p^0 K^p$. For each locally noetherian R-scheme S, we consider quadruples $(A, \lambda, \iota, \eta)$ over S where

- A is an abelian scheme over S of relative dimension n;
- λ is a prime-to-p polarization of A;
- $\iota : \mathcal{O} \to \mathrm{End}_S(A)$ is a homomorphism giving A the structure of an abelian scheme with CM by $\mathcal{O}$ of type (n,n) (in this special case this is equivalent to the determinant condition '(det)' in [Hi99, Ko92]) and satisfying $\iota(a)^\vee \circ \lambda = \lambda \circ \iota(\bar a)$;
- $\bar\eta$ is a K-level structure (relative to λ).

We define an equivalence relation on such quadruples by setting $(A, \lambda, \iota, \eta) \sim (A', \lambda', \iota', \eta')$ if there exists a prime-to-p isogeny $A \xrightarrow{f} A'$ such that $\lambda = rf^\vee \circ \lambda' \circ f$ for some $r \in \mathbf{Z}_{(p),>0}^\times$, $f \circ \iota(a) = \iota'(a) \circ f$ for all $a \in \mathcal{O}$, and $f \circ \bar\eta = \bar\eta'$. Recall that a prime-to-p isogeny is an invertible element of $\mathrm{Hom}_S(A, A') \otimes \mathbf{Z}_{(p)}$ and a prime-to-p polarization of A is a prime-to-p isogeny $\phi \in \mathrm{Hom}_S(A, A^\vee)$ such that ϕ induces a polarization of A_s for each geometric point $s \in S$.

For S an $R[1/p]$-scheme we can also consider quadruples as above but without requiring K_p to be hyperspecial and allowing λ to be any polarization. The equivalence relation is similarly adjusted to allow f to be any isogeny and merely requiring $r \in \mathbf{Q}_{>0}$. For $K = K_p^0 K^p$ there is an obvious map from the first collection of equivalence classes to the second; this is a bijection.

5.3.5. *Kottwitz models.* We continue to let $R = \mathcal{O}_{(p)}$. Assume $K = K_p^0 K^p$ is neat. For each locally noetherian R-scheme S, let $\mathcal{F}_K(S)$ be the set of equivalence classes of quadruples $(A, \lambda, \iota, \eta)_{/S}$ for the relation $\sim$. The association $S \mapsto \mathcal{F}_K(S)$ defines a contravariant functor from the category of locally noetherian R-schemes to the category of sets. By a theorem of Kottwitz [Ko92] the functor $\mathcal{F}_K$ is representable by a smooth, quasi-projective scheme $S_G(K)_{/R}$ whose $\mathbf{C}$-points are isomorphic to $Sh_G(K)(\mathbf{C})$ as a $\mathbf{C}$-analytic variety: in the notation of *loc. cit.* one takes $B = \mathcal{K}$, $\mathcal{O}_B = \mathcal{O}$, $*$ to be the non-trivial automorphism of $\mathcal{K}$, $V = \mathcal{K}^{2n}$, $(-,-)$ the trace of the Hermitian pairing associated to $\delta_{\mathcal{K}} w_n$ (so $(x, y) = \mathrm{trace}_{\mathcal{K}/\mathbf{Q}}(x \delta_{\mathcal{K}} w_n {}^t\bar{y})$), and $\Lambda_0 = \mathcal{O}_p^{2n} \subset V \otimes \mathbf{Q}_p$ (this is a self-dual $\mathbf{Z}_p$-lattice as p splits in $\mathcal{K}$); then $C = \mathrm{End}_{\mathcal{K}}(V) = M_{2n}(\mathcal{K})$, and the G of *loc. cit.* is G_n; one takes for the $*$-homomorphism $\mathbf{C} \to C \otimes \mathbf{R}$ the $\mathbf{R}$-linear extension of the map $z \mapsto h(z)$, h being as in 5.2.2. Note that K_p^0 is the stabilizer in $G(\mathbf{Q}_p)$ of Λ_0.

The scheme $S_G(K)$, being the solution to a PEL moduli problem, is equipped with a universal abelian scheme $\mathcal{A}_K$, indeed a universal quadruple $(\mathcal{A}_K, \lambda^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \eta^{\mathrm{univ}})/S_G(K)$ (up to equivalence). Furthermore, the scheme $S_G(K)_{/R}$ is the canonical model of the Shimura variety associated to (G, h^{-1}).

We also consider the similar functor on $R[1/p]$-schemes for any neat K. This, too, is representable by an $R[1/p]$-scheme $S_G(K)_{/R[1/p]}$ (by the same argument of Kottwitz as before). Of course, if $K = K_p^0 K^p$ this scheme is just the base change to $R[1/p]$ of $S_G(K)_{/R}$.

One may, of course, replace (V_n, ψ_n) with another skew-Hermitian pairing (W, ψ_W) of signature (n, n) such that $GU(W)(\mathbf{Q}_p)$ has a hyperspecial maximal compact $K_{p,W}^0$ ($GU(W)$ being the unitary similitude group of (W, ψ_W)). Then, as above, associated with an open compact subgroup $K = K_{p,W}^0 K^p \subseteq GU(W)(\mathbf{A}_f)$ is a smooth, quasi-projective scheme $S_{GU(W)}(K)_{/R}$ (the lattice Λ_0 depends on $K_{p,W}$). Of course, an isomorphism $(W, \psi_W) \cong (V_n, \psi_n)$ induces an isomorphism $GU(W) \cong G_n$. Assume we are given an such an isomorphism that identifies $K_{p,W}^0$ with K_p^0. If $K' \subset GU(W)(\mathbf{A}_f)$ is identified with $K \subset G_n(\mathbf{A}_f)$, then there is a canonical isomorphism $S_{GU(W)}(K') \cong S_{G_n}(K)$.

5.3.6. *Level structures at p.* For an integer $s \geq 0$, we let $I_s = I_{0,s} \subset K_p^0 = G(\mathbf{Z}_p)$ be the Iwahori subgroup of depth s: this is the set of matrices $g \in G(\mathbf{Z}_p)$ such that $g \pmod{p^s}$ belongs to $B(\mathbf{Z}/p^s\mathbf{Z}) \times (\mathbf{Z}/p^s\mathbf{Z})^\times$. Let B^u be the unipotent radical of B. We let

$I_{1,s} \subset I_{0,s}$ be the subgroup of g such that $g \pmod{p^s}$ belongs to $B^u(\mathbf{Z}/p^s\mathbf{Z}) \times (\mathbf{Z}/p^s\mathbf{Z})^\times$. Then for any open compact subgroup $K = K_p^0 K^p$ we let $K_t(p^s) := I_{t,s}K^p$ $(t = 0, 1)$.

Let $\mathcal{O}_{(\mathfrak{p})}$ denote the localization of $\mathcal{O}$ at the prime $\mathfrak{p}$ (we write $\mathcal{O}_{(\mathfrak{p})}$ for the localization to distinguish it from the completion at $\mathfrak{p}$, which we denote $\mathcal{O}_\mathfrak{p}$). We define $S_G(K_0(p^s))_{/\mathcal{O}_{(\mathfrak{p})}}$ to be the scheme over $S_G(K)_{/\mathcal{O}_{(\mathfrak{p})}}$ classifying equivalence classes of 5-tuples $(A, \lambda, \iota, \eta, F_\bullet)$ where $(A, \lambda, \iota, \eta)$ is a S-quadruple as above and $F_\bullet : 0 = F_0 \subset \cdots \subset F_n$ is a filtration of subgroups of $A[p^s]$ such that each F_i is isotropic for the Weil pairing and $F_i/F_{i-1} \cong \mu_{p^s}$. Equivalence is defined by requiring the isogeny defining the equivalence of S-quadruples to identify the filtrations. We define $S_G(K_1(p^s))_{/\mathcal{O}_{(\mathfrak{p})}}$ to be the finite scheme over $S_G(K_0(p^s))_{/\mathcal{O}_{(\mathfrak{p})}}$ classifying equivalence classes of 6-tuples $(A, \lambda, \iota, \eta, F_\bullet, \alpha_p)$ where α_p is an isomorphism of the graded module attached to $F_\bullet$ with $\mu_{p^s}^s$; the notion of equivalence is extended in the obvious way. One can easily check that $S_G(K_t(p^s))$ is smooth over $\mathcal{O}_{(\mathfrak{p})}$ and that $S_G(K_t(p^s))$ is quasi-finite (but not finite) over $S_G(K)_{/\mathcal{O}_{(\mathfrak{p})}}$. Furthermore, over $\mathcal{K} = \mathcal{O}_{(\mathfrak{p})}[1/p]$ there is a canonical isomorphism of $S_G(K_t(p^s))$ with the scheme so-denoted in 5.3.5 (so the notation should cause no confusion); the isomorphism is induced by the obvious map on moduli problems.

5.4. **Compactifications.** Following the methods of Faltings and Chai [FC], Lan [Lan08] has constructed arithmetic toroidal and minimal compactifications of Shimura varieties of PEL-type, so in particular of the $S_G(K)$'s from the preceding paragraphs (there is work of Fujiwara [Fu] that predates [Lan08]). In what follows we sketch the main points of these constructions[3], recalling properties of these compactifications needed for our exposition of Hida theory for G. As the arithmetic minimal compatification is deduced from the toroidal compactification, we start by recalling the latter.

In what follows, $K = K_p^0 K^p \subset G(\mathbf{A}_f)$ is a neat open compact subgroup. We choose a decomposition $G(\mathbf{A}_f) = \sqcup_i G(\mathbf{Q})g_iK$ with the g_i such that $g_{i,p} \in K_p^0$ and $\mu(g_i) \in \widehat{\mathbf{Z}}^\times$.

5.4.1. *Local charts.* Let $r, s \geq 0$ be integers such that $n = r + s$. Let $L_0 := \mathcal{O}^{2n}$ be the standard $\mathcal{O}$-lattice in $V = V_n$. Let $\Psi_V := \mathrm{Tr}_{\mathcal{K}/\mathbf{Q}}\delta_\mathcal{K}^{-1}\psi_n$; this is a symplectic pairing on V. Under Ψ_V, L_0 is self-dual. Let $v_1, ..., v_{2n}$ be the standard basis of L and let W_s be the s-dimensional isotropic $\mathcal{K}$-space spanned by $v_{n+r+1}, ..., v_{2n}$. Let $W_{s,1}$ be the $\mathcal{K}$ span of $v_1, .., v_r, v_{n+1}, ..., v_{n+r}$ and let W_s' be the span of $v_{r+1}, ..., v_n$. Then $W_s^\perp := W_{s,1} \oplus W_s$ is the annihilator of W_s under Ψ_V. The restriction $\psi_{W_{s,1}}$ of ψ_n to $W_{s,1}$ is a skew-Hermitian pairing and there is a canonical isomorphism $(W_{s,1}, \psi_{W_{s,1}}) \cong (V_r, \psi_r)$ and so a canonical identification of the unitary similitude group $GU(W_{s,1})$ with G_r. The decomposition $V = W_s' \oplus W_{s,1} \oplus W_s$ induces an embedding $GU(W_{s,1}) \times \mathrm{GL}_\mathcal{K}(W_s) \hookrightarrow G = G_n$: $(g, h) \mapsto (\mu(g)h^*, g, h)$ with $\mu(g)$ the similitude factor of g (this defines h^*). Let $P_r \subset G$ be the stabilizer of W_s. Then $GU(W_{s,1}) \times \mathrm{GL}_\mathcal{K}(W_s) = G_r \times \mathrm{GL}_\mathcal{K}(W_s)$ is a Levi factor of P_r.

[3]This section was written before [Lan08] was available. We made no effort to reconcile our notation with that in *loc. cit.*

Let $g \in G(\mathbf{A}_f)$ with $g_p \in K_p^0$ and write $g = \gamma g_i k$ with $\gamma \in G(\mathbf{Q})$ and $k \in K$. Let $W := W_s\gamma$, $W_1 := W_{s,1}\gamma$, and $W' := W_s'\gamma$. Then $W^\perp := W_1 \oplus W$ is the annihilator of W under Ψ_V. Let $P := \gamma^{-1}P_r\gamma$ be the stabilizer of the isotropic space W. The isomorphism $W_1 \stackrel{\sim}{\to} W_{s,1}$, $w \mapsto w\gamma^{-1}$ determines an isomorphism of skew-Hermitian spaces $(W_1, \psi_{W_1}) \cong (W_{s,1}, \psi_{W_{s,1}})$, where ψ_{W_1} is the restriction of ψ_n to W_1, and so an isomorphism $GU(W_1) \cong G_r$. The group $GU(W_1) \times \mathrm{GL}_{\mathcal{K}}(W) = \gamma^{-1}(GU(W_{s,1}) \times \mathrm{GL}_{\mathcal{K}}(W_s))\gamma \subset G$ is a Levi factor of P.

Let $L := (L_0 \otimes \widehat{\mathbf{Z}})g_i^{-1} \cap V$. Then L is also a self-dual $\mathcal{O}$-lattice under Ψ_V. Let $W_L := W \cap L$, $X := \mathrm{Hom}_{\mathbf{Z}}(W_L, \mathbf{Z})$, and $Y := L/L \cap W^\perp$. Then Ψ_V determines a c-semi-linear isomorphism $\phi : Y \stackrel{\sim}{\to} X$: $y \mapsto (w \mapsto \Psi_V(x,y))$ (c-semi-linear in the sense that $\phi(ay) = \bar{a}\phi(y)$). Let $K_1 := GU(W_1; \mathbf{A}_f) \cap g_iKg_i^{-1}$. Via the isomorphism $GU(W_1) \cong G_r$, K_1 is identified with $K_{r,g} := G_r(\mathbf{A}_f) \cap gKg^{-1}$ which is of the form $K_{p,r}^0 K_1^p$. Let $S_{1/\mathcal{O}_{(p)}} := S_{GU(W_1)}(K_1) \cong S_{G_r}(K_{r,g})$.

Let $(\mathcal{A}_1, \lambda_1, \iota_1, \eta_1)$ be the universal quadruple over S_1. Let $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \mathcal{A}_1)$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, \mathcal{A}_1^\vee)$, and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \mathcal{A}_1^\vee)$ be the obvious sheaves over the big étale site over S_1 (viewing X and Y as constant group schemes with $\mathcal{O}$-actions). These are representable by abelian schemes. Let $\mathbf{c}$ and $\mathbf{c}^\vee$ be the universal morphisms over $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, \mathcal{A}_1^\vee)$ and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \mathcal{A}_1)$, respectively. Then

$$Z := \underline{\mathrm{Hom}}_{\mathcal{O}}(X, \mathcal{A}_1^\vee) \times_{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \mathcal{A}_1^\vee)} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \mathcal{A}_1)$$

is the sheaf such that for an S_1-scheme S, $Z(S)$ is the set of commutative squares

$$\begin{array}{ccc} X_{/S} & \xrightarrow{c} & \mathcal{A}^\vee_{1\,/S} \\ \phi \uparrow & & \uparrow \lambda_1 \\ Y_{/S} & \xrightarrow{c^\vee} & \mathcal{A}_{1/S} \end{array}$$

with c and $c^\vee$ both $\mathcal{O}$-linear. As ϕ is an isomorphism, it is easy to see that Z is isomorphic to $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, \mathcal{A}_1) \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \mathcal{A}_1)$ and so representable by an abelian scheme. The associated universal pair of morphisms over Z is just $(\mathbf{c}, \mathbf{c}^\vee) = (\lambda_1 \circ \mathbf{c}^\vee \circ \phi^{-1}, \mathbf{c}^\vee)$.

Let N_P be the unipotent radical of P. Let $Z(N_P(\mathbf{Q}))$ be the center of $N_P(\mathbf{Q})$ and let $H := Z(N_P(\mathbf{Q})) \cap g_iKg_i^{-1}$. This latter group can be identified with a lattice of Hermitian $s \times s$-matrices: for each $h \in H$ we let $b_h : Y \times Y \to \mathfrak{d}^{-1}$ be the unique Hermitian pairing such that $\mathrm{Tr}_{\mathcal{K}/\mathbf{Q}} b_h(y, y') = \psi_V(y(h-1), y')$ and we identify h with a Hermitian element of $\mathrm{End}_{\mathcal{K}}(W') \cong \mathrm{End}_{\mathcal{K}}(Y \otimes \mathbf{Q})$ such that $b_h(y, y') = yh{}^t\bar{y}'$. Let $S := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$. This is identified with the lattice of Hermitian $s \times s$-matrices h such that $\mathrm{Tr}\,(hh') \in \mathbf{Z}$ for all $h' \in H$. Let $S^+ \subset S$ be the subset of positive semi-definite matrices. Let $\Gamma := \mathrm{GL}_{\mathcal{K}}(W) \cap g_iKg_i^{-1}$. The group Γ acts naturally on H (via conjugation); this induces an action of Γ on S and S^+. Under the identification of S with a lattice of Hermitian matrices this action is just $\gamma.h = {}^t\bar{\gamma}^* h \gamma^*$. The group Γ also acts on Z through its natural actions on X and Y.

Let T_H be the split torus over $\mathbf{Z}$ defined by $T_H := H \otimes_{\mathbf{Z}} \mathbf{G}_m$. Then S is canonically identified with the character group of T_H.

Let $\mathcal{P}_{\mathcal{A}_1}$ be the Poincaré sheaf over the fiber product $\mathcal{A}_1^\vee \times \mathcal{A}_{1/Z}$ and $\mathcal{P}_{\mathcal{A}_1}^\times$ its associated $\mathbf{G}_m$-torsor. For any $y, y' \in Y$, let $[y \otimes y']$ be the character of T_H corresponding to the element of S defined by $h \mapsto b_h(y, y')$. We denote by Ξ the T_H-torsor over Z such that for any $y, y' \in Y$, the push-forward of Ξ by $[y \otimes y']$ is equal to $(\lambda_1 \circ \mathbf{c}^\vee(y) \times \mathbf{c}^\vee(y'))^* \mathcal{P}_{\mathcal{A}_1}^\times$ as a $\mathbf{G}_m$-torsor over Z. For any $h \in S$, we let $\mathcal{L}(h)$ be the $\mathbf{G}_m$-torsor over Z obtained by pushing forward Ξ by the character h. The pull-back of $\mathcal{L}(h)$ under the natural action of $\gamma \in \Gamma$ is just $\mathcal{L}(\gamma.h)$. We note that by construction there is a tautological trivialization of the $\mathbf{G}_m$-torsor $(\mathbf{c} \times \mathbf{c}^\vee)^* \mathcal{P}_{\mathcal{A}_{1/\Xi}}^\times$:

$$(\mathbf{c} \times \mathbf{c}^\vee)^* \mathcal{P}_{\mathcal{A}_{1/\Xi}}^\times \cong \mathbf{G}_{m/(X \times Y)/\Xi}. \tag{5.4.1.a}$$

For $h \in S$ let $\Gamma(h) \subset \Gamma$ be the stabilizer of h. Then $\Gamma(h)$ acts naturally on $H^0(Z, \mathcal{L}(h))$.

Lemma 5.4.2. *The group $\Gamma(h)$ acts trivially on $H^0(Z, \mathcal{L}(h))$.*

Proof. Let d be the rank of h. We assume $d > 0$ since the case $d = 0$ is trivial. Let $Y = Y_h \oplus Y^h$ be an $\mathcal{O}$-decomposition with Y_h the kernel of h. Then Y^h has $\mathcal{O}$-rank d. We have $Z \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \mathcal{A}_1) = \underline{\mathrm{Hom}}_{\mathcal{O}}(Y_h, \mathcal{A}_1) \times \underline{\mathrm{Hom}}_{\mathcal{O}}(Y^h, \mathcal{A}_1)$. Let π denote the projection to $Z^h := \underline{\mathrm{Hom}}_{\mathcal{O}}(Y^h, \mathcal{A}_1)$. As $\Gamma(h)$ fixes h and so stabilizes Y_h, it follows from the neatness of K that $\Gamma(h)$ acts trivially on Y/Y_h. Therefore there is a commutative diagram

$$\begin{array}{ccc} Z \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \mathcal{A}_1) & \xrightarrow{\pi} & Z^h = \underline{\mathrm{Hom}}(Y^h, \mathcal{A}_1) \\ \gamma \big\downarrow & & id \big\downarrow \\ Z \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \mathcal{A}_1) & \xrightarrow{\pi} & Z^h = \underline{\mathrm{Hom}}(Y^h, \mathcal{A}_1). \end{array}$$

We will show that $\mathcal{L}(h) = \pi^* \tilde{\mathcal{L}}$ for some $\tilde{\mathcal{L}}$ on Z^h, from which the lemma follows by the above diagram and the properness of π.

We have $h = \sum_{i=1}^m [y_i \otimes y_i']$ with $y_i, y_i' \in Y^h$. For each i, the map $\mathbf{c}(\phi(y_i)) \times \mathbf{c}^\vee(y_i') : Z \to \mathcal{A}_1^\vee \times \mathcal{A}_1$ factors through π since $y_i, y_i' \in Y^h$. Therefore $\mathcal{L}([y_i \otimes y_i'])$ is the pull-back by π of a sheaf on Z^h. As $\mathcal{L}(h) = \mathcal{L}([y_1 \otimes y_1']) \otimes \cdots \otimes \mathcal{L}([y_m \otimes y_m'])$ it follows that $\mathcal{L}(h)$ is also a pull-back by π of a sheaf on Z^h. ∎

Torus embeddings. Let $H_{\mathbf{R}}^+ \subset (H \otimes \mathbf{R})^+$ denote the cone of positive semi-definite Hermitian forms on $Y \otimes \mathbf{R}$ whose radical is the $\mathbf{R}$-span of a $\mathbf{Q}$-subspace of $Y \otimes \mathbf{Q}$. Let $\Sigma := \{\sigma\}$ be a rational polyhedral cone decomposition of $H_{\mathbf{R}}^+$. Associated with each $\sigma \in \Sigma$ is a torus embedding

$$T_H \hookrightarrow T_{H,\sigma} := \operatorname{Spec} \mathbf{Z}[S \cap \sigma^\vee],$$

where $\sigma^\vee \subset S \otimes \mathbf{R}$ is the dual cone of σ. As σ varies in Σ, the $T_{H,\sigma}$ glue to form $T_{H,\Sigma}$, yielding the torus embedding associated with Σ: $T_H \hookrightarrow T_{H,\Sigma}$. If Σ is Γ-admissible (that is, the action of Γ on H extends to a permutation of the cones σ in Σ), then functoriality yields an action of Γ on $T_{H,\Sigma}$ with respect to which the torus embedding $T_H \hookrightarrow T_{H,\Sigma}$ is equivariant. It is known that smooth Γ-admissible rational cone decompositions of $H^+_{\mathbf{R}}$ exists.

Local compactification data. We now describe the degeneration datum associated with g and a smooth Γ-admissible rational polyhedral cone decomposition Σ of $H^+_{\mathbf{R}}$. Let

$$\overline{\Xi}_\Sigma := \Xi \times^{T_H} T_{H,\Sigma},$$

and let $\mathcal{S}$ be the formal completion of $\overline{\Xi}_\Sigma$ along the complement $\partial T_{H,\Sigma} := T_{H,\Sigma}\backslash T_H$. From the data $\mathcal{A}_1, \lambda_1, X, Y, \mathbf{c}, \mathbf{c}^\vee$ and the trivialization (5.4.1.a) we obtain over any formal affine subscheme $\mathrm{Spf}\,\mathcal{R} \subset \mathcal{S}$ a degeneration datum (in the sense of [FC] but with CM by $\mathcal{O}$). From Mumford's construction (cf. [FC] and also [Lan08]) we therefore obtain a formal semi-abelian scheme $\mathcal{G}_{/\mathrm{Spf}\,\mathcal{R}}$ together with polarization and CM by $\mathcal{O}$. These glue into a formal semi-abelian scheme $\mathcal{G}_{/\mathcal{S}}$. The local compactification datum associated with g and Σ is then defined as the algebraization of $\mathcal{S}/\Gamma$ (cf. [FC] and [Lan08] for the notion of algebraization). The toroidal compactification of $S_G(K)_{/\mathcal{O}_{(p)}}$ is defined by glueing together a suitable collection of such local compactification data.

The case $r = 0$. Suppose $r = 0$ (so $s = n$). Let $\sigma \in \Sigma$ be a cone contained in the interior of $H^+_{\mathbf{R}}$. Let $R_\sigma := \mathcal{O}_{(p)}[\![q^{\sigma^\vee \cap S}]\!]$ (the completion of $\mathcal{O}_{(p)}[q^{\sigma^\vee\cap S}]$ at the ideal I_σ generated by $q^{\sigma^\vee \cap S/\{0\}}$). As $\overline{\Xi}_\Sigma = T_{H,\Sigma}$ in this case, $\mathrm{Spf} R_\sigma$ is canonically the formal completion along the boundary of the σ-stratum $\overline{\Xi}_\sigma = T_{H,\sigma}$, so there is a canonical map $\mathrm{Spf} R_\sigma \to \mathcal{S}$. The Mumford construction gives a semi-abelian scheme $\mathcal{G}_\sigma/R_\sigma$ (together with polarization and CM by $\mathcal{O}$) that is an abelian variety over the fraction field of R_σ and completely toric over $R_{\sigma,0} := R_\sigma/I_\sigma R_\sigma$; tautologically, this is the semi-abelian scheme over R_σ associated with the pull-back to $\mathrm{Spf} R_\sigma$ of the formal semi-abelian scheme $\mathcal{G}_{/\mathcal{S}}$.

5.4.3. *Genus $2r$ cusps.* The set of 'cusp labels' of genus $2r$ for $S_G(K)$ is defined to be

$$C_r(K) := (G_r(\mathbf{A}_f) \times \mathrm{GL}_\mathcal{K}(W_s))N_r(\mathbf{A}_f)\backslash G(\mathbf{A}_f)/K,$$

where N_r is the unipotent radical of P_r. Here we use the identification $GU(W_{s,1}) = G_r$. The class of some $g \in G(\mathbf{A}_f)$ in $C_r(K)$ is denoted $[g]_r$, or even $[g]$ if r is implied by the context or not important.

5.4.4. *The toroidal compactification.* We denote by $\overline{S}_G(K)_{/\mathcal{O}_{(p)}}$ the arithmetic toroidal compactification over $\mathcal{O}_{(p)}$ associated with a (fixed once and for all) well-chosen smooth rational polyhedral cone decomposition of $\prod_g H^+_{g,\mathbf{R}}$, where $H^+_{g,\mathbf{R}} = H^+_{\mathbf{R}}$ is the set of semi-definite Hermitian matrices associated with g as in 5.4.1. Here g is running over representatives of the various cusp labels $[g]_r \in C_r(K)$ for $0 \le r \le n$ (which we always take so that $g_p \in K^0_p$). The polyhedral cone decomposition Σ_g of $H^+_{g,\mathbf{R}}$ is taken to be Γ_g-stable (Γ_g being the group Γ associated with g and r above). Furthermore, if $[g]_r = [g']_{r'}$, $r' < r$, then Σ_g is required to be identified (in the obvious way) with the cone decomposition in $\Sigma_{g'}$ of the corresponding face of $H^+_{g,\mathbf{R}}$. The toroidal compactification is obtained by gluing the local compactification data attached to the various g. (For the details of the gluing procedure see [FC] or [Lan08].)

For a well-chosen Σ, the toroidal compactification $\overline{S}_G(K)_{/\mathcal{O}_{(p)}}$ is a proper smooth scheme over $\mathcal{O}_{(p)}$ containing $S_G(K)$ as an open subscheme. The complement of $S_G(K)$ is a relative Cartier divisor with normal crossings. The scheme $\overline{S}_G(K)$ is equipped with a quadruple $(\mathcal{G}, \lambda, \iota, \eta)$ with $\mathcal{G}$ a semi-abelian scheme, a homomorphism $\lambda : \mathcal{G} \to \mathcal{G}^\vee$, and an injective homomorphism $\iota : \mathcal{O} \to \mathrm{End}(\mathcal{G})$ satisfying $\iota(a)^\vee \circ \lambda = \lambda \circ \iota(\bar{a})$ such that $(\mathcal{G}|_{S_G(K)}, \lambda|_{S_G(K)}, \iota, \eta)$ is the universal quadruple over $S_G(K)$ (η is just the universal K-level structure on $\mathcal{G}|_{S_G(K)}$). The toroidal compactification $\overline{S}_G(K)$ is stratified by locally closed subschemes:

$$\overline{S}_G(K) = \sqcup_{0\le r\le n} \sqcup_{[g]\in C_r(K)} \sqcup_{\sigma\in\Sigma_{H^+_{g,\mathbf{R}}}/\Gamma_g} Z([g], \sigma),$$

where for a given g and r, $\Sigma_{H^+_{g,\mathbf{R}}}$ is the fixed polyhedral cone decomposition of $H^+_{g,\mathbf{R}}$ and Γ_g is the group associated with g and r in 5.4.1. If $r = 0$ and $\sigma \in \Sigma_{H^+_{g,\mathbf{R}}}$ is a cone in the interior of $H^+_{g,\mathbf{R}}$, then the formal completion of $\overline{S}_G(K)$ along the stratum $Z([g], \sigma)$ is canonically $\mathrm{Spf}(R_\sigma^\Gamma)$. Via the induced map $\mathrm{Spec} R_\sigma \to \overline{S}_G(K)$, the semi-abelian scheme $\mathcal{G}_\sigma/R_\sigma$ (together with its polarization and CM-by-$\mathcal{O}$ structure) is just the pull-back of the quadruple $(\mathcal{G}, \lambda, \iota, \eta)$.

Level structure at p. Of course if $\mathcal{O}_{(p)}$ is replaced with $\mathcal{O}_{(p)}[1/p] = \mathcal{K}$, then the construction of the toroidal compactification can be done even for K not of the form $K_p^0 K^p$. These compactifications were known before [FC] and [Lan08]: see [Ha89]. In particular toroidal compactifications of $S_G(K_1(p^s))$ and $S_G(K_0(p^s))$ exist over $\mathcal{K}$.

5.4.5. *The minimal compactification.* Let $\underline{\omega}$ be the pull back of $\Omega_{\mathcal{G}/\overline{S}_G(K)}$ by the zero section of $\mathcal{G}/\overline{S}_G(K)$. The minimal compactification of $S_G(K)$ over $\mathcal{O}_{(p)}$ is $S_G^*(K) := \mathrm{Proj}(\oplus_m H^0(\overline{S}_G(K), \det(\underline{\omega})^m))$. There is a canonical morphism $\pi : \overline{S}_G(K) \to S^*_G(K)$, and the minimal compactification is equipped with a line bundle, also denoted $\det(\underline{\omega})$, and defined to be the direct image of $\det(\underline{\omega})$ under π. By the arguments in [FC, V] (see also [Lan08, 7]) we have the following.

Theorem 5.4.6.

(i) $\det(\underline{\omega})$ *is ample on* $S^*_G(K)$.
(ii) *The scheme* $S^*_G(K)$ *is a normal projective scheme of finite type over* $\mathcal{O}_{(p)}$. *It has a natural stratification by locally closed subschemes indexed by* $\sqcup_{r=0}^n C_r(K)$. *The stratum* $S_{[g]}$ *indexed by* $[g] \in C_r(K)$ *is naturally isomorphic to* $S_{GU(W_1)}(K_1) \cong S_{G_r}(K_{r,g})$, *where* W_1, K_1, $K_{r,g}$, *and the isomorphism are associated with* g *as in 5.4.1. In particular, there is a natural decomposition*

$$S^*_G(K) = \bigsqcup_{r=0}^{n} \bigsqcup_{[g]_r\in C_r(K)} S_{G_r}(K_{r,g}), \quad K_{r,g} = G_r(\mathbf{A}_f) \cap gKg^{-1}.$$

For any $[g]_r \in C_r(K)$, the Zariski closure of the stratum $S_{[g]_r}$ in $S^*_G(K)$ is

$$\bigsqcup_{r' \leq r} \bigsqcup_{\substack{[h] \in C_{r'}(K) \\ [h]_r = [g]_r}} S_{[h]},$$

which is naturally isomorphic to the minimal compactification of $S_{[g]_r} \cong S_{G_r}(K_{r,g})$.

We will write $S_{[g]}$ for the locally closed subscheme of $S^*_G(K)$ attached to some class $[g] \in C_r(K)$ by this theorem and will write $S^*_{[g]}$ for the Zariski closure of $S_{[g]}$ in $S^*_G(K)$. Such a component will be called a rational component of genus $2r$. The local structure of the minimal compactification at the boundary component $S_{[g]}$ is given by the following theorem whose proof is similar to [FC, V.2.7] (see also [Lan08, 7.2.3.11]); the $r = 0$ case has already been noted.

Theorem 5.4.7. *Let $r \in \{0, \dots, n-1\}$. Let $\bar{x}$ be a geometric point of $S_{[g]_r}$. The strict henselization $\mathcal{O}_{S^*_G(K),\hat{\bar{x}}}$ of the structure sheaf at $\bar{x}$ is canonically isomorphic to the ring of formal power series*

$$\left\{ \sum_{h \in S^+} a(h) q^h \; : \; a(h) \in H^0(Z, \mathcal{L}(h))_{\bar{x}} \right\}^{\Gamma},$$

where Z, S^+, Γ, and $\mathcal{L}(h)$ ($h \in S^+$) are associated with g as in 5.4.1. The invariance under Γ is equivalent to $a(\gamma.h) = \gamma^* a(h)$ for all $\gamma \in \Gamma$.

Remark 5.4.8. We identify $S_{[g]}$ with $S_{G_r}(K_{r,g})$ via the natural isomorphism between the two shemes (this depends on g). Then the isomophism in the preceding theorem can be written as follows. Let $s = n - r$. Let $\Gamma_{[g]} := \mathrm{GL}_{\mathcal{K}}(W_s) \cap gKg^{-1}$ and let $S^+_{[g]}$ comprise those Hermitian matrices h in $M_n(\mathcal{K})$ such that $\mathrm{Tr}\, hh' \in \mathbf{Z}$ for all Hermitian matrices h' such that $\left(\begin{smallmatrix} 1 & h' \\ & 1 \end{smallmatrix}\right) \in Z(N_r(\mathbf{Q})) \cap gKg^{-1}$ (N_r begin the unipotent radical of P_r). If $g = \gamma g_i k$, then $\Gamma_{[g]} = \gamma \Gamma \gamma^{-1}$ and $S^+_{[g]}$ is identified with a similar conjugate of S^+. Let $Z_{[g]}$ be the abelian scheme over $S_{G_r}(K_{r,g})$ corresponding via the natural isomorphism $S_{[g]} \cong S_{G_r}(K_{r,g})$ to the scheme Z over $S_{[g]}$ associated with g. For $h \in S^+_{[g]}$ we write $\mathcal{L}(h)$ for the line bundle on $Z_{[g]}$ defined as in 5.4.1. It is then easy to see that the theorem implies an isomorphism (canonical up to the chosen identifications) of $\mathcal{O}_{S^*_G(K),\hat{\bar{x}}}$ with

$$\left\{ \sum_{h \in S^+_{[g]}} a(h) q^h \; : \; a(h) \in H^0(Z_{[g]}, \mathcal{L}(h))_{\bar{x}} \right\}^{\Gamma_{[g]}},$$

where invariance by $\alpha \in \Gamma_{[g]}$ means $\alpha a(h) = a(\alpha.h) = a({}^t\bar{\alpha}^* h \alpha^*)$.

5.4.9. *The stratification of the minimal compactification.* Let $\mathcal{S} := \overline{S}_G(K)$ and $\mathcal{S}^* := S^*_G(K)$. For $q \in \{0, \dots, n-1\}$ we denote by $\partial^q \mathcal{S}^*$ the union of the genus $2r$ components of the boundary of $\mathcal{S}^*$ for those r such that $r < n - q$. In particular, $\mathcal{S}^* \backslash \partial^0 \mathcal{S}^* = S_G(K)$. By Theorem 5.4.6, $\partial^q \mathcal{S}^*$ is a closed subscheme of $\mathcal{S}^*$. We denote by $\mathcal{I}^q_{\mathcal{S}^*}$ the corresponding sheaf of ideals in $\mathcal{O}_{\mathcal{S}^*}$ and we put $\mathcal{I}^q_{\mathcal{S}} := \pi^*\mathcal{I}^q_{\mathcal{S}^*}$ where $\pi : \mathcal{S} \to \mathcal{S}^*$ is the canonical projection. By definition, for any $q, q' \in \{0, \ldots, n-1\}$ with $q' > q$ there is an exact sequence

$$0 \to \mathcal{I}^q_{\mathcal{S}^*} \to \mathcal{I}^{q'}_{\mathcal{S}^*} \to \bigoplus_{[g]\in C_{n-q-1}(K)} \iota_{[g],*}\mathcal{I}^{q'-q-1}_{S^*_{[g]}} \to 0, \tag{5.4.9.a}$$

where $\iota_{[g]} : S^*_{[g]} \hookrightarrow \mathcal{S}^*$ is the inclusion map.

5.5. **Automorphic forms.** We denote by $Q = Q_n$ the Siegel parabolic of U_n; this is the stabilizer in U_n of the rank n sublattice of $L_0 = \mathcal{O}^{2n}$ spanned by the last n elements of the canonical basis. It has a standard Levi factor M_Q that is isomorphic to $Res_{\mathcal{O}/\mathbf{Z}}GL_n$ via the diagonal embedding $g \mapsto \mathrm{diag}({}^t\bar{g}^{-1}, g)$. We denote by H the base change of M_Q to $R = \mathcal{O}_{(p)}$. Then $H \cong \mathrm{GL}_{n/R} \times \mathrm{GL}_{n/R}$, the isomorphism being induced by $R \otimes_{\mathbf{Z}} R \xrightarrow{\sim} R \times R$, $x \otimes y \mapsto (xy, \bar{x}y)$.

5.5.1. *Automorphic Sheaves.* Let e be the zero section of the 'universal' semi-abelian scheme $\mathcal{G}$ over $\overline{S}_G(K)_{/R}$. As before, we let $\underline{\omega} := e^*\Omega^1_{\mathcal{G}/\overline{S}_G(K)_{/R}} = \omega_{\mathcal{G}/\overline{S}_G(K)_{/R}}$. This is a locally free coherent sheaf on $\overline{S}_G(K)_{/R}$ of rank $2n$. Recall that the complex multiplication by $\mathcal{O}$ induces a decomposition

$$\underline{\omega} = \underline{\omega}^+ \oplus \underline{\omega}^-$$

of type (n, n), where $\underline{\omega}^+$ and $\underline{\omega}^-$ are locally free of rank n. Let

$$\mathcal{E}^\pm := \underline{\mathrm{Isom}}_{\overline{S}_G(K)}(\mathcal{O}^n_{\overline{S}_G(K)}, \underline{\omega}^\pm).$$

Then $\mathcal{E} := \mathcal{E}^+ \oplus \mathcal{E}^-$ is an H-torsor over $\overline{S}_G(K)$; the H-action can be described in the following way. For $U \subseteq \overline{S}_G(K)$, $i = (i^+, i^-) \in \mathcal{E}(U)$ with $i^\pm$ trivializations $i^\pm :\ \mathcal{O}(U)^n \cong \underline{\omega}^\pm(U)$, $h = (h^+, h^-) \in H(\mathcal{O}(U)) = \mathrm{GL}_n(\mathcal{O}(U)) \times \mathrm{GL}_n(\mathcal{O}(U))$, and $v^\pm \in \mathcal{O}(U)^n$ we set

$$(h.i)(v^+, v^-) := (i^+(v^+h^+), i^-(v^-h^-)).$$

For any algebraic representation of $H_{/R}$ on a free R-module V, we define an automorphic sheaf

$$\omega_V := \mathcal{E} \times^H V.$$

This is a locally free $\mathcal{O}_{\overline{S}_G(K)}$-module.

Let $B_H := H \cap B_n$; this is identified with the lower-triangular Borel of $H = \mathrm{GL}_n \times \mathrm{GL}_n$. We fix a parametrization of the dominant weights of the diagonal torus $T = T_n$ of $U_{n/R}$ with respect to the pair (B_H, T) as follows. Let $k_1, \ldots, k_{2n} \in \mathbf{Z}$ such that $k_1 \geq k_2 \geq \cdots \geq k_{2n}$ and $\underline{k} := (k_{n+1}, \ldots, k_{2n}; k_n, \ldots, k_1)$. This defines a dominant weight for the above pair by the rule

$$[\underline{k}] :\ \mathrm{diag}(t_1, \ldots, t_{2n}) \mapsto t_1^{k_{n+1}} \cdots t_n^{k_{2n}} \cdot t_{n+1}^{k_n} \cdots t_{2n}^{k_1}. \tag{5.5.1.a}$$

We denote by $\omega_{\underline{k}}$ the automorphic sheaf associated as above to the algebraic representation $\rho_{\underline{k}}$ of H of highest weight $[\underline{k}]$ with respect to the pair (B_H, T). Using the identification $H = \mathrm{GL}_n \times \mathrm{GL}_n$ defined previously, we can describe the algebraic representation

$\rho_{\underline{k}}$ of H as

$$\rho_{\underline{k}}(g_+,g_-) := \rho_{(k_n,\dots,k_1)}(g_+)\otimes\rho_{(-k_{n+1},\dots,-k_{2n})}(g_-),$$

where for any increasing sequence of integers $a_1 \leq \dots, \leq a_n$, we have denoted by $\rho_{(a_1,\dots,a_n)}$ the dual of the irreducible algebraic representation of GL_n with highest weight with respect to the lower-triangular Borel subgroup being the character of the diagonal torus given by $t := \mathrm{diag}(t_1,\dots,t_n) \mapsto t^{\underline{a}} := t_1^{a_1}\cdots t_n^{a_n}$. Recall that $\rho_{\underline{a}}$ can be realized as the space of algebraic maps $\phi : \mathrm{GL}_n \to \mathbf{A}^1$ such that $\phi(n^-tg) = t^{\underline{a}}\phi(g)$ for all $g \in \mathrm{GL}_n$, t diagonal, and n^- lower-triangular unipotent. Note that $t = \mathrm{diag}(t_1,....,t_{2n}) \in U_{n/R} = \mathrm{GL}_{2n/R}$ is identified with $\mathrm{diag}(t_{n+1},....,t_{2n})\times\mathrm{diag}(t_1^{-1},...,t_n^{-1}) \in H$.

If we take R a $\mathcal{K}$-algebra, then these constructions apply to the toroidal compactifications $\overline{S}_G(K_t(p^s))$, $t = 0,1$.

5.5.2. *Complex uniformization of $\omega_{\underline{k}}$.* We recall the classical definition of the above vector bundles $\omega_{\underline{k}}$ over $S_G(K)_{/\mathbf{C}}$. For $\alpha = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in G(\mathbf{R})$ and $Z \in \mathbf{H}_n$, put

$$\mu_\alpha(Z) := CZ + D, \quad \kappa_\alpha(Z) := \overline{C}Z + \overline{D}$$

and

$$J(\alpha, Z) := (\mu_\alpha(Z), {}^t\kappa_\alpha(Z)^{-1})$$

Then $k_\infty \mapsto J(k_\infty, \mathbf{i})$ defines a group homomorphism from $C_\infty := K_\infty^+ Z_\infty = K_\infty^+\mathbf{R}^\times$ to $H(\mathbf{C}) = \mathrm{GL}_n(\mathbf{C})\times\mathrm{GL}_n(\mathbf{C})$. The automorphic vector bundle $\omega_{\underline{k}}$ on $S_G(K)_{/\mathbf{C}}$ can be described as the sheaf of holomorphic sections of the projection

$$G^+(\mathbf{Q})\backslash X^+ \times G(\mathbf{A}_f)\times V_{\underline{k}}(\mathbf{C})/KC_\infty \to G^+(\mathbf{Q})\backslash X^+\times G(\mathbf{A}_f)/KC_\infty = Sh_G(K)(\mathbf{C}),$$

where $V_{\underline{k}}(\mathbf{C})$ is the space of the representation

$$k_\infty \mapsto \rho_{\underline{k}}\circ J(k_\infty, i) \tag{5.5.2.a}$$

of C_∞. This follows from the complex uniformization of the universal abelian variety over $Sh_G(K)(\mathbf{C})$ (as explained for example in [Sh00, I]). At least if $n > 1$, the global sections of this sheaf are the holomorphic $V_{\underline{k}}(\mathbf{C})$-valued Hermitian modular forms. This follows from the Koecher principle. When $n = 1$ the modular forms are those sections that extend holomorphically over the toroidal compactification (that is, are sections of $\omega_{\underline{k}}$ over $\overline{S}_G(K)_{/\mathbf{C}}$). The classical scalar-valued modular forms of weight $k \geq 0$ correspond to $\underline{k} = (0,\dots,0;k,\dots,k)$. The action of the center $\mathbf{R}^\times \subset Z_\infty$ via the representation (5.5.2.a) is $t \mapsto t^{|\underline{k}|}$ with $|\underline{k}| := k_1 + \cdots + k_{2n}$.

Remark. In the notation of [Ha00], our vector bundle $\omega_{\underline{k}}$ is the one of weight $\tau := (k_{n+1},\dots,k_{2n};k_1,\dots,k_n;|\underline{k}|)$.

Let $f \in H^0(S_G(K)_{/\mathbf{C}}, \omega_{\underline{k}})$ be a modular form. Then f defines a function $f : \mathbf{H}_n \times G(\mathbf{A}_f) \to V_{\underline{k}}(\mathbf{C})$ that is holomorphic in the first variable. For any such function, for $\gamma \in G^+(\mathbf{R})$ put

$$(f|_{\underline{k}}\gamma)(Z) := \mu(\gamma)^{(\sum_{i=1}^{2n} k_i)/2}\rho_{\underline{k}}(J(\gamma,Z))^{-1}f(\gamma(Z)).$$

The condition that f be a modular form is then

$$f(-,g)|_{\underline{k}}\gamma = f(-,g) \quad \forall\gamma \in \Gamma_{K,g} := gKg^{-1} \cap G^+(\mathbf{Q})$$

together with the condition that $f|_{\underline{k}}\gamma$ be bounded as a function of Z for all $\gamma \in G^+(\mathbf{Q})$ (this is automatic if $n \geq 2$ by the Koecher priniciple).

q-expansions of classical forms. Any modular form f of weight $\underline{k}$ has Fourier expansions

$$f(Z,g) = \sum_{h \in L_{K,g}^\vee, h \geq 0} a(h,g)e(\operatorname{Tr} hZ), \quad a(h,g) \in V_{\underline{k}}(\mathbf{C}), \tag{5.5.2.b}$$

with h running over the positive semi-definite Hermitian matrices in the lattice $L_{K,g}^\vee$ of Hermitian matrices $h \in M_2(\mathcal{K})$ such that tr $hh' \in \mathbf{Z}$ for all Hermitian matrices h' such that $\left(\begin{smallmatrix} 1 & h' \\ & 1 \end{smallmatrix}\right) \in \Gamma_{K,g}$. Note that the positive semi-definite matrices in $L_{K,g}^\vee$ comprise the set denoted $S_{[g]_0}^+$ in 5.4.8.

5.5.3. *Spaces of automorphic forms.* As before, let $\mathcal{S} := \overline{S}_G(K)_{/\mathcal{O}_{(p)}}$ and $\mathcal{S}^* := S_G^*(K)_{/\mathcal{O}_{(p)}}$. For a general weight $\underline{k}$ and any $\mathcal{O}_{(p)}$-algebra R put

$$\mathbf{M}_{\underline{k}}^n(K,R) := H^0(\mathcal{S}_{/R}, \omega_{\underline{k}})$$

and for $q \in \{0,...,n-1\}$ put

$$\mathbf{M}_{\underline{k}}^{n,q}(K,R) := H^0(\mathcal{S}_{/R}, \omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \mathcal{I}_\mathcal{S}^q).$$

We also have

$$\begin{aligned} \mathbf{M}_{\underline{k}}^n(K,R) &= H^0(\mathcal{S}_{/R}^*, \pi_*\omega_{\underline{k}}) \\ \mathbf{M}_{\underline{k}}^{n,q}(K,R) &= H^0(\mathcal{S}_{/R}^*, \pi_*\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}^*}} \mathcal{I}^q). \end{aligned} \tag{5.5.3.a}$$

It is often more convenient to work with the minimal compactification. The drawback is that the sheaf $\pi_*\omega_{\underline{k}}$ is generally not locally free and this causes some complications. But on the other hand, this sheaf is independant of the choice of the toroidal compactification. In fact, we have the following description of its stalks, analogous to Theorem 5.4.7.

Proposition 5.5.4. *Let $\bar{x}$ be a geometric point of a stratum $S_{[g]}$ of $S_G^*(K)_{/R}$ at a cusp label $[g] \in C_r(K)$.*

(i) *The completion of the stalk $(\pi_*\omega_{\underline{k}})_{\bar{x}}$ is canonically isomorphic to the module of formal power series*

$$\left\{ \sum_{h \in S^+} a(h)q^h \ : a(h) \in H^0(Z, \mathcal{L}(h))_{\bar{x}} \otimes_R \rho_{\underline{k}}(R) \right\}^\Gamma,$$

where Z, S^+, $\Gamma \subset \gamma H(\mathcal{K})\gamma^{-1}$, and $\mathcal{L}(h)$ ($h \in S^+$) are associated with $g = \gamma g_i k$ and r as in 5.4.1. The Γ-invariance is equivalent to $a(\alpha.h) = \rho_{\underline{k}}(\gamma^{-1}\alpha\gamma)^{-1}a(h)$ for all $\alpha \in \Gamma$ and $h \in S^+$.

(ii) *Let $q \in \{0,...,n-1\}$. The completion of the stalk $(\pi_*\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{I}^q)_{\bar{x}}$ is canonically isomorphic to the module of formal power series*

$$\left\{ \sum_{\substack{h\in S^+ \\ \mathrm{rank}(h)\geq n-r-q}} a(h)q^h \ : a(h) \in H^0(Z, \mathcal{L}(h))_{\bar{x}} \otimes_R \rho_{\underline{k}}(R) \right\}^{\Gamma}.$$

Remark 5.5.5. Let $\Gamma_{[g]}$ and $S^+_{[g]}$ be as in 5.4.8, and for $h \in S^+_{[g]}$ let $\Gamma_{[g]}(h) \subseteq \Gamma$ be the stabilizer of h. It follows from the preceding proposition that there is a canonical isomorphism

$$(\pi_*\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}^*}} \mathcal{I}^q)_{\bar{x}} \cong \left\{ \sum_{\substack{h\in S^+_{[g]} \\ \mathrm{rank}(h)\geq n-r-q}} a(h)q^h \ : \ a(h) \in H^0(Z_{[g]}, \mathcal{L}(h))_{\bar{x}} \otimes_R \rho_{\underline{k}}(R)) \right\}^{\Gamma_{[g]}}.$$

When p is invertible in R (so R is a $\mathcal{K}$-algebra) the above definitions and results also hold for the automorphic sheaves on the compactifications $\overline{S}_G(K_t(p^s))_{/R}$, $t = 0,1$.

Functorial character of automorphic forms. From the definition of the spaces of automorphic forms it follows that an automorphic form in $\mathbf{M}^{n,q}_{\underline{k}}(K,R)$ functorially associates an element of $\rho_{\underline{k}}(S)$ to a tuple $(A_{/S}, \lambda, \iota, \eta, (\boldsymbol{\omega}^+, \boldsymbol{\omega}^-))$ with $(A_{/S}, \lambda, \iota, \eta)$ an S-quadruple for an R-algebra S and $\boldsymbol{\omega}^{\pm}$ an ordered S-basis of the global sections of $\underline{\omega}^{\pm}_{/S}$.

The Koecher principle. The (algebraic) Koecher principle is the observation that if R is flat over some normal $\mathcal{O}_{(p)}$-algebra and if $n > 1$ then the canonical map

$$H^0(\mathcal{S}^*_{/R}, \pi_*\omega_{\underline{k}}) \to H^0(S_G(K)_{/R}, \pi_*\omega_{\underline{k}})$$

is an isomorphism. In particular, the functorial character of automorphic forms alluded to above actually characterizes them. The Koecher principle can be proved by the arguments given for the symplectic case in [FC] (see also [Lan08, 7.2.3.9]).

5.5.6. *Automorphic forms with nebentypus.* For any character $\psi : T(\mathbf{Z}/p^n\mathbf{Z}) \to \overline{\mathbf{Q}}^\times$ we let $\mathcal{O}_{(p)}(\psi) \subset \overline{\mathbf{Q}}$ be the extension of $\mathcal{O}_{(p)}$ generated by the values of ψ. The canonical projection $\phi : S_G(K_1(p^s)) \to S_G(K_0(p^s))$ is a Galois cover with Galois group $T(\mathbf{Z}/p^s\mathbf{Z})$. We define automorphic sheaves

$$\omega_{\underline{k},\psi/S_G(K_0(p^s))\times\mathcal{O}_{(p)}(\psi)} = \phi_*\omega_{\underline{k}/S_G(K_1(p^s))\times\mathcal{O}_{(p)}(\psi)}[\psi]$$

to be the sheaves associate to the presheaves of sections that transform by ψ under the action of $T(\mathbf{Z}/p^n\mathbf{Z})$. For any $\mathcal{O}_{(p)}(\psi)$-algebra R in which p is not a zero divisor in R, we define $\mathbf{M}^{n,q}_{\underline{k}}(K_1(p^s), \psi, R)$ to be the inverse image of $\mathbf{M}^{n,q}_{\underline{k}}(K_1(p^s), R[1/p])$ in $H^0(S_G(K_0(p^s))_{/R}, \omega_{\underline{k},\psi})$. Clearly, the elements of $\mathbf{M}^{n,q}_{\underline{k}}(K_0(p^s), \psi, R)$ have a functorial character similar to those of $\mathbf{M}^{n,q}_{\underline{k}}(K,R)$.

5.5.7. *Siegel operators.* Let $[g] \in C_r(K)$ and let $i : S^*_{[g]} \hookrightarrow S^*_G(K)$ be the corresponding closed immersion. The Siegel operator for $[g]$ is the restriction map

$$\Phi_{[g]} : H^0(S^*_G(K), \pi_*\omega_{\underline{k}}) \stackrel{res}{\to} H^0(S^*_{[g]}, i^*\pi_*\omega_{\underline{k}}).$$

By (5.5.3.a), $\mathbf{M}^{n,q}_{\underline{k}}(K,R)$ is the submodule of forms $f \in \mathbf{M}^n_{\underline{k}}(K,R)$ such that $\Phi_{[g]}f = 0$ for all $[g] \in C_{n-q-1}(K)$.

A better understanding of the Siegel operator requires a better description of its target. We give such a description in the case $[g] \in C_r(K)$ for $r = n-1$. This requires additional notation. For $[g] \in C_{n-1}(K)$ let W, P, and Γ be associated with g as in 5.4.1. From W being one-dimensional, $\mathcal{K}$ imaginary quadratic, and K neat, it follows that Γ is trivial. In what follows we make use of the identification $S^*_{[g]} = S^*_{G_r}(K_{r,g})$.

Let $N_{[g]} := N_r(\mathbf{Q}) \cap gKg^{-1} \cap H(\mathcal{K})$, where N_r is the unipotent radical of the parabolic P_r and, as before, H is the base change to $\mathcal{K}$ of the standard Levi factor of the Siegel parabolic of U. Recall that there is a canonical isomorphism $P_r/N_r \cong G_r \times \mathrm{GL}_{\mathcal{K}}(W_s)$ ($s = n-r$). Let $H_{[g]}$ be the image of $P_r(\mathbf{Q}) \cap gKg^{-1}$ in $G_r(\mathbf{Q})$ via the canonical projection $P_r(\mathbf{Q}) \to P_r(\mathbf{Q})/N_r(\mathbf{Q}) \to G_r(\mathbf{Q})$ intersected with $H_r(\mathcal{K})$ (H_r denoting the base change to $\mathcal{O}$ of the standard Levi factor of the Siegel parabolic of G_r). As K is neat and $r = n-1$, $(P_r(\mathbf{Q}) \cap gKg^{-1} \cap H(\mathcal{K}))/N_{[g]} \stackrel{\sim}{\to} H_{[g]}$. Therefore, if ρ is a representation of $H(\mathcal{K})$ then the coinvariant module $\rho_{N_{[g]}}$ is a representation of $H_{[g]}$.

Proposition 5.5.8. *Let $[g] \in C_{n-1}(K)$. For any $\mathcal{O}_{\mathfrak{p}}$-algebra R there is a canonical isomorphism*

$$i^*(\pi_*\omega_{\underline{k}} \otimes_{\mathcal{S}^*} \mathcal{I}^1) \cong (\pi_{[g]})_*(\mathcal{E}_{[g]} \times^{H_{[g]}} \rho_{\underline{k}}(R)_{N_{[g]}}) \otimes_{\mathcal{O}_{S^*_{G_{n-1}}(K_{n-1,g})}} \mathcal{I}^0_{S^*_{G_{n-1}}(K_{n-1,g})},$$

where $\mathcal{E}_{[g]}$ is the H_{n-1}-torsor on a toroidal compactification $\overline{S}_{G_{n-1}}(K_{n-1,g})$ defined just as $\mathcal{E}$ in 5.5.1 and $\pi_{[g]} : \overline{S}_{G_{n-1}}(K_{n-1,g}) \to S^*_{G_{n-1}}(K_{n-1,g})$ is the canonical morphism. In particular, if L is a field extension of $\mathcal{K}_{\mathfrak{p}}$ then there is a canonical isomorphism

$$i^*(\pi_*\omega_{\underline{k}/L} \otimes_{\mathcal{O}_{\mathcal{S}^*}} \mathcal{I}^1) \cong (\pi_{[g]})_*\omega_{\underline{k}'/L} \otimes_{\mathcal{O}_{S^*_{G_{n-1}}(K_{n-1,g})}} \mathcal{I}^0_{S^*_{G_{n-1}}(K_{n-1,g})}$$

with $\underline{k}'$ the highest weight of the algebraic representation $H^0(N_{n-1/\mathcal{O}_{\mathfrak{p}}} \cap H, \rho_{\underline{k}})$ of $H_{n-1} = \mathrm{GL}_{n-1} \times \mathrm{GL}_{n-1/\mathcal{O}_{\mathfrak{p}}}$. Especially, there is a canonical identification

$$H^0(S^*_{[g]}, i^*(\pi_*\omega_{\underline{k}/L} \otimes_{\mathcal{O}_{\mathcal{S}^*}} \mathcal{I}^1)) = \mathbf{M}^{n-1,0}_{\underline{k}'}(K_{n-1,g}, L).$$

Proof. To prove the first isomorphism it suffices to show that there is a canonical isomorphism between the completions of the stalks of the two sheaves. Let $\bar{x}$ be a geometric point attached to a genus $2r \le 2n-2$ cusp label $[g']$ such that $S_{[g']} \subset S^*_{[g]}$. Without loss of generality we may assume $g' = g$. We apply Proposition 5.5.4 as intepreted in Remark 5.5.5. The completion of the stalk of $\pi_*\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}^*}} \mathcal{I}^1$ at $\bar{x}$ is canonically identified with the module of formal power series $\sum_{h \in S^+_{[g]_r}} a(h)q^h$ such that

(i) $a(h) \in H^0(Z_{[g]_r}, \mathcal{L}(h)) \otimes_R \rho_{\underline{k}}(R)$ which are $\Gamma_{[g]_r}$-invariant;

(ii) $a(h) = 0$ if $\mathrm{rank}(h) < n - r - 1$.

The restriction to $S^*_{[g]}$ (with $[g] = [g]_{n-1}$) corresponds to reducing modulo the ideal of formal power series with coefficients supported on those h with $W'_1 \cap \ker h = 0$ ($W'_1 = W'_s$ with $s = 1$) and projecting from $\rho_{\underline{k}}$ to the $N_{[g]}$-coinvariant module $(\rho_{\underline{k}})_{N_{[g]}}$. This means that the stalk at $\bar{x}$ of the restriction is identified with the module of formal power series

$$\left\{ \sum_{\substack{h \in S^+_{[g]_r} \\ \mathrm{rank}(h) > n-r-2, W'_1 \subseteq \ker h}} a(h) q^h \ : \ a(h) \in \rho_{\underline{k}}(R)_{N_{[g]}} \otimes_R H^0(Z_{[g]_r}, \mathcal{L}(h)) \right\}^{\Gamma_{[g]_r} \cap P_{n-1}(\mathbf{Q})} .$$

(Note that $\Gamma_{[g]_r}(h) \subseteq N_{[g]}$.) The rank condition together with the condition on the kernel forces $\mathrm{rank}(h) = n - r - 1$ in these series.

Let $[g]'_r$ be the genus $2r$ cusp label for $S_{G_{n-1}}(K_{n-1,g})$ attached to g, and let $S^+_{[g]'_r}$ be the lattice of associated Hermitian matrices in $M_{n-1}(\mathcal{K})$. This is naturally identified with the sublattice of $h \in S^+_{[g]_r}$ with $W'_1 \subseteq \ker h$. Let $\Gamma_{[g]'_r} \subseteq G_{n-1}(\mathbf{Q})$ be the subgroup of $H_{n-1}(\mathcal{K})$ associated to $[g]'_r$. Then the inclusion $S^+_{[g]'_r} \hookrightarrow S^+_{[g]_r}$ described above induces a bijection of $S^+_{[g]'_r}/\Gamma_{[g]'_r}$ with the classes in $S^+_{[g]_r}/\Gamma_{[g]_r}$ represented by some h with $\ker h \supseteq W'_1$; that is, with the set of classes of $h \in S^+_{[g]_r}$ with $\ker h = W'_1$ modulo $\Gamma_{[g]_r} \cap P_{n-1}(\mathbf{Q}) = \Gamma_{[g]'_r}$. Let $Z_{[g]'_r}$ be the abelian scheme over $S_{G_r}(K_{r,g})$ associated to $[g]'_r$, and for $h \in S^+_{[g]'_r}$ let $\mathcal{L}(h)'$ be the associated line bundle on $Z_{[g]'_r}$. It is easily seen that there is a canonical identification $H^0(Z_{[g]_r}, \mathcal{L}(h)) = H^0(Z'_{[g]_r}, \mathcal{L}(h)')$ (this is immediate from $W'_1 \subseteq W'_r$). Therefore, the completion of the stalk at $\bar{x}$ of the restriction $i^*(\pi_* \omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}^*}} \mathcal{I}^1)$ is canonically identified with the module of power series

$$\left\{ \sum_{\substack{h \in S^+_{[g]'_r} \\ \mathrm{rank}(h) = n-r-1}} a(h) q^h \ : \ a(h) \in \rho_{\underline{k}}(R)_{N_{[g]}} \otimes_R H^0(Z_{[g]'_r}, \mathcal{L}(h)') \right\}^{\Gamma_{[g]'_r}} .$$

But by Remark 5.5.5, this is just the completion of the stalk at $\bar{x}$ of $(\pi_{[g]})_*(\mathcal{E}_{[g]} \times^{H_{[g]}} \rho_{\underline{k}}(R)_{N_{[g]}}) \otimes_{\mathcal{O}_{S_{G_{n-1}}(K_{n-1,g})}} \mathcal{I}^0_{S_{G_{n-1}}(K_{n-1,r})}$. The rest of the proposition is clear. ■

Corollary 5.5.9. *For any weight $\underline{k}$ there is an exact sequence:*

$$0 \to \mathbf{M}^{n,0}_{\underline{k}}(K, \mathbf{C}) \to \mathbf{M}^{n,1}_{\underline{k}}(K, \mathbf{C}) \to \bigoplus_{[g] \in C_{n-1}(K)} \mathbf{M}^{n-1,0}_{\underline{k}'}(K_{n-1,g}, \mathbf{C}),$$

where the last arrow is the direct sum of the Siegel operators $\Phi_{[g]}$.

Proof. The exact sequence (5.4.9.a) gives rise to an exact sequence

$$0 \to \pi^*\mathcal{I}^0_{\mathcal{S}^*} \to \pi^*\mathcal{I}^1_{\mathcal{S}^*} \to \bigoplus_{[g]\in C_{n-1}(K)} \pi^* i_{[g],*}\mathcal{I}^0_{S^*_{[g]}} \to 0$$

where $i_{[g]} : S^*_{[g]} \hookrightarrow \mathcal{S}^*$ is the canonical inclusion. Since $\omega_{\underline{k}}$ is locally free over $\mathcal{S}$, there is then an exact sequence

$$0 \to \pi^*\mathcal{I}^0_{\mathcal{S}^*} \otimes_{\mathcal{O}_\mathcal{S}} \omega_{\underline{k}} \to \pi^*\mathcal{I}^1_{\mathcal{S}^*} \otimes_{\mathcal{O}_\mathcal{S}} \omega_{\underline{k}} \to \bigoplus_{[g]\in C_{n-1}(K)} \pi^* i_{[g],*}\mathcal{I}^0_{S^*_{[g]}} \otimes_{\mathcal{O}_\mathcal{S}} \omega_{\underline{k}} \to 0.$$

The desired exact sequence follows easily from taking global sections and from the preceding proposition. ∎

5.5.10. *q-expansions.* Let $[g] \in C_0(K)$. The q-expansion at $[g]$ of some $f \in \mathbf{M}^n_{\underline{k}}(K,R)$ is its image in the completion of the stalk $(\pi_*\omega_{\underline{k}})_{\bar{x}}$ at the geometric point $\bar{x}$ corresponding to $[g]$ ($S_{[g]}$ is a point) under the identification of this stalk with the module of formal power series as in Proposition 5.5.4(i). Equivalently, let $\sigma \subset \Sigma_{H^+_{g,\mathbf{R}}}$ be an interior cone in the rational polyhedral cone decomposition used to define $\overline{S}_G(K)$. Recall that there is a canonical map $\operatorname{Spec} R_\sigma \to \overline{S}_G(K)$. The q-expansion of f at g can also be defined as the evaluation of f on the tuple obtained by pulling back the quadruple $(\mathcal{G},\lambda,\iota,\eta)$ to $\operatorname{Spec} R_\sigma \otimes_{\mathcal{O}_{(p)}} R$; the pull-back $\mathcal{G}_\sigma/R_\sigma$ of $\mathcal{G}$ has a canonical basis of $\omega^{\pm}_{/R_\sigma}$ and the choice of a representative g determines an ordering. The q-expansion of f at $[g]$ is independent of the chosen cone σ and so actually takes values in $\mathcal{O}_{(p)}[\![q^{S^+_{[g]}}]\!] \otimes_{\mathcal{O}_{(p)}} \rho_{\underline{k}}(R)$. Note that $S^+_{[g]} = \cap_{\sigma\in\Sigma_{H^+_{g,\mathbf{R}}}}(\sigma^\vee \cap S_{[g]})$. The equivalence of these definitions is essentially formal (the identification of the stalk with a subring of a ring of power series is via Mumford's construction). The reader can consult [FC] for details in the symplectic case, at least for scalar weights. We will write $f_g(q)$ for this q-expansion of f at the cusp $[g] \in C_0(K)$ (note that it depends on the choice of the representative g).

There are injective maps

$$\mathbf{M}^n_{\underline{k}}(K,R), \mathbf{M}^{n,q}_{\underline{k}}(K,R) \hookrightarrow \bigoplus_{g\in G(\mathbf{Q})\backslash G(\mathbf{A}_f)/K} \mathcal{O}_{(p)}[\![q^{S^+_{[g]}}]\!] \otimes_{\mathcal{O}_{(p)}} \rho_{\underline{k}}(R), \quad f \mapsto (f_g(q)),$$

and if $A \subseteq R$ then the submodules $\mathbf{M}^n_{\underline{k}}(K,A)$ and $\mathbf{M}^{n,q}_{\underline{k}}(K,A)$ consist precisely of those f with each $f_g(q)$ having coefficients in A (this is the q-expansion principle). These facts can be established by the same arguments given in [FC] for the symplectic case and scalar weights (see also [Lan08, 7.1.2.15]).

If $R = \mathbf{C}$, then replacing q^h with $e(\operatorname{Tr} hZ)$, $Z \in \mathbf{H}_n$, in the q-expansion of $f \in M_{\underline{k}}(K,\mathbf{C})$ at a cusp $[g]$ yields the Fourier expansion of $f(Z,g)$ defined in (5.5.2.b); a detailed explanation of this for the analytic and algebraic expansions for the general PEL case can be found in [Lan10]. In particular, if R is a subring of $\mathbf{C}$, then $f \in M_{\underline{k}}(K,\mathbf{C})$ belongs to $M_{\underline{k}}(K,R)$ if and only if its Fourier expansion at each genus 0 cusp $[g]$ (equivalently, one such cusp on each connected component of $S^*_G(K)$) has coefficients in R. Similarly, if p is not a zero-divisor in R, then $\mathbf{M}^n_{\underline{k}}(K_0(p^s),\psi,R)$ consists of forms in $\mathbf{M}^n_{\underline{k}}(K_0(p^s),\psi,R[1/p])$ with q-expansion coefficients in R at all genus 0 cusps in $C_0(K_0(p^s))$ (equivalently, one such cusp on each connected component of $S^*_G(K)$).

The image of the Siegel operators can be seen on the q-expansions of modular forms as follows:

$$\Phi_{[g]}(f)(W,g') = \sum_{h=\left(\begin{smallmatrix} h' & 1 \\ 1 & 0\end{smallmatrix}\right)\in S^+_{[g'g]}} a(h,g'g))e(\mathrm{Tr}\, h'W).$$

This is immediate from the definitions.

5.5.11. *Hecke algebras.* Let S be a finite set of primes containing the primes that ramify in $\mathcal{K}$. For $\ell \notin S$ we let $R_\ell := \mathcal{C}^\infty_c(K^0_\ell\backslash G(\mathbf{Q}_\ell)/K^0_\ell,\mathbf{Z})$. This is the local spherical Hecke algebra at ℓ. For any weight $\underline{k}$, there is a natural action of R_ℓ on $\mathbf{M}_{\underline{k}}(K,\mathbf{C})$, defined via the usual action of correspondances. When $n=2$, in 9.5.1 below R_ℓ is denoted $\tilde{\mathcal{H}}_{K^0_\ell}$ and we define a subalgebra $\mathcal{H}'_{K^0_\ell}$ with specified generators. For later use we let $R'_\ell = \mathcal{H}'_{K^0_\ell}$ if $n=2$, and otherwise we let $R'_\ell = R_\ell$. We define a global Hecke algebra $R^S := \otimes' R'_\ell$ to be the restricted tensor product taken with respect to the unit elements of each R'_ℓ. This acts on $\mathbf{M}^{n,q}_{\underline{k}}(K,\mathbf{C})$ through the action of the R'_ℓs. We define $h^{S,q}_{\underline{k}}(K,\mathbf{Z})$ to be the image of R^S in $\mathrm{End}_{\mathbf{C}}(\mathbf{M}^{n,q}_{\underline{k}}(K,\mathbf{C}))$.

Let π be a holomorphic cuspidal representation of $G(\mathbf{A})$ of weight $\underline{k}$ and level K such that $k_n - k_{n-1} \geq 2n$. This means that $\pi = \pi_\infty \otimes \pi_f$ is such that $\pi_f^K \neq 0$ and $\pi_\infty = \Pi_{\underline{k}}$ is the holomorphic discrete series representation with lowest $K^+_\infty Z_\infty$-type given by $k_\infty \mapsto \rho_{\underline{k}} \circ J(k_\infty,\mathbf{i})$. We denote by λ_π the algebra homomorphism $h^{S,0}_{\underline{k}}(K) \to \mathbf{C}$ giving the eigenvalues of the Hecke algebra acting on π_f^K. We will say that λ_π is associated with π. The link between λ_π and the Langlands parameters of an unramified local constituent of π can be described as follows. Let ℓ be unramified in $\mathcal{K}$ and $v|\ell$. Let χ_π be the central character of π (a character of $\mathbf{A}^\times_{\mathcal{K}}$) and put $\psi := \chi^c_\pi$. There exists a specific degree $2n$ polynomial $Q_v(X) \in R_\ell[X]$ such that

$$\lambda_\pi(Q_v)(q_v^{-s}) = L(\pi_v \otimes \psi_v, s-(n+1)/2)^{-1}, \qquad (5.5.11.a)$$

where $L(\pi_v\otimes\psi_v,s)$ is the twist by ψ_v of the standard L-function associated with the v-constituent of the (formal) base change to $\mathrm{GL}_{2n}(\mathbf{A}_{\mathcal{K}})$ of some irreducible $U_n(\mathbf{A})$ constituent of π. The connection with the Langlands parameters follows upon recalling that the Langlands parameters of a spherical representation σ of GL_m over a local field with residue field of cardinality q are the complex numbers $(a_1,\dots,a_m)$ such that $L(\sigma,s) = \prod_{i=1}^m (1-a_i q^{-s})^{-1}$.

Remark 5.5.12. In 9.6 we make explicit the polynomials $Q_v(X)$ in the case $n=2$.

6. Hida theory for unitary groups

In this section, we review some aspects of Hida theory for the groups G_n. This theory is nowadays well understood (see [Hi99],[Hi04],[Ur04]), but since there is no complete reference for the case we need, we give an account, sketching proofs, of what is necessary for our purposes.

6.1. **The Igusa tower and p-adic automorphic forms.** For each $m > 0$ we fix an identification of group schemes $\boldsymbol{\mu}_{p^m} = \operatorname{Spec} \mathbf{Z}[x, x^{-1}]/(x^{p^m} - 1)$ that is compatible with varying m. This fixes a compatible family of $\mathcal{O}_S$-bases $x\frac{d}{dx}$ of $\operatorname{Lie}(\mu_{p^m}/S)$ for any scheme S.

6.1.1. *Some preliminaries.*

Lemma 6.1.2. *Let S be a $\mathbf{Z}_{(p)}$-scheme with non-empty special fiber $S_p := S \times_{\mathbf{Z}_{(p)}} \mathbf{F}_p$. Let $G \to S$ be a semi-abelian scheme over S of relative dimension m such that $G \times_S S_p$ is ordinary. Then for any integer $t > 0$ such that $p^t \mathcal{O}_S = 0$ there is an isomorphism*

$$\operatorname{Hom}_{grp-sch/S}(G[p^t]^\circ, \boldsymbol{\mu}_{p^t}) \otimes_{\mathbf{Z}} \mathcal{O}_S \xrightarrow{\sim} \omega_{G/S} = \operatorname{Hom}_{\mathcal{O}_S}(\operatorname{Lie}(G/S), \mathcal{O}_S)$$

that is canonical up to the chosen identification $\operatorname{Lie}(\boldsymbol{\mu}_{p^t}/S) \cong \mathcal{O}_S$.

Proof. Let $\phi : G[p^t]^\circ \to \boldsymbol{\mu}_{p^t}$ be a homomorphism of group schemes over S. Then ϕ induces a homomorphism $\operatorname{Lie}(\phi) : \operatorname{Lie}(G[p^t]^\circ/S) \to \operatorname{Lie}(\mu_{p^t}/S) \cong \mathcal{O}_S$, the last isomorphism being the fixed one. Since $p^t\mathcal{O}_S = 0$ we have canonical identifications $\operatorname{Lie}(G/S) = \operatorname{Lie}(G[p^t]/S) = \operatorname{Lie}(G[p^t]^\circ/S)$, and so the map $\phi \mapsto \operatorname{Lie}(\phi)$ defines a homomorphism from $\operatorname{Hom}_{grp-sch/S}(G[p^t]^\circ, \boldsymbol{\mu}_{p^t})$ to $\omega_{G/S}$. We extend it $\mathcal{O}_S$-linearly to get a homomorphism

$$\operatorname{Hom}_{grp-sch/S}(G[p^t]^\circ, \boldsymbol{\mu}_{p^t}) \otimes_{\mathbf{Z}} \mathcal{O}_S \to \omega_{G/S}. \tag{6.1.2.a}$$

If $G[p^t]^\circ_{/S} \cong \boldsymbol{\mu}^m_{p^t}{}_{/S}$, then this map is an isomorphism. Since $G \times_S S_p$ is ordinary, such an isomorphism holds étale locally over S. By faithfully flat descent, (6.1.2.a) is therefore an isomorphism.∎

Proposition 6.1.3. *Let S and $G \to S$ be as in the preceding lemma with $m = 2n$ and assume that G has CM by $\mathcal{O}$. Assume also that $\mathcal{O}_S$ is a sheaf of $\mathcal{O}_{\mathfrak{p}}$-algebras and that $\omega_{G/S}$ has a decomposition of the form*

$$\omega_{G/S} = \omega^+_{G/S} \oplus \omega^-_{G/S}$$

as an $\mathcal{O}_S \otimes_{\mathbf{Z}} \mathcal{O} = \mathcal{O}_S \otimes_{\mathcal{O}} \mathcal{O} \times \mathcal{O}_S \otimes_{\mathcal{O},c} \mathcal{O}$-module with $\omega^\pm_{G/S}$ locally free of rank n. If $p^t\mathcal{O}_S = 0$, then there is an étale cover S' of S such that

$$\operatorname{Hom}_{grp-sch}(G[p^t]^\circ, \boldsymbol{\mu}_{p^t})_{/S'} \cong (\mathcal{O}/\mathfrak{p}^t)^n_{/S'} \times (\mathcal{O}/\overline{\mathfrak{p}}^t)^n_{/S'}$$

as schemes with $\mathcal{O}$-actions.

Proof. We may assume that S is connected. Since $\mathrm{Hom}_{grp-sch}(G[p^t]^\circ, \boldsymbol{\mu}_{p^t})$ is étale over S, there exists an étale cover S'/S such that $\mathrm{Hom}_{grp-sch}(G[p^t]^\circ, \boldsymbol{\mu}_{p^t})_{/S'} \cong (\mathcal{O}/\mathfrak{p}^t)^r_{/S'} \times (\mathcal{O}/\overline{\mathfrak{p}}^t)^s_{/S'}$ as group schemes with an $\mathcal{O}\otimes\mathbf{Z}/p^t\mathbf{Z} = \mathcal{O}/\mathfrak{p}^t \times \mathcal{O}/\overline{\mathfrak{p}}^t$-action. Since $r+s = \dim G/S = 2n$, it suffices to check that $r = n$. As $\mathcal{O}_{S'}$ is a sheaf of $\mathcal{O}_{\mathfrak{p}}$-algebras, $\omega^-_{G/S'}[\mathfrak{p}^t] = 0$ and $\omega^+_{G/S'}[\mathfrak{p}^t]$ is locally isomorphic to $(\mathcal{O}_{S'}\otimes_{\mathcal{O}}\mathcal{O}/\mathfrak{p}^t)^n$. From Lemma 6.1.2, applied with S' in place of S, we deduce that $r = n$.■

6.1.4. *The Igusa tower.* Let K^p be an open compact subgroup of $G(\mathbf{A}_f^p)$ such that $K = K_p^0 K^p$ is neat. We denote by $\mathcal{S} = \mathcal{S}_K$ a fixed toroidal compactification of $S_G(K)$ over $\mathcal{O}_{\mathfrak{p}}$ as in §5. We let $\mathcal{G}/\mathcal{S}$ be the corresponding semi-abelian scheme. We write $\mathcal{S}^*$ for the minimal compactification of $S_G(K)$ over $\mathcal{O}_{\mathfrak{p}}$. The latter is flat over $\mathcal{O}_{\mathfrak{p}}$.

We recall the definition of the Hasse invariant H. Let T be a scheme of characteristic p and let F be the absolute Frobenius map. Let $(A,\lambda,\iota,\eta)_{/T}$ be a T-quadruple as in 5.3.4 such that $\omega_{A/T}$ is free over $\mathcal{O}_T$. Let $(\omega_i)_i := (\omega_1,\dots,\omega_{2n})$ be an $\mathcal{O}_T$-basis of $\underline{\omega}$ and $(\eta_1,\dots,\eta_n)$ its dual basis. Then $F^*(\eta_1\wedge\cdots\wedge\eta_n) = H(A,\lambda,\iota,\eta,(\omega_i)_i).(\eta_1\wedge\cdots\wedge\eta_n)$ for some $H(A,\lambda,\iota,\eta,(\omega_i)_i)) \in \Gamma(T,\mathcal{O}_T)$. One checks easily that the rule defining $H(A,\lambda,\iota,\eta,(\omega_i)_i)$ defines a global section of $\det(\omega)^{p-1}_{S_G(K)/\mathbf{F}_p} = \det(\omega_{\mathcal{G}/S_G(K)/\mathbf{F}_p})^{p-1}$. The definition of the Hasse invariant extends to semi-abelian schemes, so the Hasse invariant H extends to a section over $\mathcal{S}/\mathbf{F}_p$ (and hence to a section over the minimal compactification over $\mathbf{F}_p$). As is well-known, the complement of the zero locus of H is the ordinary locus of $\mathcal{S}_{/\mathbf{F}_p}$.

Since $\det(\omega)$ is ample on the minimal compactification $\mathcal{S}^*$, some power of H can be lifted[4] over $\mathcal{O}_{\mathfrak{p}}$. We denote by E a fixed lifting of a power H^m such that $\det(\omega)^{\otimes m}$ is very ample. Then $\mathcal{S}^*[1/E]$ is affine. For any positive integer m, let $S_m := \mathcal{S}[1/E]\times_{\mathcal{O}_{\mathfrak{p}}}\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^m$ and $S_m^* := \mathcal{S}^*[1/E]\times_{\mathcal{O}_{\mathfrak{p}}}\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^m$. Note that S_m^* is affine and contains all the genus 0 cusps. For any positive integer s, we set

$$P_s/\mathcal{S} := \mathrm{Hom}_{fppf-ab-sheaves/S}(\mathcal{G}[p^s]^0, \boldsymbol{\mu}_{p^s}),$$

and similary define $P_s^+/\mathcal{S}$ and $P_s^-/\mathcal{S}$ by replacing $\mathcal{G}[p^s]^0$ with $\mathcal{G}[\mathfrak{p}^s]^0$ and $\mathcal{G}[\bar{\mathfrak{p}}^s]^0$, respectively (so $P_s = P_s^+\oplus P_s^-$).

For $s\geq m$, we write $T_{s,m}/S_m$ for the étale scheme over S_m that represents the étale sheaf over S_m

$$\underline{T}_{s,m} := \underline{\mathrm{Isom}}_{S_m}(P_s,(\mathcal{O}/\mathfrak{p}^s)^n\times(\mathcal{O}/\bar{\mathfrak{p}}^s)^n). \qquad (6.1.4.a)$$

Note that $\underline{T}_{,sm} = \underline{T}^+_{s,m}\oplus\underline{T}^-_{s,m}$, where $\underline{T}^+_{s,m} := \underline{Isom}_{S_m}(P_s^+,(\mathcal{O}/\mathfrak{p}^s))$ and $\underline{T}^-_{s,m} := \underline{Isom}_{S_m}(P_s^-,(\mathcal{O}/\bar{\mathfrak{p}}^s))$. Here all isomorphisms are required to be $\mathcal{O}$-linear. It follows from Proposition 6.1.3 that $T_{s,m}/S_m$ is an étale Galois cover (but not an irreducible cover) with Galois group $\mathrm{GL}_n(\mathcal{O}/\mathfrak{p}^s)\times\mathrm{GL}_n(\mathcal{O}/\bar{\mathfrak{p}}^s) = H(\mathcal{O}/\mathfrak{p}^s)$. The action of $g = (g^+,g^-)\in H(\mathcal{O}/\mathfrak{p}^s)$ on $\phi = (\phi^+,\phi^-)\in T_{s,m}$ is $g\phi = (g^+\phi^+, g^-\phi^-)$.

[4]This follows from Theorem 5.4.7, which shows that the minimal compactification over $\mathbf{F}_p$ (over which H is defined) is the same as the base change to $\mathbf{F}_p$ of the minimal compactification over $\mathcal{O}_{\mathfrak{p}}$.

The étale sheaves $\bigwedge^n T^+_{s,m}$ and $\bigwedge^n T^-_{s,m}$ are constant and even isomorphic to $\bigwedge^n(\mathcal{O}/\mathfrak{p}^s)^n$ and $\bigwedge^n(\mathcal{O}/\bar{\mathfrak{p}}^s)^n$, respectively. Therefore,

$$\underline{Isom}_{S_m}(\bigwedge^n T^+_{s,m} \otimes_{\mathbf{Z}_p} \bigwedge^n T^-_{s,m}, \bigwedge^n(\mathcal{O}/\mathfrak{p}^s)^n \otimes_{\mathbf{Z}_p} \bigwedge^n(\mathcal{O}/\bar{\mathfrak{p}}^s)^n) \cong (\mathbf{Z}/p^s)^\times.$$

We choose isomorphisms compatible with varying s and m. There is an induced map

$$\det : T_{n,s} \to (\mathbf{Z}/p^s)^\times$$

which sends $\phi = (\phi^+, \phi^-)$ to $\det(\phi) := \det\phi^+ \otimes \det\phi^-$. In particular, $\det(g\phi) = \det(g+)\det(g^-)\det(\phi)$.

For an irreducible component S of S_m and a $v \in (\mathbf{Z}/p^s)^\times$, let

$$T^{(v)}_{s,m}/S := \det{}^{-1}(v)/S.$$

The action of $g = (g^+, g^-) \in H(\mathcal{O}/\mathfrak{p}^s)$ on $T_{s,m}/S$ maps $T^{(v)}_{s,m}$ isomorphically onto $T^{(\det(g^+)\det(g^-)v)}_{s,m}$, and each $T^{(v)}_{s,m}$ is stable under the action of

$$H_1(\mathcal{O}/\mathfrak{p}^s) := \{g = (g^+, g^-) \in H(\mathcal{O}/\mathfrak{p}^s) \ : \ \det(g^+)\det(g^-) = 1\}.$$

Theorem 6.1.5. *Each $T^{(v)}_{s,m}/S$ is an irreducible component of $T_{s,m}/S$.*

Proof. This result is due independently to several people. The first version of this theorem is due to Igusa and Ribet (for GL$_2$). It was generalized by Faltings-Chai [FC, §V.7] in the Siegel modular case; by Hida in [Hi04, §8.4] for general PEL-type Shimura varieties by establishing an arithmetic version of Shimura's reciprocity law, with another proof for the unitary case (the case here) given in [Hi08]; and also by Chai [Ch08] for the general PEL case by an argument using Igusa's result for GL$_2$. ∎

Let $\pi_m : T_{s,m} \to S_m$ be the canonical projection. For positive integers s, m, and q with $0 \le q \le n$ we put

(6.1.5.a) $$V^q_{s,m} := \Gamma(T_{s,m}, \mathcal{O}_{T_{s,m}} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{I}^q_{\mathcal{S}}).$$

Lemma 6.1.6. *Let $\underline{k}$ be a dominant weight. Then for all integers $s \ge m > 0$ and any subgroup $J \subseteq \mathrm{Gal}(T_{s,m}/S_m)$ we have a canonical isomorphism:*

$$H^0(J, V^q_{s,m} \otimes_{\mathcal{O}_\mathfrak{p}} \rho_{\underline{k}/\mathcal{O}_\mathfrak{p}/\mathfrak{p}^s}) \xrightarrow{\sim} \Gamma(T_{s,m}/J, \pi_m^*(\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{I}^q_{\mathcal{S}})),$$

the action of J on $V^q_{S,m}$ being via the isomorphism

$$\mathrm{Gal}(T_{s,m}/S_m) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}/\mathfrak{p}^s) \times \mathrm{GL}_n(\mathcal{O}/\bar{\mathfrak{p}}^s) = H(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^s).$$

The canonicalness in this lemma is subject to the same caveat as in Lemma 6.1.2.

Proof. Over $T_{s,m}$ there is a universal isomorphism $P_{s/T_{s,m}} \cong (\mathcal{O}/\mathfrak{p}^s)^n_{/T_{s,t}} \times (\mathcal{O}/\bar{\mathfrak{p}}^s)^n_{/T_{s,t}}$. From Lemma 6.1.2 we deduce a canonical $\mathrm{Gal}(T_{s,m}/S_m)$-equivariant isomorphism

$$((\mathcal{O}/\mathfrak{p}^s)^n_{/T_{s,m}} \times (\mathcal{O}/\bar{\mathfrak{p}}^s)^n_{/T_{s,m}}) \otimes_{\mathbf{Z}} \mathcal{O}_{T_{s,m}} \cong \omega_{\mathcal{G}/T_{s,m}}.$$

The lemma is now an easy consequence of the definition of $\omega_{\underline{k}}$. ■

6.1.7. *p-adic automorphic forms.* Let $I_{j,s}^H := I_{j,s} \cap H(\mathcal{O}_\mathfrak{p}) \subset \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}) \times GL_n(\mathcal{O}_{\overline{\mathfrak{p}}})$ for $j = 0, 1$, where $I_{j,s}$ is as in 5.3.6. We also write I_s^H for $I_{0,s}^H$. For q between 0 and n we put

$$W_{s,m}^q := H^0(I_{1,s}^H, V_{s,m}^q) \text{ and } \mathcal{W}^q := \varinjlim_m (\varinjlim_s W_{s,m}^q).$$

There is a natural action of $I_{0,s}^H / I_{1,s}^H \cong T_H(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^s) \cong (\mathcal{O}_\mathfrak{p}/\mathfrak{p}^s)^{\times,n} \times (\mathcal{O}_{\overline{\mathfrak{p}}}/\overline{\mathfrak{p}}^s)^{\times,n}$ on $W_{s,m}^q$ and hence on $\mathcal{W}^q$, where T_H is the diagonal torus of H.

The module of p-adic automorphic forms on G of weight $\underline{k}$ and level $K = K_p^0 K^p$ with p-divisible coefficients is defined to be the direct limit

$$V_{\underline{k}}^q(K, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p}) := \varinjlim_m \Gamma(S_m, \omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \mathcal{I}_\mathcal{S}^q).$$

Similarly, if A is an $\mathcal{O}_\mathfrak{p}$-algebra, then the modules of p-adic forms with coefficients in A are defined as the inverse limits:

$$V_{\underline{k}}^q(K, A) := \varprojlim_m \Gamma(S_m, (\omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \mathcal{I}_\mathcal{S}^q) \otimes_{\mathcal{O}_\mathfrak{p}} A).$$

More generally, let ψ be a finite order character of $T(\mathbf{Z}_p) \cong T_H(\mathcal{O}_\mathfrak{p})$ (this is the canonical isomorphism). We say that ψ is of level s if it is trivial on the kernel of $T(\mathbf{Z}_p) \to T(\mathbf{Z}/p^s\mathbf{Z})$. For any weight $\underline{k}$ and any finite order character ψ, we denote by $\psi_{\underline{k}}$ the character of $T(\mathbf{Z}_p)$ defined by $t \mapsto t^{\underline{k}}\psi(t)$. Let $\mathcal{O}_\mathfrak{p}(\psi)$ be the extension of $\mathcal{O}_\mathfrak{p}$ generated by the values of ψ. Let $\pi_m : T_{m,m} \to S_m$ be the canonical projection. For any p-adically complete $\mathcal{O}_\mathfrak{p}(\psi)$-algebra R we define

$$V_{\underline{k}}^q(K^p I_s, \psi, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} R) := \varinjlim_m \Gamma(T_{m,m}/I_s^H, \pi_m^*(\omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \mathcal{I}_\mathcal{S}^q) \otimes_{\mathcal{O}_\mathfrak{p}} R)[\psi]$$

and

$$V_{\underline{k}}^q(K^p I_s, \psi, R) := \varprojlim_m \Gamma(T_{m,m}/I_s^H, \pi_m^*(\omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \mathcal{I}_\mathcal{S}^q) \otimes_{\mathcal{O}_\mathfrak{p}} R)[\psi]$$

to be the indicated limits over m of the submodules of global sections that transform by ψ under the natural action of $T(\mathbf{Z}_p/p^s\mathbf{Z})$.

Automorphic forms as p-adic automorphic forms. There is a canonical injection

$$\mathbf{M}_{\underline{k}}^{n,q}(K, R) \hookrightarrow V_{\underline{k}}^q(K, R). \tag{6.1.7.a}$$

This follows from Corollary 6.1.6. Similarly, there is also an injection

$$\mathbf{M}_{\underline{k}}^{n,q}(K^p I_s, \psi, R) \hookrightarrow V_{\underline{k}}^q(K^p I_s, \psi, R). \tag{6.1.7.b}$$

This last injection can seen by considering (p-adic) modular forms as functions of suitable test objects.

q-expansions of p-adic automorphic forms. For $g \in G(\mathbf{A}_f)$ with $g_p \in Q(\mathbf{Z}_p)$ we define a q-expansion map

$$\Gamma(T_{s,m}, \mathcal{O}_{T_{s,m}}) \to \mathbf{Z}/p^m\mathbf{Z}[\![q^{S^+_{[g]}}]\!], \quad f \mapsto f_g(q).$$

The semi-abelian scheme $\mathcal{G}_\sigma/R_\sigma$ for an interior cone $\sigma \in \Sigma_{H^+_{g,\mathbf{R}}}$ (for $r = 0$) comes equipped with a trivialization $\omega_{\mathcal{G}} \cong ((\mathcal{O}/\mathfrak{p}^s)^n \times (\mathcal{O}/\bar{\mathfrak{p}}^s)^n) \otimes R_{\sigma,m}$ over $R_{\sigma,m} := R_\sigma \otimes_{\mathcal{O}_{(p)}} \mathcal{O}_\mathfrak{p}/\mathfrak{p}^m$ that depends on g and so gives rise to an S_m-map $\operatorname{Spec} R_{\sigma,m} \to T_{s,m}$ (that depends also on the chosen isomorphism $\mathrm{Lie}(\boldsymbol{\mu}_{p^s}/R_{\sigma,m}) \cong R_{\sigma,m}$). The q-expansion at g of $f \in \Gamma(T_{s,m}, \mathcal{O}_{T_{s,m}})$ is just its pull-back under this map. The q-expansion maps are compatible with varying m and s and so by taking limits we obtain maps

$$V^q_{\underline{k}}(K, R),\ V^q_{\underline{k}}(K^pI_s, \psi, R) \to R[\![q^{S^+_{[g]}}]\!].$$

When $\underline{k}$ is a parallel weight these are just the usual q-expansion maps at the genus 0 cusp $[g]$ on the spaces of modular forms $\mathbf{M}^{n,q}_{\underline{k}}(K, R)$ and $\mathbf{M}^{n,q}_{\underline{k}}(K^pI_s, \psi, R)$. When $\underline{k}$ is not parallel, these are the compositions of the usual q-expansions with a projection to a highest weight vector (this will not be needed).

Let $T^*_{s,m}$ be the normalization of S^*_m in $T_{s,m}$. Then $T^*_{s,m}/S^*_m$ is again an étale Galois cover with Galois group $H(\mathcal{O}/\mathfrak{p}^s)$, and there are well-defined cusps on $T^*_{s,m}$. The group $H(\mathcal{O}/\mathfrak{p}^s)$ acts transitively on the set of cusps over a given cusp on S^*_m. The q-expansions $f_g(q)$ are naturally identified with the images of f in the completions of the stalks of $\pi^*_m(\omega_{\underline{k}} \otimes_{\mathcal{O}_S} \mathcal{I}^q_{\mathcal{S}}) \otimes_{\mathcal{O}_\mathfrak{p}} R$ at the geometric points corresponding to these cusps.

The q-expansion principle. The geometrically irreducible components of $\mathcal{S}^*/\mathcal{K}$ (which are identified with the connected components of $Sh_G(K)(\mathbf{C})$) are the generic fibres of the irreducible components of $\mathcal{S}^*$, which are in bijection with the irreducible components of the special fiber $\mathcal{S}^*/\mathbf{F}_p$ (this follows from the same result for $\mathcal{S}$ in place of $\mathcal{S}^*$, and for this see [Lan08, 6.4.1.2,6.4.1.4]). In particular each irreducible component of S^*_m contains a genus 0 cusp (the Hasse invariant does not vanish at the cusps of $S^*/\mathbf{F}_p$), and hence so does each irreducible component of $T^*_{s,m}$.

Let S be an irreducible component of S^*_m. It follows from Theorem 6.1.5 that the irreducible components of $T^*_{s,m}/S$ are just the normalizations of S in the $T^{(v)}_{s,m}/S$ for $v \in (\mathbf{Z}/p^s)^\times$. It then follows that a p-adic modular in $V^q_{\underline{k}}(K, R)$ or $V_{\underline{k}}(K^pI_t, \psi, R)$ is zero if and only if its q-expansions $f_g(q)$ vanish for all g in a set $X \subset G(\mathbf{A}^p_f)Q(\mathbf{Z}_p)$ that contains at least one g corresponding to a cusp on each irreducible component of S^*_m. Here we use that the orbit of the action of $T_H(\mathcal{O}/\mathfrak{p}^s)$ on a cusp of $T^*_{s,m}$ over one on S contains a cusp on each irreducible component of $T^*_{s,m}/S$. In particular, if $X(K)$ is a (finite) set of representatives of $G(\mathbf{Q})\backslash G(\mathbf{A}_f)/K$ with $x_p \in Q(\mathbf{Z}_p)$ for each $x \in X(K)$, then

$$V^q_{\underline{k}}(K^pI_s, \psi, R) \hookrightarrow \oplus_{x\in R} R[\![q^{S^+_{[x]}}]\!], \quad f \mapsto (f_x(q))_{x\in X(K)},$$

is injective.

6.2. **Ordinary automorphic forms.**

6.2.1. *Hida's ordinary idempotent.* Let $\mathcal{U}_p$ be the subalgebra of $C_c^\infty(I_1\backslash G(\mathbf{Q}_p)/I_1)$ generated by the characteristic functions of the double classes $u_t := I_1tI_1$ with $t = \mathrm{diag}(t_1,\dots,t_{2n}) \in T(\mathbf{Z}_p)$ satisfying the contraction property

$$t^{-1}B(\mathbf{Z}_p)t \subset B(\mathbf{Z}_p), \tag{6.2.1.a}$$

or, equivalently,

$$t_2/t_1, ..., t_n/t_{n-1}, t_{n+1}/t_n, t_{n+1}/t_{n+2}, ..., t_{2n-1}/t_{2n} \in \mathbf{Z}_p. \tag{6.2.1.b}$$

Recall that T is the diagonal torus of $U_{n/\mathcal{O}} = \mathrm{GL}_n$ and we identify $\mathbf{Z}_p$ with $\mathcal{O}_\mathfrak{p}$.

If M is a compact $\mathbf{Z}_p$-module equipped with a continuous action of $\mathcal{U}_p$, we will denote by M_{ord} the maximal submodule of M on which the operators $u_t \in \mathcal{U}_p$ are invertible. This is a $\mathbf{Z}_p$-direct summand of M and the projector e^{ord} of M onto M_{ord}, called Hida's ordinary idempotent, satisfies

$$e_{\mathrm{ord}} = \lim_{n\to\infty} u_{t^+}^{n!}$$

for any t^+ in the set of T^+ of elements in $T(\mathbf{Q}_p)\cap M_{2n}(\mathcal{O}_\mathfrak{p})$ satisfying (6.2.1.b) but with the ratios all belonging to $p\mathbf{Z}_p$.

6.2.2. *Ordinary p-adic modular forms.* There is a natural action of $\mathcal{U}_p$ on the spaces of mod p^m modular forms $W^q_{s,m}$ and on $V^q_{\underline{k}}(K^pI_s, \psi, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} A)$ for any weight $\underline{k}$ and any $\mathcal{O}_\mathfrak{p}(\psi)$-algebra A. We do not recall the definition of this action, refering to [SU06] or [Hi04] for a definition using correspondences on the Igusa tower. For f in $M^q_{\underline{k}}(K, A)$ or $M^q_{\underline{k}}(K^pI_s, \psi, A)$ with $A \subset \mathbf{C}$ we have

$$u_t.f = |[\underline{k}^*](t)|_p^{-1} \cdot f|_{\underline{k}}u_t \tag{6.2.2.a}$$

where $[\underline{k}^*] = [\underline{k} + (n,\dots,n;-n,\dots,-n)]$ is the algebraic character of T defined in 5.5.1 and $f|_{\underline{k}}u_t$ denotes the usual Hecke action (for $u_t = \sqcup I_t u_i$ with $u_i \in G^+(\mathbf{Q})$, $(f|_{\underline{k}}u_t)(Z, g) = \sum_i f(Z, gu_i^{-1})$). Note that if $f \in M^q_{\underline{k}}(K, A)$ then $u_t \cdot f \in M^q_{\underline{k}}(K^pI_1, A)$.

We use the subscript 'ord' to denote the ordinary parts of the various modules we consider. As $V^q_{\underline{k}}(K, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} A) \subseteq \mathcal{W}^q$ we can also define $V^q_{\underline{k},\mathrm{ord}}(K, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} A) := e_{\mathrm{ord}}.V^q_{\underline{k}}(K, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} A) \subseteq \mathcal{W}^q_{\mathrm{ord}}$.

Lemma 6.2.3. *For any weight $\underline{k}$, we have canonical isomorphisms*

$$V^q_{\underline{k},\mathrm{ord}}(K, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p}) \cong \mathcal{W}^q_{\mathrm{ord}}[\underline{k}] := \{w \in \mathcal{W}^q_{\mathrm{ord}} \ : \ t.w = t^{\underline{k}}w \ \forall t \in T_H(\mathcal{O}_\mathfrak{p})\}$$

and

$$\begin{aligned} V^q_{\underline{k},\mathrm{ord}}(K^pI_s, \psi, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} A) &\cong (\mathcal{W}^q_{\mathrm{ord}} \otimes_{\mathcal{O}_\mathfrak{p}} A)[\psi_{\underline{k}}] \\ &:= \{w \in \mathcal{W}^q_{\mathrm{ord}} \otimes_{\mathcal{O}_\mathfrak{p}} A \ : \ t.w = \psi_{\underline{k}}(t)w \ \forall t \in T_H(\mathcal{O}_\mathfrak{p})\} \end{aligned}$$

for any $\mathcal{O}_\mathfrak{p}(\psi)$-algebra A.

Proof. This follows from an argument due to Hida. We sketch a proof of the first isomorphism. The second is obtained in an identical way. Notice that it is sufficient to show that

$$e_{\mathrm{ord}}.H^0(S_m, \omega_{\underline{k}} \otimes \mathcal{I}^q) \cong W^q_{s,m,\mathrm{ord}}[\underline{k}] \tag{6.2.3.a}$$

for $s \geq m$. By Corollary 6.1.6, the left-hand side of (6.2.3.a) is canonically isomorphic to $e_{\mathrm{ord}}.H^0(H(\mathcal{O}/\mathfrak{p}^s), V^q_{s,m} \otimes \rho_{\underline{k}/\mathcal{O}_\mathfrak{p}/\mathfrak{p}^m})$. On the other hand, $\rho_{\underline{k}}$ can be realized as the space of algebraic functions f from H to the affine line $\mathbf{A}^1$ such that $f(tug) = t^{-\underline{k}}f(g)$ for any $t \in T_H$, $g \in H$, and u a lower unipotent matrix, the action of $g \in H$ on f being defined by $(h.f)(g) = f(gh)$. There is therefore an evaluation map $ev:\ f \mapsto f(id)$ from $\rho_{\underline{k}}$ to $\mathbf{A}^1$ satisfying $ev(t.f) = t^{\underline{k}}ev(f)$. This evaluation map induces a map

$$H^0(H(\mathcal{O}/\mathfrak{p}^s), V^q_{s,m} \otimes \rho_{\underline{k}}) \to W^q_{s,m}[\underline{k}] \tag{6.2.3.b}$$

which is easily seen to be an isomorphism of ordinary parts. Indeed, $V^q_{s,m} \otimes \rho_{\underline{k}/\mathcal{O}_\mathfrak{p}/\mathfrak{p}^m}$ can be identified with the algebraic functions f from H to $V^q_{s,m}$ satisfying $f(tng) = t^{-\underline{k}}f(g)$ with the action of $H(\mathcal{O}/\mathfrak{p}^s)$ given by $(h.f)(g) = h.f(gh)$. The inverse of (6.2.3.b) is then defined by $w \mapsto f_w$ with $f_w(h) = h^{-1}.w$. If f is ordinary (i.e., $e_{\mathrm{ord}}.f = f$), it follows from the contraction properties of the u_t-operators that $h \mapsto f_w(h)$ is algebraic. This implies that this map is an isomorphism of the ordinary parts. ∎

6.2.4. *A base change proposition.*

Proposition 6.2.5. *Let $q = 0$ or 1. For any sufficiently regular weight $\underline{k} \geq 0$, the base-change morphism*
(6.2.5.a)
$$e_{\mathrm{ord}}.\Gamma(\mathcal{S}^*[1/E], \pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \pi^*\mathcal{I}^q) \otimes \mathbf{Z}/p^m\mathbf{Z}) \to e_{\mathrm{ord}}.\Gamma(\mathcal{S}^*[1/E], \pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \pi^*\mathcal{I}^q \otimes \mathbf{Z}/p^m\mathbf{Z}))$$
is an isomorphism.

The right hand side is the image under e_{ord} of

$$\Gamma(\mathcal{S}^*[1/E], \pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \pi^*\mathcal{I}^q \otimes \mathbf{Z}_p/\mathfrak{p}^m\mathbf{Z}_p)) = \Gamma(\mathcal{S}_m, \omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \pi^*\mathcal{I}^q) \subseteq \mathcal{W}^q_m.$$

Similarly, the left hand side is the image under e_{ord} of

$$\begin{aligned}\Gamma(\mathcal{S}^*[1/E], \pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \pi^*\mathcal{I}^q) \otimes \mathbf{Z}_p/p^m\mathbf{Z}_p) &= \Gamma(\mathcal{S}^*[1/E], \pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \pi^*\mathcal{I}^q)) \otimes \mathbf{Z}_p/p^m\mathbf{Z}_p \\ &= \Gamma(\mathcal{S}[1/E], \omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \pi^*\mathcal{I}^q) \otimes \mathbf{Z}_p/p^m\mathbf{Z}_p,\end{aligned}$$

(the second of these equalities follows from $\mathcal{S}^*[1/E]$ being affine) which is canonically identified with a submodule of $\Gamma(\mathcal{S}_m, \omega_{\underline{k}} \otimes_{\mathcal{O}_\mathcal{S}} \pi^*\mathcal{I}^q) \subseteq \mathcal{W}^q_m$. Note that the left hand side of (6.2.5.a)is canonically a submodule of the right hand side and the map is the canonical injection.

The following is an immediate corollary of the above proposition.

Corollary 6.2.6. *Assume $q = 0$ or 1. For any sufficiently regular weight $\underline{k}$ the module $V^q_{\underline{k},\mathrm{ord}}(K, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p})$ is divisible.*

The case $q = 0$ of Proposition 6.2.5 is proved in [Hi04]. In this case an analysis of stalks yields an isomorphism $\pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_S} \mathcal{I}^0) \otimes \mathbf{Z}_p/p^m\mathbf{Z}_p \cong \pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_S} \mathcal{I}^0 \otimes \mathbf{Z}_p/p^m\mathbf{Z}_p)$ and there is no need of recourse to the ordinary projector. However, it turns out that the corresponding statement is not true in general for $q > 0$ if one does not take the ordinary part.

Turning to the proof of the general situation we note that it is readily seen that it suffices to prove the case $m = 1$. In this case, provided $\underline{k}$ is sufficiently regular, the action of each u_t agrees with an action of a Hecke operator (the usual action of KtK twisted by a power of p depending on t; this is explained in [Hi04]) and so we have a natural action of u_t and hence of e_{ord} on the global section of the sheaves, and this action can be described for the images of global sections in stalks as follows.

To simplify matters - and because it is sufficient for our needs - we assume $n = 2$. We fix $\bar{x}$ a geometric point of S_1^* and assume that $\bar{x}$ is a cusp of genus 0. That is, $\bar{x}$ is the geometric point of some $S_{[g]}$ attached to a class $[g] \in C_0(K)$. As recorded in Proposition 5.5.4, the Mumford construction over the cusp $\bar{x}$ yields a canonical isomorphism of the completion of the stalk $(\pi_*\omega_{\underline{k}})_{\bar{x}}$ with

$$H^0(\Gamma_{[g]}, \rho_{\underline{k}}(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}) \otimes_{\mathcal{O}_{\mathfrak{p}}} R_g),$$

where

$$R_g := \{f = \sum_{h \in S^+_{[g]}} a(f,h)q^h, a(f,h) \in \bar{\mathcal{O}}_{\mathfrak{p}}\},$$

and $\Gamma_{[g]} = \Gamma$ and $S^+_{[g]} = S^+$ are as in 5.4.8.

Let t satisfy (6.2.1.a) and consider a decomposition:

$$I_1^H t I_1^H = \sqcup_i I_1^H \gamma_i$$

with $\gamma_i = \mathrm{diag}({}^t\bar{a}_i^{-1}, a_i) \in M_Q(\mathbf{Q}) \cap K$. Then there is a decomposition

$$u_t = I_1 t I_1 = \sqcup_i \sqcup_{n \in N_i} I_1 \gamma_i n$$

with $N_i \subset N_Q(\mathbf{Q}) \cap K$ a full set of representatives of $N_Q(\mathbf{Z}_p)/\gamma_i^{-1} N_Q(\mathbf{Z}_p)\gamma_i$. It is easy to check that the sets N_i all have the same cardinality $[N_Q(\mathbf{Z}_p) : t^{-1}N_Q(\mathbf{Z}_p)t]$ (and this is the denominator used to define the action of u_t on the space of p-adic modular forms). For any $\mathcal{O}_{\mathfrak{p}}$-algebra A, we define an action of u_t on $\tilde{R}_{g,\underline{k}/A} := H^0(\Gamma_{[g]}, \rho_{\underline{k}}(A) \otimes_{\mathcal{O}_{\mathfrak{p}}} R_g)$ as follows. For $f \in \tilde{R}_{g,\underline{k}/A}$ we put

$$u_t.f := \sum a(h, u_t.f)q^h, \quad a(h, u_t.f) = \sum_i \rho_{\underline{k}}(\gamma_i)^{-1}.a(a_i.h, f).$$

This coincides with the action of u_t on global sections when $A = \mathcal{O}_{\mathfrak{p}}$.

If $h \in S^+_{[g]}$ is positive definite then $\Gamma_{[g]}(h)$ is trivial and we set $N_h = 1$. Suppose then that h is rank 1 and let N_h be the unipotent radical of the Borel subgroup of H that stabilizes the kernel of h (that is, of the Hermitian form defined by h). As $\mathcal{K}$ is imaginary quadratic and K is neat, $\Gamma_{[g]}(h) \subset N_h(\mathcal{K})$. In both cases we write $\rho_{\underline{k}}^{N_h}$ for the polynomial functions that are invariant under the action of the algebraic group N_h. Then for an

$\mathcal{O}_{\mathfrak{p}}$-algebra A, $H^0(\Gamma_{[g]}(h), \rho_{\underline{k}}(A))$ contains $\rho_{\underline{k}}^{N_h}(A)$. In general this containment is not an equality in the rank one case, though it is if A has characteristic zero. (This is why the proposition is false in general without taking ordinary parts.) There is an identification

$$\tilde{R}_{g,\underline{k}/A} = \prod_h H^0(\Gamma_{[g]}(h), \rho_{\underline{k}}(A)),$$

where the product runs over a complete set of representatives of the equivalence classes of non-zero matrices in $\Gamma_{[g]}\backslash S^+_{[g]}$. Let $\bar{R}_{g,\underline{k}/A} \subset \tilde{R}_{g,\underline{k}/A}$ consist of those f identified with an element in $\prod_h \rho_{\underline{k}}^{N_h}(A)$. What is useful about $\bar{R}_{g,\underline{k}/A}$ is that its formation commutes with base change: for any A-algebra B we have

$$\bar{R}_{g,\underline{k}/A} \otimes B = \bar{R}_{g,\underline{k}/B}.$$

Lemma 6.2.7. *For any $\mathcal{O}_{\mathfrak{p}}$-algebra A in which p is topologically nilpotent,*

$$e_{\mathrm{ord}}.\tilde{R}_{g,\underline{k}/A} = e_{\mathrm{ord}}.\bar{R}_{g,\underline{k}/A}.$$

In particular, the formation of $e_{\mathrm{ord}}.\tilde{R}_{g,\underline{k}/A}$ commutes with base change to any $\mathcal{O}_{\mathfrak{p}}$-algebra in which p is topologically nilpotent.

Proof. We assume p is nilpotent in A. We need to prove that for any $f \in \tilde{R}_{g,\underline{k}/A}$ there exists t satisfying (6.2.1.a) such that $u_t.f \in \bar{R}_{g,\underline{k}/A}$. Let m be such that $p^m A = 0$ and choose t such that $t^{-1}N_H(\mathbf{Z}_p)t \subset N_H(p^m\mathbf{Z}_p)$ with N_H denoting the unipotent radical of B_H. We need to show that $a(h, u_t.f) \in \rho_{\underline{k}}^{N_h}(A)$. If h is positive definite this is trivial. Let h be of rank one. Since N_h is a conjugate of the unipotent radical N_H of B_H, we can choose the system of representatives of h modulo the action of $\Gamma_{[g]}$ such that either (a) $N_h = N_H$ or (b) $N_h = N_H^-$ is the opposite unipotent subgroup.

Recall that

$$a(h, u_t.f) = \sum_i \rho_{\underline{k}}(\gamma_i^{-1}).a(a_i.h)$$

where the γ_i are a full set of representatives of $I_1^H t I_1^H / I_1^H$ that can be chosen of the form $\gamma_i = n_i t$ with $n_i \in N_H(\mathbf{Z}_p)$. Then an easy computation shows that $a(a_i.h)$ is independent of i in case (a), which implies that $a(h, f|e_{\mathrm{ord}}) = 0$ in that case. In case (b) a similar computation shows that $a(a_i.h, f) = 0$ unless $\gamma_i = t$. This implies that $a(h, u_t.f) = \rho_{\underline{k}}(t)^{-1}.a(t.h, f)$. By the contraction property satisfied by t, this easily implies the desired result. ∎

Proof of Proposition 6.2.5

We treat the case $q = 1$ and $n = 2$ since the case $q = 0$ is proved by Hida in [Hi04]. The general case $n \geq 3$ is also true by similar arguments, but we omit the proof here as it is not needed for the main results of this paper. Recall that it suffices to prove the case $m = 1$.

Let $f \in e_{\rm ord}.\Gamma(S^*[1/E], \pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}}} \pi^*\mathcal{I}^1 \otimes \mathbf{Z}/p\mathbf{Z}))$. It suffices to show that f is in the image of the base change map

$$\Gamma(\mathcal{S}^*[1/E], \pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}}} \pi^*\mathcal{I}^q) \otimes \mathbf{Z}/p\mathbf{Z}) \to \Gamma(\mathcal{S}^*[1/E], \pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}}} \pi^*\mathcal{I}^q \otimes \mathbf{Z}/p\mathbf{Z})).$$

As $S^*[1/E]$ is affine, to prove that f is in this image, it suffices to show that the image of f in the completion of the stalk $\pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}}} \pi^*\mathcal{I}^q \otimes \mathbf{Z}/p\mathbf{Z})_{\bar{x}}$ at each geometric point $\bar{x}$ of $\mathcal{S}^*[1/E]$ is in the image of the map coming from the base change map on stalks. If $\bar{x}$ is of genus 4 then f is clearly in this image as π is an isomorphism on the Zariski open genus 4 locus $S_G(K)$. The same holds if $\bar{x}$ is of genus 2. For by Proposition 5.5.4 and Remark 5.5.5, the completion of the stalk $\pi_*(\omega_{\underline{k}} \otimes \mathcal{I}^1)_{\bar{x}}$ is identified with

$$\prod_h H^0(\Gamma_{[g]}(h), H^0(Z_{[g]}, \mathcal{L}(h))_{\bar{x}} \otimes_{\mathcal{O}_{\mathfrak{p}}} \rho_{\underline{k}}(\mathcal{O}_{\mathfrak{p}})),$$

where h runs over the set of equivalence classes modulo $\Gamma_{[g]}$ of non-zero elements of $S_{[g]}^+$. But because K is neat and the rank of the units of $\mathcal{K}$ is zero[5], the groups $\Gamma_{[g]}(h)$ are all trivial. This implies that the reduction modulo p map is an isomorphism on the stalks at $\bar{x}$ since for each h the formation of $H^0(Z_{[g]}, \mathcal{L}(h))_{\bar{x}}$ commutes with base change. Finally, we note that if $\bar{x}$ is of genus 0 then, as noted above, the ordinary part of the completion of the stalk $\pi_*(\omega_{\underline{k}} \otimes \mathcal{I}^1)_{\bar{x}}$ is identified with $e_{\rm ord}.\tilde{R}_{g,\underline{k}/\mathbf{Z}_p}$, which commutes with the mod p base change map by Lemma 6.2.7. As f is assumed ordinary, it is therefore in the image of the base change map on stalks at $\bar{x}$. This completes the proof of the proposition.

6.2.8. *Ordinary and classical forms.* For $t \in T(\mathbf{Q}_p)$ satisfying (6.2.1.a) we define an action of the Hecke operator $T_t := K_p^0 t K_p^0$ on $M_{\underline{k}}^q(K, A)$, A a subring of $\mathbf{C}$, by $T_t.f := |[k^*](t)|_p^{-1} \cdot f|_{\underline{k}} T_t$ with $f|_{\underline{k}} T_t$ the usual Hecke action. For $t \in T^+$ we can define a projector $e_{\rm ord}^0 = \varinjlim_m T_t^{m!}$ on any compact $\mathbf{Z}_p$-module on which T_t acts (this is independent of t). In particular, as $M_{\underline{k}}^q(K, \mathbf{C}) = M_{\underline{k}}^q(K, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{C}$ there is an action of $e_{\rm ord}^0$ on $M_{\underline{k}}^q(K, \mathbf{C})$.

Lemma 6.2.9. *Assume $q = 0$ or 1. The dimension of $e_{\rm ord}^0.\mathbf{M}_{\underline{k}}^{n,q}(K, \mathbf{C})$ is bounded independent of $\underline{k}$ as the weight $\underline{k}$ varies.*

Proof. For $q = 0$ this is due to Hida. For $q = 1$ this follows from the $q = 0$ case and the exact sequence

$$0 \to e_{\rm ord}^0.\mathbf{M}_{\underline{k}}^{n,0}(K, \mathbf{C}) \to e_{\rm ord}^0.\mathbf{M}_{\underline{k}}^{n,1}(K, \mathbf{C}) \to \bigoplus_{[g] \in C_{n-1}(K)} e_{\rm ord}^0.\mathbf{M}_{\underline{k'}}^{n-1,0}(K_{n-1,g}, \mathbf{C})$$

obtained from Corollary 5.5.9.■

[5]Proposition 6.2.5 is no longer true if $\mathcal{K}$ is of degree > 2 over $\mathbf{Q}$ since the rank of the group of units is positive.

Theorem 6.2.10. *Assume $q = 0$ or 1. For any sufficiently regular weight $\underline{k}$ there is a constant $C(\underline{k}) > 0$ depending on $\underline{k}$ such that for any integer $l > C(\underline{k})$, the canonical map*

$$e_{\mathrm{ord}}.\mathbf{M}^q_{\underline{k}+l(p-1)\underline{t}}(K, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p}) \hookrightarrow V^q_{\underline{k}+l(p-1)\underline{t},\mathrm{ord}}(K, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p}) \tag{6.2.10.a}$$

with $\underline{t} = (0, ..., 0; 1, \dots, 1)$ is an isomorphism.

Proof. This is proved in [Hi04] for $q = 0$. The same proof works for $q = 1$ using Lemma 6.2.9 and Corollary 6.2.6 and the fact that $e_{\mathrm{ord}}e^0_{\mathrm{ord}} = e_{\mathrm{ord}}$.∎

6.3. Λ-adic ordinary automorphic forms.

6.3.1. *Weight algebras and arithmetic characters.* Let $T = T_n$ be the diagonal torus of U_n. The identification of $U_n(\mathbf{Z}_p)$ with $\mathrm{GL}_{2n}(\mathbf{Z}_p)$ identifies $T(\mathbf{Z}_p)$ with $(\mathbf{Z}_p^\times)^{2n}$. In particular, the pro-p-Sylow subgroup $\Gamma_n \subset T(\mathbf{Z}_p)$ has rank $2n$ over $\mathbf{Z}_p$. We let $\Lambda_n := \mathbf{Z}_p[\![\Gamma_n]\!]$ and $\underline{\Lambda}_n := \mathbf{Z}_p[\![T(\mathbf{Z}_p)]\!]$. Letting Δ_n be the torsion subgroup of $T(\mathbf{Z}_p)$, we have canonical isomorphisms $T(\mathbf{Z}_p) = \Delta_n \times \Gamma_n$ and $\underline{\Lambda}_n = \Lambda_n[\Delta_n]$. For any dominant algebraic weight $\underline{k}$, the algebraic character $[\underline{k}]$ of T defined in (5.5.1.a) defines a continuous $\mathbf{Z}_p$-valued character of Γ_n and extends by continuity to a homomorphism of Λ_n that we continue to denote by $[\underline{k}]$. A $\overline{\mathbf{Q}}_p$-valued character ψ of Γ_n (or $\underline{\Lambda}_n$) will be called arithmetic if it has a decomposition $\psi = \psi_0[\underline{k}]$ with ψ_0 of finite order and some dominant weight $\underline{k}$. We say that such a character ψ is of level p^r if ψ_0 factors through the canonical map $\Gamma_n \to T(\mathbf{Z}/p^r\mathbf{Z})$. For any arithmetic character ψ we denote also by ψ the induced homomorphism from Λ_n to $\overline{\mathbf{Q}}_p$ and write P_ψ for the kernel of this homomorphism. Note that the image of ψ is $\mathcal{O}_\mathfrak{p}(\psi_0)$. We will write $P_{\underline{k}}$ for $P_{[\underline{k}]}$. For ψ of finite order we write $\psi_{\underline{k}}$ for the character $\psi[\underline{k}]$.

For any $\underline{a} = (a_1, \dots, a_n; a_{n+1}, \dots, a_{2n}) \in (\mathbf{Z}/(p-1)\mathbf{Z})^{2n}$ we denote by $\omega^{\underline{a}}$ the character of Δ_n defined by $\mathrm{diag}(t_1, \dots, t_{2n}) \mapsto t_1^{a_1} \cdots t_{2n}^{a_{2n}}$.

For $i = 1, \dots, 2n$, let $\delta_i : \mathbf{Z}_p^\times \to T(\mathbf{Z}_p)$ be the co-character defined by $\prod_{i=1}^{2n}(\delta_i(x_i)) = \mathrm{diag}(x_{n+1}, \dots, x_{2n}, x_1, \dots, x_n)$. We write $\boldsymbol{\delta}_i$ for the projection of δ_i to Γ_n.

6.3.2. *Freeness over Λ_n.* Let $\mathcal{V}^q_{\mathrm{ord}} := e_{\mathrm{ord}}.\mathcal{W}^q$ and let $\mathbf{V}^q_{\mathrm{ord}}$ be its Pontrjagin dual. These carry an action of $T(\mathbf{Z}_p)$ and therefore of $\underline{\Lambda}_n$. For any $\underline{a} \in \mathbf{Z}^{2n}$ we let $\mathbf{V}^q_{\underline{a},\mathrm{ord}}$ be the subspace of $\mathbf{V}^q_{\mathrm{ord}}$ on which Δ_n acts via $\omega^{\underline{a}}$. This is a Λ_n-direct summand as Δ_n has order prime to p.

Theorem 6.3.3. *For $q = 0$ or 1, $\mathbf{V}^q_{\underline{a},\mathrm{ord}}$ is free of finite rank over Λ_n. In particular, if $\psi : \Gamma_n \to \overline{\mathbf{Q}}_p^\times$ has finite order and is of level s, then $V^q_{\underline{k},\mathrm{ord}}(K^pI_s, \psi\omega^{\underline{a}-\underline{k}}, \mathcal{K}_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} \mathcal{O}_\mathfrak{p}(\psi))^*$ is free of finite rank over $\mathcal{O}_\mathfrak{p}(\psi)$ for all $\underline{k}$, and this rank is independent of ψ and $\underline{k}$.*

Recall that the superscript '$*$' denotes the Pontrjagin dual.

Proof. For any weight $\underline{k}$, by Lemma 6.2.3 there is an isomorphism

$$\mathbf{V}^q_{\underline{a},\mathrm{ord}} \otimes_{\Lambda_n} \Lambda_n/P_{\psi[\underline{k}]} \cong V^q_{\underline{k},\mathrm{ord}}(K, \psi\omega^{\underline{a}-\underline{k}}, \mathcal{K}_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p}}} \mathcal{O}_{\mathfrak{p}}(\psi))^*. \tag{6.3.3.a}$$

In particular, if $\underline{k}$ is sufficiently regular and congruent to $\underline{a}$ modulo $p-1$, this implies by Theorem 6.2.10 that $V^q_{\underline{k},\mathrm{ord}}(K^pI_1, \omega^{\underline{a}-\underline{k}}, \mathcal{K}_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}})^*$ is finite over $\mathcal{O}_{\mathfrak{p}}$. Therefore by compactness $\mathbf{V}^r_{\underline{a},\mathrm{ord}}$ is finite over Λ_n. This together with Corollary 6.2.6 implies that $\mathbf{V}^r_{\mathrm{ord}} \otimes_{\Lambda_n} \Lambda_n/P_{\underline{k}}$ is free of the same finite rank for all suficiently regular weights $\underline{k}$. Since the ideals $P_{\underline{k}}$ form a Zariski dense set of Spec$\,(\Lambda_n)$ as $\underline{k}$ runs over the sufficiently regular weights, we deduce easily that $\mathbf{V}^r_{\mathrm{ord}}$ is free of finite rank over Λ_n. The rest of the theorem then follows from (6.3.3.a).∎

6.3.4. *Λ_n-adic forms.* Let $\mathcal{M}^q_{\underline{a},\mathrm{ord}}(K^p, \Lambda_n) := \mathrm{Hom}_{\Lambda_n}(\mathbf{V}^q_{\underline{a},\mathrm{ord}}, \Lambda_n)$; this is a free Λ_n-module. For a ψ of level s we have a canonical isomorphism

$$\mathcal{M}^q_{\underline{a},\mathrm{ord}}(K^p, \Lambda_n) \otimes_{\Lambda_n} \Lambda_n/P_{\psi_{\underline{k}}} \cong V^q_{\underline{k},\mathrm{ord}}(K^pI_s, \psi\omega^{\underline{a}-\underline{k}}, \mathcal{O}_{\mathfrak{p}}(\psi)). \tag{6.3.4.a}$$

For any Λ_n-algebra A, we put $\mathcal{M}^q_{\underline{a},\mathrm{ord}}(K^p, A) := \mathcal{M}^q_{\underline{a},\mathrm{ord}}(K^p, \Lambda_n) \otimes_{\Lambda_n} A$ and

$$\mathcal{M}^q_{\mathrm{ord}}(K^p, A) := \bigoplus_{\underline{a} \in (\mathbf{Z}/(p-1)\mathbf{Z})^{2n}} \mathcal{M}^q_{\underline{a},\mathrm{ord}}(K^p, A)$$

6.3.5. *The Λ_n-adic q-expansion principle.* Recall that for each $x \in G(\mathbf{A}_f)$ with $x \in K^0_p$, we have a q-expansion map

$$H^0(T_{s,m}, \mathcal{O}_{T_{s,m}}) \longrightarrow \mathbf{Z}/p^s\mathbf{Z}[\![q^{S^+_x}]\!]$$

where $S^+_x := S^+_{[x]}$ is as in 5.4.8 for $[x] = [x]_0 \in C_0(K)$. We deduce by passing to the limits over s and m that there is an injective morphism

$$\mathcal{V}^q_{\mathrm{ord}} \hookrightarrow \bigoplus_{x \in X(K)} \mathbf{Q}_p/\mathbf{Z}_p[\![q^{S^+_{n,x}}]\!],$$

where $X(K)$ is a (finite) set of representatives x of $G(\mathbf{Q})\backslash G(\mathbf{A}_f)/K$ with $x_p \in Q(\mathbf{Z}_p)$. For each x and each $h \in S^+_x$, the map $f \mapsto a(h, f_x)$ is an element of $\mathbf{V}^q_{\mathrm{ord}}$ and we let $a(h, F_x) \in \Lambda_n$ be the image of this element under $F \in \mathcal{M}^q_{\underline{a},\mathrm{ord}}(K^p, \Lambda_n) := \mathrm{Hom}_{\Lambda_n}(\mathbf{V}^r_{\underline{a},\mathrm{ord}}, \Lambda_n)$. We obtain therefore a Λ_n-adic q-expansion by forming the formal sum

$$F_x(q) := \sum_{h \in S^+_x} a(h, F_x)q^h \in \Lambda_n[\![q^{S^+_{n,x}}]\!].$$

For each pair $(\underline{k}, \psi)$ with $\underline{k}$ dominant and ψ finite of level s, the reduction modulo $P_{\psi_{\underline{k}}}$ of the Λ_n-adic q-expansion is the q-expansion map

$$V^q_{\underline{k},\mathrm{ord}}(K^pI_s, \psi\omega^{\underline{a}-\underline{k}}, \mathcal{O}_{\mathfrak{p}}(\psi)) \to \bigoplus_{x \in X(K)} \mathcal{O}_{\mathfrak{p}}(\psi)[\![q^{S^+_x}]\!].$$

Lemma 6.3.6. *(Λ_n-adic q-expansion principle). The map*

$$\mathcal{M}^q_{\underline{a},\mathrm{ord}}(K^p,\Lambda_n) \hookrightarrow \bigoplus_{x\in X(K)} \Lambda_n[\![q^{S_x^+}]\!]$$

defined by $F \mapsto (F_x(q))_x$ is injective.

Proof. To show the injectivity it is sufficient to prove that the map is injective after reducing modulo $P_{\psi_{\underline{k}}}$ for all pairs $(\underline{k},\psi)$ since the ideals $P_{\psi_{\underline{k}}}$ are Zariski dense in $\mathrm{Spec}\,(\Lambda_n)$. For each such pair the injectivity modulo $P_{\psi_{\underline{k}}}$ follows from the q-expansion principle for p-adic modular forms (see 6.1.7). This proves the lemma.■

Let A be a finite torsion-free Λ_n-algebra and let $Z \subset \mathrm{Spec}\,(A)$ be a Zariski dense subset of primes Q such that $Q \cap \Lambda_n = P_{\psi_{\underline{k}}}$ for some pair $(\underline{k},\psi)$. Let $\mathcal{N}^r_{\underline{a},Z,\mathrm{ord}}(A)$ be the set of elements $(F_x)_x \in \bigoplus_x A[[q^{S_x^+}]]$ such that for each $Q \in Z$ above $P_{\psi_{\underline{k}}}$ the reduction of $(F_x)_x$ is the q-expansion of some element $f \in V^r_{\underline{k},\mathrm{ord}}(K^pI_s,\psi,A/Q)$. The Λ_n-adic q-expansion principle gives a natural inclusion

$$\mathcal{M}^r_{\underline{a},\mathrm{ord}}(K^p,A) \hookrightarrow \mathcal{N}^r_{\underline{a},Z,\mathrm{ord}}(A) \tag{6.3.6.a}$$

Lemma 6.3.7. *The inclusion* (6.3.6.a) *is an equality.*

Proof. This can proved similarly to [Ur04, Prop 2.4.23]. The proof rests on the finiteness over Λ_n of $\mathcal{N}^r_{\underline{a},Z,\mathrm{ord}}(A)$ and the q-expansion principle in characteristic p. ■

6.3.8. *The fundamental exact sequence II.* It follows from Theorem 5.4.6 and the exact sequence (5.4.9.a) that there is an exact sequence

$$0 \to \pi^*\mathcal{I}^0_{\mathcal{S}^*} \to \pi^*\mathcal{I}^1_{\mathcal{S}^*} \to \bigoplus_{[g]\in C_{n-1}(K)} \pi^*\iota_{[g],*}\mathcal{I}^0_{S^*_{[g]}} \to 0.$$

We need to generalize this exact sequence for the Igusa tower. For any pair (s,t) of positive integers let $Y_{s,t} := T_{s,t}/I_s^H$ and consider the Stein factorization of the composite map $f_{s,t} : Y_{s,t} \to S_t \to S_t^*$. This yields a commutative square:

$$\begin{array}{ccc} Y_{s,t} & \xrightarrow{\pi_{s,t}} & Y^*_{s,t} \\ \downarrow{\scriptstyle \phi_{s,t}} & & \downarrow{\scriptstyle \psi_{s,t}} \\ S_t & \xrightarrow{\pi} & S^*_t \end{array}$$

where $Y^*_{s,t} := \mathbf{Spec}_{S_t^*}(f_{s,t})_*\mathcal{O}_{Y_{s,t}} \overset{\psi_{s,t}}{\to} S_t^*$ is finite and étale of the same degree as $\phi_{s,t}$ and $Y_{s,t} \overset{\pi_{s,t}}{\to} Y^*_{s,t}$ has geometrically connected fibers. In particular, $Y^*_{s,t}$ is affine and $(\pi_{s,t})_*\mathcal{O}_{Y_{s,t}} = \mathcal{O}_{Y^*_{s,t}}$. Let $\mathcal{I}^q_{Y^*_{s,t}} := \psi^*_{s,t}\mathcal{I}^q$. Since $\psi_{s,t}$ is flat, this is a sheaf of ideals of $\mathcal{O}_{Y^*_{s,t}}$. We consider the exact sequence

$$0 \to \mathcal{I}^0_{Y^*_{s,t}} \to \mathcal{I}^1_{Y^*_{s,t}} \to \bigoplus_{[g]\in C_{n-1}(K)} \psi^*\iota_{[g],*}\mathcal{I}^0_{S^*_{[g]}} \to 0$$

obtained from (5.4.9.a). Since $Y^*_{s,t}$ is affine we deduce that the following sequence is exact:

(6.3.8.a)
$$0 \to H^0(Y^*_{s,t}, \mathcal{I}^0_{Y^*_{s,t}}) \to H^0(Y^*_{s,t}, \mathcal{I}^1_{Y^*_{s,t}}) \to \bigoplus_{[g]\in C_{n-1}(K)} H^0(\psi^{-1}_{s,t}(S^*_{[g]}), \psi^*_{s,t}\iota_{[g],*}\mathcal{I}^0_{S^*_{[g]}}) \to 0.$$

We also have

(6.3.8.b) $H^0(Y^*_{s,t}, \mathcal{I}^q_{Y^*_{s,t}}) = H^0(Y_{s,t}, (\pi_{s,t})^*\mathcal{I}^q_{Y^*_{s,t}}) = H^0(Y_{s,t}, (\phi_{s,t})^*\pi^*\mathcal{I}^q_{S^*_t}) = W^q_{s,t},$

the first equality following easily from the definition of $Y^*_{s,t}$ (which immediately gives the analogous equality for the fibers of $\pi_{s,t}$).

We write $T_{[g],s,t}$ for the Igusa tower over the compactification $\bar{S}_{[g]} = \bar{S}_{G_{n-1}}(K_{n-1,g})$ of the rational boundary component $S_{[g]}$ (recall that there is a natural identification of $S_{[g]}$ with $S_{G_{n-1}}(K_{n-1,g})$). This is the étale Galois cover of $S_{G_{n-1}}(K_{n-1,g})[1/E] \times_{\mathcal{O}_\mathfrak{p}} \mathcal{O}_\mathfrak{p}/\mathfrak{p}^t$ with Galois group $\mathrm{GL}_{n-1}(\mathcal{O}/\mathfrak{p}^s) \times \mathrm{GL}_{n-1}(\mathcal{O}/\bar{\mathfrak{p}}^s)$ constructed in the same way as $T_{s,t}$. Let $Y_{[g],s,t} := T_{[g],s,t}/I_s^{H_{n-1}}$ where we have written $I_s^{H_{n-1}}$ for the Iwahori subgroup of $H_{n-1}(\mathcal{O}_\mathfrak{p}) = \mathrm{GL}_{n-1}(\mathcal{O}_\mathfrak{p}) \times GL_{n-1}(\mathcal{O}_{\bar{\mathfrak{p}}})$ (exactly analogous to I_s^H). Similarly, we can define $Y^*_{[g],s,t}$ and $\mathcal{I}^q_{Y^*_{[g],s,t}}$ in the same way we have defined $Y^*_{s,t}$ and $\mathcal{I}^q_{Y^*_{s,t}}$. Then we have the following lemma.

Lemma 6.3.9. *For any $[g] \in C_{n-1}(K)$, there is a canonical isomorphism*
$$H^0(\psi^{-1}_{s,t}(S^*_{[g]}), \psi^*_{s,t}\iota_{[g],*}\mathcal{I}^0_{S^*_{[g]}}) \cong \mathrm{Ind}^{T_n(\mathbf{Z}/p^s\mathbf{Z})}_{T_{n-1}(\mathbf{Z}/p^s\mathbf{Z})} H^0(Y^*_{[g],s,t}, \mathcal{I}^0_{Y^*_{[g],s,t}})$$
which commutes with the action of $T_n(\mathbf{Z}/p^s\mathbf{Z})$. Here $T_{n-1} \hookrightarrow T_n$ is defined by
$$\mathrm{diag}(t_1, \ldots, t_{n-1}, t_{n+1}, \ldots, t_{2n-2}) \mapsto \mathrm{diag}(t_1, \ldots, t_{n-1}, 1, t_{n+1}, \ldots, t_{2n-2}, 1).$$

Proof. The lemma follows easily from the isomorphism
$$T_{s,t} \times_{S_t} \pi^{-1}(S_{[g]}) \cong \bigsqcup_{x\in H_n(\mathbf{Z}/p^s\mathbf{Z})/H_{n-1}(\mathbf{Z}/p^s\mathbf{Z})} x.T_{[g],s,t}$$
where we have embedded H_{n-1} in H_n via the embedding of GL_{n-1} in GL_n given by $x \mapsto \mathrm{diag}(x, 1)$. This isomorphism can be obtained by the same argument used to prove the top isomorphism of [Hi99, p.33].∎

After passing to the inductive limit over s and t and taking the ordinary part, it follows from (6.3.8.a), (6.3.8.b), and Lemma 6.3.9 that there is an exact sequence
$$0 \to \mathcal{V}^0_{\mathrm{ord}}(K^p) \to \mathcal{V}^1_{\mathrm{ord}}(K^p) \xrightarrow{\Phi} \bigoplus_{[g]\in C_{n-1}(K)} \mathrm{Ind}^{T_n(\mathbf{Z}_p)}_{T_{n-1}(\mathbf{Z}_p)} \mathcal{V}^0_{\mathrm{ord}}(K^p_{n-1,g}) \to 0.$$

This map is clearly equivariant for the action of $T_n(\mathbf{Z}_p)$.

For $\underline{a} \in \mathbf{Z}^{2n}$, we write $\underline{a}' = (a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_{2n-2})$. Taking Pontrjagin duals and Λ_n-duals, we deduce from the previous exact sequence another exact sequence:

$$0 \to \mathcal{M}^0_{\underline{a},\mathrm{ord}}(K^p) \to \mathcal{M}^1_{\underline{a},\mathrm{ord}}(K^p) \xrightarrow{\Phi} \bigoplus_{[g]\in C_{n-1}(K)} \mathcal{M}^0_{\underline{a}',\mathrm{ord}}(K^p_{n-1,g}) \otimes_{\Lambda_{n-1}} \Lambda_n \to 0,$$

where the tensor product on the right hand side is defined for the map $\Lambda_{n-1} \to \Lambda_n$ deduced from the inclusion $T_{n-1}(\mathbf{Z}_p) \hookrightarrow T_{n-1}(\mathbf{Z}_p)$ as in Lemma 6.3.9. The map Φ is just the sum over the $[g] \in C_{n-1}(K)$ of the Λ_n-adic versions of the Siegel maps $\Phi_{[g]}$; modulo primes of the form $P_{\psi_{\underline{k}}}$ these are the Siegel maps from 5.5.7. In particular, $\Phi_{[g]}$ sends an element of $\mathcal{M}^1_{\underline{a},\mathrm{ord}}(K^p, A)$ to its constant term along the cusp of genus $2n-2$ attached to $[g] \in C_{n-1}(K)$. At the level of q-expansions attached to the genus 0 cusps $[g]_0$ and $[g]_0'$ this is

$$\Phi_{[g]}(F_x(q)) = \sum_{h=\begin{pmatrix} h' & 1 \\ 1 & 0\end{pmatrix} \in S^+_{[g]_0}} a(h, F_x) q^{h'} \in A[\![q^{S^+_{n-1,[g]_0'}}]\!].$$

We summarize this observation in the following theorem.

Theorem 6.3.10. *Let $\underline{a} \in \mathbf{Z}^{2n}$. For any Λ_n-algebra A there is a short exact sequence*

$$0 \to \mathcal{M}^0_{\underline{a},\mathrm{ord}}(K^p, A) \to \mathcal{M}^1_{\underline{a},\mathrm{ord}}(K^p, A) \xrightarrow{\Phi} \bigoplus_{[g]\in C_{n-1}(K)} \mathcal{M}^0_{\underline{a}',\mathrm{ord}}(K^p_{n-1,g}, \Lambda_{n-1}) \otimes_{\Lambda_{n-1}} A \to 0$$

with $\Phi = \oplus_{[g]} \Phi_{[g]}$,

Proof. We have proved the theorem for $A = \Lambda_n$. It then follows easily for any Λ_n-algebra A since the modules of Λ_n-adic forms are free over Λ_n.∎

6.4. **Universal ordinary Hecke algebras.** Let S be a finite set of primes such that $K^p = \prod_{\ell \neq p} K_\ell$ is maximal outside of S and let $R^{S\cup\{p\}}$ be as in 5.5.11. Let $R_{S,p} := R^{S\cup\{p\}} \otimes \mathcal{U}_p$. For any finite torsion-free Λ_n-algebra A we denote by $\mathbf{h}^{S,q}(K^p; A)$ the A-algebra generated by the image of $R_{S,p}$ in $\mathrm{End}_A(\mathcal{M}^{n,q}_{\mathrm{ord}}(K^p, A))$. Similarly, for a dominant weight $\underline{k}$ we write $h^{S,q}_{\underline{k},\mathrm{ord}}(K)$ for the $\mathbf{Z}_p$-algebra generated by the image of $R_{S,p}$ in $\mathrm{End}_{\mathbf{C}}(M^{n,q}_{\underline{k},\mathrm{ord}}(K, \mathbf{C}))$. The following is a consequence of Theorem 6.2.10 and the isomorphism (6.3.4.a).

Theorem 6.4.1. *Let $q = 0$ or 1. For any sufficiently regular algebraic dominant character $\underline{k}$, the canonical surjective homomorphism*

$$\mathbf{h}^{S,q}(K^p; \Lambda_n) \otimes_{\Lambda_n} \Lambda_n / P_{\underline{k}} \twoheadrightarrow h^{S,q}_{\underline{k},ord}(K)$$

has nilpotent kernel. In particular, any $\overline{\mathbf{Q}}_p$-valued character of the left-hand side is the compostion of this surjection with a $\overline{\mathbf{Q}}_p$-valued character of the right-hand side.

6.4.2. *p-stabilizations.* Let $\lambda : \mathcal{U}_p \to \overline{\mathbf{Q}}_p$ be a character. Assume that $\lambda(u_t) \neq 0$ for any $t \in T(\mathbf{Q}_p)$ satisfying (6.2.1.a). Then there exists an n-tuple $(\alpha_1, \ldots, \alpha_{2n}) \in \mathbf{C}^n$ such that

$$\lambda(u_t) = \prod \alpha_i^{\mathrm{ord}_p(t_i)}. \qquad (6.4.2.a)$$

Let $\pi = \pi_{\underline{k}} \otimes \pi_f$ be a cuspidal representation of $G_n(\mathbf{A})$ which is unramified at p. If λ is associated (via the natural action) to some eigenvector of $\mathcal{U}_p$ in $\pi_f^{K^pI_1}$, then $(\alpha_1.p^{n-1/2}, \ldots, \alpha_i.p^{1/2+n-i}, \ldots, \alpha_{2n}.p^{1/2-n})$ is an ordering of the Langlands parameters of the spherical representation π_p. The choice (up to multiplication by a scalar) of an eigenvector imposing this ordering[6] in $\pi_p^{I_1}$ is called a p-stabilization of π. One says that this p-stabilization is ordinary (with respect to the chosen embedding ι_p) if the eigenvalues for the action (6.2.2.a) of the operators u_t are p-adic units. This means that the p-adic valuations (slopes) of the corresponding ordered roots $(\alpha_1.p^{2n-1}, \ldots, \alpha_i.p^{2n-i}, \ldots, \alpha_{2n}.p)$ of $\lambda_\pi(Q_p(X))$ are given by $\underline{k}^* + (2n-1, 2n-2, \ldots, 1, 0) = (k_1+n-1, \ldots, k_n, k_{n+1}+2n-1, \ldots, k_{2n}+n)$. As we will see later, these slopes are equal to the Hodge-Tate weights of the p-adic Galois representation attached to π.

When an ordinary p-stabilization exists it is unique, and we say that π is p-ordinary. We then write $\lambda_\pi^{\mathrm{ord}}$ for the character of $R_{S,p}$ giving the eigenvalues of the Hecke operators on the ordinary p-stabilized vector in $\pi_f^{K^pI_1}$.

6.5. **The Eisenstein ideal for $GU(2,2)$.** In this section we define the Eisenstein ideal for the group $GU(2,2)$; this is the image in the cuspidal Hecke algebra of the ideal generated by the operators that annihilate a certain Eisenstein series studied later in this paper. This ideal plays a central role in the proofs of the main theorems of this paper.

In what follows n is 2.

6.5.1. *p-adic Eisenstein data.* We freely use the notation from 3.3.8-3.3.10 and 3.4.5.

A p-adic Eisenstein datum is a 6-tuple $\mathbf{D} = (A, \mathbb{I}, \mathbf{f}, \psi, \xi, \Sigma)$ consisting of

- the ring of integers A of a finite extension of $\mathbf{Q}_p$;
- a domain $\mathbb{I}$ that is a finite integral extension of $\Lambda_{W,A}$;
- an ordinary $\mathbb{I}$-adic newform $\mathbf{f}$ of some tame level M with associated A-valued Dirichlet character $\chi_{\mathbf{f}}$;
- a finite order A-valued idele class character ψ of $\mathbf{A}_\mathcal{K}^\times/\mathcal{K}^\times$ such that $\psi|_{\mathbf{A}^\times} = \chi_{\mathbf{f}}$;
- a finite order A-valued idele class character ξ of $\mathbf{A}_\mathcal{K}^\times/\mathcal{K}^\times$;
- a finite set Σ of primes containing those that divide $MpD_\mathcal{K}$ as well as those ℓ such that ψ_ℓ or ξ_ℓ is ramified.

[6] Not every ordering may occur as a p-stabilization.

We let $\Lambda_{\mathbf{D}} := \mathbb{I}[\![\Gamma_{\mathcal{K}}^- \times \Gamma_{\mathcal{K}}]\!] = \mathbb{I}_{\mathcal{K}}[\![\Gamma_{\mathcal{K}}^-]\!]$. We give $\Lambda_{\mathbf{D}}$ the structure of a Λ_2-algebra as follows. Let

$$\alpha : A[\![\Gamma_{\mathcal{K}}]\!] \to \mathbb{I}[\![\Gamma_{\mathcal{K}}^-]\!], \quad \alpha(\gamma_+) = (1+p)(1+W)^{1/2}, \ \alpha(\gamma_-) = (1+p)(1+W)^{1/2}\gamma_-,$$

and

$$\beta : A[\![\Gamma_{\mathcal{K}}]\!] \to \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!], \quad \beta(\gamma_+) = (1+W)^{-1}\gamma_+, \quad \beta(\gamma_-) = \gamma_-.$$

These define a homomorphism $\alpha \otimes \beta : A[\![\Gamma_{\mathcal{K}} \times \Gamma_{\mathcal{K}}]\!] \to \mathbb{I}[\![\Gamma_{\mathcal{K}}^-]\!]\hat{\otimes}_{\mathbb{I}}\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] = \Lambda_{\mathbf{D}}$. We define a homomorphism from $\Gamma_2 = (1+p\mathbf{Z}_p)^4 \subset T_2(\mathbf{Z}_p)$ to $\mathbf{A}[\![\Gamma_{\mathcal{K}} \times \Gamma_{\mathcal{K}}]\!]$ by

$$(t_1, t_2, t_3, t_4) \mapsto \psi\Psi_{\mathcal{K}}^{-1}(t_3t_4, t_1^{-1}t_2^{-1}) \times \xi\Psi_{\mathcal{K}}(t_4^{-1}, t_2),$$

using the fixed identification $\mathcal{O}_p^\times = \mathbf{Z}_p^\times \times \mathbf{Z}_p^\times$. This defines a $\Lambda_2 = \mathbf{Z}_p[\![\Gamma_2]\!]$-algebra structure on $A[\![\Gamma_{\mathcal{K}} \times \Gamma_{\mathcal{K}}]\!]$ and so on $\Lambda_{\mathbf{D}}$ via composition with $\alpha \otimes \beta$. Note that $\Lambda_{\mathbf{D}}$ is a finite integral local reduced Λ_2-algebra.

Remark. In 12.1 below we define a set of arithmetic homomorphisms $\mathcal{X}_{\mathbf{D}}^a \subset \mathcal{X}_{\Lambda_{\mathbf{D}},A}$ and explain how to associate an Eisenstein series to $\mathbf{D}$ and $\phi \in \mathcal{X}_{\mathbf{D}}^a$. The Eisenstein series are naturally associated to $\mathbf{f}_{\phi|\mathbb{I}}$, $\phi \circ \alpha \circ \mathrm{rec}_{\mathcal{K}}$, and $\phi \circ \beta \circ \mathrm{rec}_{\mathcal{K}}$. This partially explains our terminology and notation.

Let $\chi_{\mathbf{f},0}$ be the unique A-valued Dirichlet character such that $\chi_{\mathbf{f},0}|_{\widehat{\mathbf{Z}}^\times} = \prod_{\ell \neq p} \chi_{\mathbf{f},\ell}$.

6.5.2. *The Galois representation associated with* $\mathbf{D}$. Define two $\Lambda_{\mathbf{D}}^\times$-valued characters $\boldsymbol{\sigma}_\psi$ and $\boldsymbol{\sigma}_\xi$ of $G_{\mathcal{K}}$:

$$\boldsymbol{\sigma}_\psi := \alpha \circ \sigma_{\omega^{-1}\psi}\boldsymbol{\varepsilon}_{\mathcal{K}}^{-1} \qquad \boldsymbol{\sigma}_\xi := \beta \circ \sigma_{\chi_{\mathbf{f}}\xi}\boldsymbol{\varepsilon}_{\mathcal{K}} = \sigma_{\chi_{\mathbf{f}}\xi}\boldsymbol{\varepsilon}_W^{-1}\boldsymbol{\varepsilon}_{\mathcal{K}}.$$

Put $\boldsymbol{\sigma}_{\xi'} := \boldsymbol{\sigma}_\xi\boldsymbol{\sigma}_\xi^c$. Let $F_{\mathbf{D}}$ be the field of fractions of $\Lambda_{\mathbf{D}}$. We define a semisimple representation $\boldsymbol{\rho}_{\mathbf{D}} : G_{\mathcal{K}} \to \mathrm{GL}_4(F_{\mathbf{D}})$ by

$$\boldsymbol{\rho}_{\mathbf{D}} := \sigma_{\bar{\chi}_{\mathbf{f},0}}\boldsymbol{\sigma}_\psi^c\epsilon^{-3} \oplus (\rho_{\mathbf{f}} \otimes \sigma_{\bar{\chi}_{\mathbf{f},0}}\boldsymbol{\sigma}_\xi^{-c}\boldsymbol{\sigma}_\psi^c\epsilon^{-2}) \oplus \sigma_{\bar{\chi}_{\mathbf{f},0}}\epsilon^{-1}\det\rho_{\mathbf{f}}\boldsymbol{\sigma}_{\xi'}^{-1}\boldsymbol{\sigma}_\psi^c. \tag{6.5.2.a}$$

This is unramified away from Σ. Note that for each $g \in G_{\mathcal{K}}$, $\det(1 - \boldsymbol{\rho}_{\mathbf{D}}(g)X) \in \Lambda_{\mathbf{D}}[X]$. For a finite place v of $\mathcal{K}$ not dividing a prime in Σ let $Q_{v,\mathbf{D}}(X) := \det(1 - \boldsymbol{\rho}_{\mathbf{D}}(\mathrm{frob}_v)X)$.

Remark. For $\phi \in \mathcal{X}_{\mathbf{D}}^a$ such that there is an associated Eisenstein series (see the remark in 6.5.1), the specialization of $\boldsymbol{\rho}_{\mathbf{D}}$ under such a ϕ is the usual p-adic Galois representation associated with the Eisenstein series. In particular, the specialization of $\det(1 - \boldsymbol{\rho}_{\mathbf{D}}(\mathrm{frob}_v)X)$, v a finite place of $\mathcal{K}$ not dividing a prime in Σ, is the Hecke polynomial later denoted Q_v (which gives the v-Euler factor of the standard L-function of the Eisenstein series).

6.5.3. *The Eisenstein ideal.* Recall that we have given $\Lambda_{\mathbf{D}}$ the structure of a Λ_2-algebra. Let $K' = K'_\Sigma K^\Sigma \subset G(\mathbf{A}_f^p)$ be an open compact subgroup with $K^\Sigma = G(\widehat{\mathbf{Z}}^\Sigma)$ and such that $K'K_p^0$ is neat. Let $\mathbf{h}_{\mathbf{D}} = \mathbf{h}_{\mathbf{D}}(K')$ be the universal ordinary cuspidal Hecke algebra $\mathbf{h}^{\Sigma,0}(K'; \Lambda_{\mathbf{D}})$ over $\Lambda_{\mathbf{D}}$. This is a finite reduced $\Lambda_{\mathbf{D}}$-algebra. For v a finite place of $\mathcal{K}$ not dividing any primes in Σ we let $Q_v(X) \in \mathbf{h}_{\mathbf{D}}(K')[X]$ be the Hecke polynomial defined in (9.6.0.a). (To be precise $Q_v(X) \in R^\Sigma$; here we take its image in $\mathbf{h_D}(K')$.) We define $I_\mathbf{D} = I_{\mathbf{D},K'} \subseteq \mathbf{h_D}(K')$ to be the ideal generated by

- the coefficients of $Q_v(X) - Q_{v,\mathbf{D}}(X)$ for all finite places v of $\mathcal{K}$ not dividing a prime in Σ;
- $u_t - \prod_{i=1}^4 \boldsymbol{\beta}_i^{a_i}$ for $t = \mathrm{diag}(p^{a_1}, p^{a_2}, p^{a_4}, p^{a_3})$ with $a_1 \leq a_2 \leq a_3 \leq a_4$ and

$(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4) = (a(p,\mathbf{f})^{-1}\boldsymbol{\psi}(\varpi^c), \chi_{\mathbf{f},0}^{-1}\boldsymbol{\psi}\boldsymbol{\xi}(\varpi^c), \chi_{\mathbf{f},0}\boldsymbol{\psi}\boldsymbol{\xi}^{-1}(\varpi), a(p,\mathbf{f})\boldsymbol{\psi}(\varpi)^{-1})$;

- $Z_{\ell,0} - \sigma_{\chi_{\mathbf{f},0}}\boldsymbol{\sigma}_\psi\boldsymbol{\sigma}_\xi^{-1}(\mathrm{frob}_v)$ for all inert $\ell \notin \Sigma$ and $v|\ell$ the place of $\mathcal{K}$;
- $Z_{\ell,0}^{(i)} - \sigma_{\chi_{\mathbf{f},0}}\boldsymbol{\sigma}_\psi\boldsymbol{\sigma}_\xi^{-1}(\mathrm{frob}_{w_i})$ for all split $\ell \notin \Sigma$, $\ell = w_1w_2$ being the factorization corresponding to the identification $\mathcal{K}_\ell = \mathbf{Q}_\ell \times \mathbf{Q}_\ell$.

Here $\boldsymbol{\psi}$ and $\boldsymbol{\xi}$ are the characters defined in 12.1, and $Z_{\ell,0}$ and $Z_{\ell,0}^{(i)}$ are the Hecke operators in R_ℓ defined and denoted Z_0 and $Z_0^{(i)}$ in 9.5.2.

The structure map $\Lambda_\mathbf{D} \to \mathbf{h_D}/I_\mathbf{D}$ is surjective by the definition of $I_\mathbf{D}$ (the elements in $\mathcal{U}_p$ together with the coefficients of the $Q_v(X)$'s and the $Z_{\ell,0}$s and $Z_{\ell,0}^{(i)}$s generate $\mathbf{h_D}$ over $\Lambda_\mathbf{D}$), and we define the Eisenstein ideal $\mathcal{E}_\mathbf{D} = \mathcal{E}_{\mathbf{D},K'} \subseteq \Lambda_\mathbf{D}$ to be the quotient of this map. So we have

(6.5.3.a) $$\Lambda_\mathbf{D}/\mathcal{E}_\mathbf{D} \xrightarrow{\sim} \mathbf{h_D}/I_\mathbf{D}.$$

Assuming $\mathcal{E}_\mathbf{D}$ is a proper ideal, we define $\mathbf{T_D} = \mathbf{T_D}(K')$ to be the local component of $\mathbf{h_D}$ associated with the maximal ideal containing $I_\mathbf{D}$.

Let $M_\mathbf{D}$ be an integer that is divisible only by primes in $\Sigma\backslash\{p\}$ and by the least common multiple of the prime-to-p parts of M, $D_\mathcal{K}$, $\mathrm{Nm}(\mathfrak{f}_\xi)$, and $\mathrm{Nm}(\mathfrak{f}_\psi)$. Suppose A contains $\mathbf{Z}[\boldsymbol{\mu}_{M_\mathbf{D}p}, i, D_\mathcal{K}^{1/2}]$. Suppose also that $\mathbf{f}$ satisfies $(\mathbf{irred})_\mathbf{f}$ and $(\mathbf{dist})_\mathbf{f}$. Assuming K' is sufficiently small (in a sense measured by $M_\mathbf{D}$), in 12.4 we prove the existence of an element $\mathbf{E_D} \in \mathcal{M}_{\mathrm{ord}}^1(K'; \Lambda_\mathbf{D})$ and a homomorphism $\lambda_\mathbf{D} = \lambda_{\mathbf{D},K'} : \mathbf{h_D}(K') \to \Lambda_\mathbf{D}$ such that $h.\mathbf{E_D} = \lambda_{\mathbf{D},K'}(h)\mathbf{E_D}$ for all $h \in \mathbf{h_D}$ and $h.\mathbf{E_D} = 0$ if $h \in I_{\mathbf{D},K'}$ (so $\lambda_\mathbf{D}(I_\mathbf{D}) = 0$). Furthermore, if $\beta \in S_2(\mathbf{Q})$ with $\beta \geq 0$ and $\det\beta = 0$ and if $x \in G(\mathbf{A}_f)$ is such that $x_p \in Q(\mathbf{Z}_p)$, then the β-Fourier coefficient $c(h, x; \mathbf{E_D})$ of $\mathbf{E_D}$ is divisible by

$$\mathcal{L}_\mathbf{D} := \mathcal{L}_{\mathbf{f},\mathcal{K},\xi}^\Sigma \mathcal{L}_{\bar\chi_\mathbf{f}\bar\chi'}^\Sigma,$$

where $\mathcal{L}_{\bar\chi_\mathbf{f}\bar\chi'}^\Sigma \in \mathbb{I}[\![\Gamma_\mathbf{Q}]\!] = A[\![\Gamma_\mathcal{K}^+]\!]$ is the image of the p-adic L-function $G_{\bar\chi_\mathbf{f}\bar\xi'}^\Sigma \in A[\![\Gamma_\mathbf{Q}]\!]$ of the Dirichlet character $\bar\chi_\mathbf{f}\bar\xi'$ as in 3.4.3 under the map $A[\![\Gamma_\mathbf{Q}]\!] \to \mathbb{I}[\Gamma_\mathcal{K}]$, $\gamma \mapsto (1+W)^{-1}\gamma_+^2$, and $\mathcal{L}_{\mathbf{f},\mathcal{K},\xi}^\Sigma \in \mathbb{I}[\![\Gamma_\mathcal{K}]\!]$ is the three-variable p-adic L-function constructed in 12.3 below. In particular, if $\xi' = \chi_\mathbf{f}$ then $\mathcal{L}_{\mathbf{f},\mathcal{K}}^\Sigma := \mathcal{L}_{\mathbf{f},\mathcal{K},\chi_\mathbf{f}}^\Sigma$ is the p-adic L-function from 3.4.5. The following theorem, which relate $\mathcal{L}_\mathbf{D}$ to the Eisenstein ideal $\mathcal{E}_\mathbf{D}$ via $\mathbf{E_D}$, is one of the key ingredients in the proof of the main result of this paper.

Theorem 6.5.4. *Assume $\mathbb{I}$ is an integrally closed domain. With the preceding notation and assumptions, if $P \subset \Lambda_\mathbf{D}$ is a height one prime such that $\mathbf{E_D}$ is non-zero modulo P then*

$$\mathrm{ord}_P(\mathcal{E}_\mathbf{D}) \geq \mathrm{ord}_P(\mathcal{L}_\mathbf{D}).$$

Proof. For each $[g] \in C_1(K'K_p^0)$, there exists $F_g \in \mathcal{M}^0_{\mathrm{ord}}(K'_{1,g};\Lambda_{\mathbf{D}})$ such that

$$\Phi_{[g]}\mathbf{E}_{\mathbf{D}} = \mathcal{L}_{\mathbf{D}}F_g.$$

By the surjectivity of the map $\Phi = \sum_{[g]}\Phi_{[g]}$ in Theorem 6.3.10, there exists $F \in \mathcal{M}^1_{\mathrm{ord}}(K';\Lambda_{\mathbf{D}})$ such that $\Phi_{[g]}F = F_g$ for all $[g] \in C_1(K'K_p^0)$. Appealing to Theorem 6.3.10 again, we find that

$$H := \mathbf{E}_{\mathbf{D}} - \mathcal{L}_{\mathbf{D}}F \in \mathcal{M}^0_{\mathrm{ord}}(K';\Lambda_{\mathbf{D}}). \tag{6.5.4.a}$$

Assume $P|\mathcal{L}_{\mathbf{D}}$ (otherwise the theorem is trivial). Then H is non-zero modulo P since $\mathbf{E}_{\mathbf{D}}$ is by assumption. Let $\beta \in S_2(\mathbf{Q})$, $\det\beta > 0$, and $x \in G(\mathbf{A}_f)$, x unramified at p, be such that the β-Fourier coefficient $c(\beta,x;H) := a(\beta;H_x)$ is non-zero modulo P. Let $r := \mathrm{ord}_P(\mathcal{L}_{\mathbf{D}})$. The surjective map $\mu : \mathbf{h}_{\mathbf{D}} \mapsto \Lambda_{\mathbf{D},P}/P^r\Lambda_{\mathbf{D},P}$ defined by $\mu(h) = c(\beta,x;h.H)/c(\beta,x;H)$ is $\Lambda_{\mathbf{D}}$-linear. If $T \in R^{\Sigma,p}$ then

$$c(\beta,x;T.H) \equiv c(\beta,x;T.\mathbf{E}_{\mathbf{D}}) \equiv \lambda_{\mathbf{D}}(T)c(\beta,x;\mathbf{E}_{\mathbf{D}}) \equiv \lambda_{\mathbf{D}}(T)a(\beta,x;H) \pmod{P^r},$$

from which it follows that $I_{\mathbf{D}} \subseteq \ker\mu$. In particular, μ defines a surjective $\Lambda_{\mathbf{D},P}$-homomorphism

$$\Lambda_{\mathbf{D},P}/\mathcal{E}_{\mathbf{D}}\Lambda_{D,P} \xrightarrow{\sim} \mathbf{h}_{\mathbf{D},P}/I_{\mathbf{D}}\mathbf{h}_{\mathbf{D},P} \twoheadrightarrow \Lambda_{\mathbf{D},P}/P^r\Lambda_{\mathbf{D},P}.$$

As $\mathrm{ord}_P(\mathcal{E}_{\mathbf{D}})$ equals the length over $\Lambda_{\mathbf{D},P}$ of $\Lambda_{\mathbf{D},P}/\mathcal{E}_{\mathbf{D}}\Lambda_{\mathbf{D},P}$, the theorem follows. ∎

7. Galois representations

7.1. **Galois representations for G_n.** Let $\pi = \pi_\infty \otimes \pi_f$ be a cuspidal automorphic representation of the unitary similitude group $G_n(\mathbf{A})$ such that π_∞ is a discrete series representation associated with a $2n$-tuple of integers $\underline{k} = (k_{n+1},...,k_{2n};k_n,...,k_1)$ with $k_1 \geq k_2 \geq \cdots \geq k_{2n}$ such that $[\underline{k}^*]$ is a dominant weight of T_n (recall that $\underline{k}^* := \underline{k} + (n,...,n;-n,...,-n)$); such $2n$-tuples define representations $\rho_{\underline{k}}$ of the maximal compact $K^+_{n,\infty}$ of $U_n(\mathbf{R})$ as explained in 5.5.2. Then $\pi_\infty = \Pi_{\underline{k}}$ is a discrete series representation with lowest $K^+_\infty Z_\infty$-type given by $k_\infty \mapsto \rho_{\underline{k}} \circ J(k_\infty,\mathbf{i})$. Let $(\kappa_1,....,\kappa_{2n})$ be the strictly increasing sequence of integers defined by $\kappa_i := k_i^* + 2n - i$. Let S_π be the set of finite places of $\mathcal{K}$ either ramified over $\mathbf{Q}$ or lying above primes of ramification for π. Let π' be an irreducible admissible automorphic $U_n(\mathbf{A})$ constituent of π. For $w \notin S_\pi$ we define $BC(\pi)_w$ to be the w-component of the (formal) base change of π' to $\mathrm{GL}_{2n}(\mathbf{A}_{\mathcal{K}})$.

In [Sk10] it is explained how the following theorem is a consequence of the works of Morel and Shin on the cohomology of Shimura varieties associated to unitary groups.

Theorem 7.1.1. *Suppose $\underline{k}^*$ is regular (i.e., $k_i* > k^*_{i+1}$). Let χ_π be the central character of π (a character of $\mathbf{A}_{\mathcal{K}}^\times$) and put $\psi := \chi_\pi^c$. There exists a finite extension*[7] *$L \subset \overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ and a continuous representation*

$$R_p(\pi):\ G_{\mathcal{K}} \longrightarrow \mathrm{GL}_{2n}(L)$$

such that

[7]In the statement of this theorem we use our fixed identification ι'_p of $\overline{\mathbf{Q}}_p$ with $\mathbf{C}$.

(i) *$R_p(\pi)^\vee(1-2n)\otimes\sigma_{\psi^{1+c}}\cong R_p(\pi)^c$;*
(ii) *$R_p(\pi)$ is unramified at all $w\notin S_\pi\cup\{v_0,\bar v_0\}$, and for such a w*

$$det(1-R_p(\pi)(\mathrm{frob}_w)q_w^{-s})=L(BC(\pi)_w\otimes\psi_w,s+1/2-n)^{-1},$$

where q_w is the order of the residue field at the place w;
(iii) *$R_p(\pi)|_{G_{\mathcal{K},v_0}}$ is Hodge-Tate with Hodge-Tate weights $\kappa_1,....,\kappa_{2n}$.*
(iv) *If π_p is unramified, then $R_p(\pi)|_{G_{\mathcal{K},v_0}}$ is crystalline and the eigenvalues of the Frobenius endomorphism of $D_{cris}(R_p(\pi))$ are $p^{n-1/2}a_1,...,p^{n-1/2}a_{2n}$ with $a_1,...,a_{2n}$ the Langlands parameters of $BC(\pi)_{v_0}\otimes\psi_{v_0}$.*

Note that χ_π has infinity type $z^{-|\underline{k}|}$ so $\rho_\psi|_{G_{\mathcal{K},v_0}}$ has Hodge-Tate weight 0. Also $\chi_\pi|_{\mathbf{A}^\times}=\chi_1|\cdot|_{\mathbf{Q}}^{-|\underline{k}|}$ for some finite-order character χ_1.

We record a consequence for ordinary π.

Lemma 7.1.2. *Let π be a weight $\underline{k}$ representation of $G_n(\mathbf{A})$ as above. Assume that π_p is ordinary (in the sense of 6.4.2) and unramified. Let $(p^{\kappa_1+1/2-n}a_1,\dots,p^{\kappa_{2n}+1/2-n}a_{2n})$ with $a_1,...,a_{2n}\in\overline{\mathbf{Z}}_p^\times$ be the Langlands parameters of $\mathrm{BC}(\pi)_{v_0}\otimes\psi_{v_0}$ (written in the order of decreasing valuation). Let $R_p(\pi)$ be as in Theorem 7.1.1. Then*

$$R_p(\pi)|_{G_{\mathcal{K},\mathfrak{p}}}\cong\begin{pmatrix}\xi_{2n,\mathfrak{p}}\epsilon^{-\kappa_{2n}} & * & \cdots & \cdots & * \\ & \xi_{2n-1,\mathfrak{p}}\epsilon^{-\kappa_{2n-1}} & * & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & * \\ & & & & \xi_{1,\mathfrak{p}}\epsilon^{-\kappa_1}\end{pmatrix}$$

and

$$R_p(\pi)|_{G_{\mathcal{K},\bar{\mathfrak{p}}}}\cong\begin{pmatrix}\xi_{1,\bar{\mathfrak{p}}}\epsilon^{\kappa_1+1-2n-|\underline{k}|} & * & \cdots & \cdots & * \\ & \xi_{2,\bar{\mathfrak{p}}}\epsilon^{\kappa_2+1-2n-|\underline{k}|} & * & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & * \\ & & & & \xi_{2n,\bar{\mathfrak{p}}}\epsilon^{\kappa_{2n}+1-2n-|\underline{k}|}\end{pmatrix},$$

where $\xi_{i,\mathfrak{p}}$ and $\xi_{i,\bar{\mathfrak{p}}}$ are, respectively, the unramified characters of $G_{\mathcal{K},\mathfrak{p}}$ and $G_{\mathcal{K},\bar{\mathfrak{p}}}$ such that

$$\xi_{i,\mathfrak{p}}(\mathrm{frob}_\mathfrak{p})=a_i\quad\text{and}\quad\xi_{i,\bar{\mathfrak{p}}}(\mathrm{frob}_{\bar{\mathfrak{p}}})=\chi_1(p)a_i^{-1}.$$

Proof. Let L be the field of coefficients of $R_p(\pi)$. Let $D:=D_{cris}(R_p(\pi)|_{G_{\mathcal{K},\mathfrak{p}}})$. By Conjecture 7.1.1(4), D is a filtered Φ-module of rank $2n$ over L, Φ being the crystalline Frobenius operator, and the eigenvalues of Φ are $p^{\kappa_1}a_1,\dots,p^{\kappa_{2n}}a_{2n}$.

Let $D_i\subset D$ be the filtered Φ-submodule generated by the Φ-eigenvectors for the eigenvalues $p^{\kappa_i}a_i,\dots,p^{\kappa_{2n}}a_{2n}$. The Newton number of D_{2n} is κ_{2n}, therefore its Hodge number is κ_{2n} since it must be less than or equal to κ_{2n} by weak admissibility[8] of D. Therefore $V_{cris}(D_{2n}) = L(\xi_{2n,\mathfrak{p}}\epsilon^{-\kappa_{2n}})$ is a $G_{\mathcal{K},\mathfrak{p}}$-subrepresentation of $R_p(\pi)|_{G_{\mathcal{K},\mathfrak{p}}}$. An easy induction argument shows that the $V_{cris}(D_i)$'s form a $G_{\mathcal{K},\mathfrak{p}}$-filtration of $R_p(\pi)|_{G_{\mathcal{K},v_0}}$ such that $V_i/V_{i+1} \cong L(\xi_{i,\mathfrak{p}}\epsilon^{-\kappa_i})$ which proves the claim for $R_p(\pi)|_{G_{\mathcal{K},\mathfrak{p}}}$. The claim for $R_p(\pi)|_{G_{\mathcal{K},\bar{\mathfrak{p}}}}$ then follows from part (i) of Theorem 7.1.1 and the relation $\chi_\pi|_{\mathbf{A}^\times} = \chi_1|\cdot|_{\mathbf{Q}}^{-|\underline{k}|}$. ■

7.2. **Families of Galois representations.** In the following we make use of pseudo-representations. For a precise definition of a pseudo-representation we refer the reader to [Ta91], contenting ourselves with recalling that an A-valued pseudo-representation of dimension d of a group G is a map $T : G \to A$ such that $T(g_1g_2) = T(g_2g_1)$ for any $g_1, g_2 \in G$ and $T(1) = d$ and such that T satisfies some other linear identities (essentially those satisfied by the trace of a genuine representation of dimension d of G). As a consequence of the theory of pseudo-representations we have the following (standard) proposition.

Proposition 7.2.1. *Let $K = K_p^0K^p$, $K^p = \prod_{\ell\neq p} K_\ell$, be a neat open compact subset of $G(\mathbf{A}_f)$ and let S be a finite set of primes containing p and those ℓ such that K_ℓ is not hyperspecial. There exists a pseudo-representation $T_{K^p}^S : G_{\mathcal{K}} \to \mathbf{h}^{S,0}(K^p)$ such that for each p-ordinary representation π of weight $\underline{k}$ with $\pi_f^{K^p} \neq 0$*

$$\mathrm{tr}\ (R_p(\pi)) = \lambda_\pi^{\mathrm{ord}} \circ T_{K^p}^S.$$

For the notion of a p-ordinary representation of weight $\underline{k}$ see 6.4.2. Here $\mathbf{h}^{S,0}(K^p)$ is the universal p-ordinary Hecke algebra from 6.4.

Proof. The proof is standard, but for the convenience of the reader we indicate the main points. To ease notation we write $\mathbf{h}$ for $\mathbf{h}^{S,0}(K^p)$. For each finite place v of $\mathcal{K}$ not dividing a prime in $S\cup\{p\}$ we let $T_v \in \mathbf{h}$ be the Hecke operator such that the coefficient of X^{2n-1} in the Hecke polynomial for v is $-T_v$ (the Hecke polynomial is the degree $2n$ polynomial in $\mathbf{h}[X]$ which specializes under each $\lambda_\pi^{\mathrm{ord}}$ to a polynomial whose whose value at q_v^{-s} is $L(BC(\pi)_v \otimes \psi_v, s + 1/2 - n)^{-1}$; the existence of the Hecke polynomial is a consequence of the Satake isomorphism (which we make explicit in 9.6 for the case $n = 2$). Let $G'_{\mathcal{K},S} \in G_{\mathcal{K},S}$ be the subset of Frobenius elements frob_v for these v's (for a given v the set of frob_v's make up a conjugacy class in $G_{\mathcal{K},S}$). By the Chebotarev density theorem, $G'_{\mathcal{K},S}$ is dense in $G_{\mathcal{K},S}$. For any $g \in G_{\mathcal{K},S}$ let $\{\mathrm{frob}_{v_n}\} \subset G'_{\mathcal{K},S}$ be a sequence converging to g in $G_{\mathcal{K},S}$. As $\mathbf{h}$ is compact, after possibly replacing v_n by a subsequence, $\{T_{v_n}\}$ converges in $\mathbf{h}$ to a value we denote $T(g)$. We claim that $T(g)$ does not depend on the sequence $\{\mathrm{frob}_{v_n}\}$, as we now show. Let $\Sigma \subset \mathrm{spec}(\mathbf{h})(\overline{\mathbf{Q}}_p)$ be a Zariski dense subset of arithmetic points with each $x \in \Sigma$ associated with a p-ordinary automorphic representations π_x (cf. 6.4.2). Then for each $x \in \Sigma$, $\lambda_{\pi_x}^{\mathrm{ord}}(T_v) = \mathrm{tr}\ (R_p(\pi)(\mathrm{frob}_v))$ for all v as above. By the continuity of $R_p(\pi)$, this implies that for all $x \in \Sigma$, $\lambda_{\pi_x}(T(g)) = \mathrm{tr}\ (R_p(\pi_x)(g))$ is independent of

[8] Recall that we use a geometric convention for Hodge -Tate weights.

the chosen sequence. Because the $\lambda^{\rm ord}_{\pi,x}$ are Zariski dense in $\mathrm{spec}(\mathbf{h})(\overline{\mathbf{Q}}_p)$, we conclude that $T(g)$ is also independent of the chosen sequence. Furthemore, since Σ is Zariski dense and since for each $x\in\Sigma$, $\lambda^{\rm ord}_{\pi_x}\circ T = \mathrm{tr}\,(R_p(\pi_x))$ is a pseudo-representation, the map $g\mapsto T(g)$ is also a pseudo-representation (see Lemma 1 of [Ta91]); it is immediate that $T^S_{K^p}:=T$ has the required properties.■

7.2.2. *The character $\boldsymbol{\omega}_{K^p}$.* For a prime $\ell\neq p$ let $U_\ell:=\mathcal{O}_\ell^\times\cap K_\ell$. The action of $\mathbf{A}_\mathcal{K}^{S,\times}$ on $\mathcal{M}_{\rm ord}(K^p;\Lambda_n)$ is unramified and factors through its image in $\mathbf{A}^\times_\mathcal{K}/\overline{\mathcal{K}^\times\mathbf{C}^\times\prod_{\ell\neq p}U_\ell}$. This last fact can be seen by specializing at a Zariski dense set of arithmetic points in $\mathrm{spec}\mathbf{h}^S(K^p)(\overline{\mathbf{Q}}_p)$ associated with p-ordinary automorphic representations. There is therefore a continuous character $\boldsymbol{\omega}_{K^p}:G_\mathcal{K}\to\mathbf{h}^S(K^p)^\times$ unramified outside S such that away from S, $\boldsymbol{\omega}_{K^p}\circ\mathrm{rec}_\mathcal{K}$ gives the above action. In particular, for a $\lambda_\pi\in\mathrm{spec}\mathbf{h}^S(K^p)$ associated with a p-ordinary automorphic representation π, $\lambda_\pi\circ\boldsymbol{\omega}_{K^p}$ is the Galois character associated with the central character of π. The polarization property - part (i) of Theorem 7.1.1 - then implies

$$T^S_{K^p}(cgc)=\epsilon^{1-2n}\boldsymbol{\omega}^{1+c}_{K^p}(g)T^S_{K^p}(g^{-1})\qquad\forall g\in G_\mathcal{K}. \tag{7.2.2.a}$$

7.2.3. *The restriction of $T^S_{K^p}$ to $G_{\mathcal{K},\mathfrak{p}}$.* Let $\underline{a}=(a_1,...,a_{2n})\in(\mathbf{Z}/(p-1)\mathbf{Z})^{2n}$. We let $\mathbf{h}^S_{\underline{a}}(K^p)$ be the component of $\mathbf{h}^S(K^p)$ such that $\Delta_n=(\mathbf{F}_p^\times)^{2n}\subset(\mathbf{Z}_p^\times)^{2n}=T_n(\mathcal{O}_\mathfrak{p})$ acts by the character $\omega^{\underline{a}}$. Let $\mathbf{B}$ be the total ring of fractions of $\mathbf{h}^S_{\underline{a}}(K^p)$ and let $R_{\underline{a},K^p}:G_\mathcal{K}\to \mathrm{GL}_{2n}(\mathbf{B})$ be the semisimple Galois representation with trace equal to $T^S_{K^p}$ composed with the projection onto $\mathbf{h}^S_{\underline{a}}(K^p)$. The existence of the representation $R^S_{K^p}$ follows from [Ta91, Theorem 1].

For $i=1,...,2n$ let $\boldsymbol{\xi}_{i,\mathfrak{p}}:G_{\mathcal{K},\mathfrak{p}}\to\mathbf{h}^S_{\underline{a}}(K^p)^\times$ be the unramified characters such that $u_t\in\mathbf{h}^S_{\underline{a}}(K^p)$ satisfies

$$u_t=\prod_{i=1}^{2n}\boldsymbol{\xi}_{i,\mathfrak{p}}(\mathrm{frob}_\mathfrak{p})^{\mathrm{ord}_p(t_i)}\boldsymbol{\omega}^{-c}_{K^p}(\mathrm{rec}_\mathcal{K}(t_i)).$$

Let $\boldsymbol{\delta}_i^*:=\boldsymbol{\delta}_i\circ\epsilon_\mathcal{K}$ with $\boldsymbol{\delta}_i:\mathbf{Z}_p^\times\to\Lambda_n^\times$, $i=1,...,2n$, as in 6.3.

Lemma 7.2.4. *The local representation $R^S_{\underline{a},K^p}|_{G_{\mathcal{K},\mathfrak{p}}}$ is equivalent over $\mathbf{B}$ to one of the form*

$$\begin{pmatrix}\boldsymbol{\xi}_{2n,\mathfrak{p}}\omega^{a_{2n}}\boldsymbol{\delta}^*_{2n} & * & \cdots & \cdots & * \\ & \boldsymbol{\xi}_{2n-1,\mathfrak{p}}\epsilon^{-1}\omega^{a_{2n-1}}\boldsymbol{\delta}^*_{2n-1} & * & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & * \\ & & & & \boldsymbol{\xi}_{1,\mathfrak{p}}\epsilon^{1-2n}\omega^{a_1}\boldsymbol{\delta}^*_1\end{pmatrix}$$

There is a similar description of $R^S_{\underline{a},K^p}|_{G_{\mathcal{K},\bar{\mathfrak{p}}}}$ (following from the polarization relation 7.2.2.a).

Proof. This follows easily from a successive application of Lemma 7.2 of [TU99]. That the hypotheses of this lemma are satisfied is a consequence of Lemma 7.1.2 together with the Zariski density of the p-ordinary classical points λ_π in $\mathrm{spec}(\mathbf{h}^S_{\underline{a}}(K^p)(\overline{\mathbf{Q}}_p)$ and the fact that the characters on the diagonal in the corollary specialize under λ_π to the characters on the diagonal in Lemma 7.1.2. ∎

7.3. **Selmer groups and Eisenstein ideals.** Throughout the rest of this section we assume that $n = 2$. Let $\mathbf{D} = (A, \mathbb{I}, f, \psi, \xi, \Sigma)$ be a p-adic Eisenstein datum as in 6.5.1; we freely use the notation of that section. Let $K' = K'_\Sigma K^\Sigma \subset G(\mathbf{A}^p_f)$, $K^\Sigma = G(\widehat{\mathbf{Z}}^\Sigma)$, be an open compact such that $K'K^0_p$ is neat. Let $\mathbf{h}_\mathbf{D} = \mathbf{h}_\mathbf{D}(K')$. Let $\mathbf{B}_\mathbf{D} = \mathbf{h}_\mathbf{D} \otimes_{\Lambda_\mathbf{D}} F_{\Lambda_\mathbf{D}}$ with $F_{\Lambda_\mathbf{D}}$ the field of fractions of $\Lambda_\mathbf{D}$; $\mathbf{B}_\mathbf{D}$ is the total ring of fractions of $\mathbf{h}_\mathbf{D}$. We denote by $T_\mathbf{D}$ the $\mathbf{h}_\mathbf{D}$-valued pseudo-representation obtained from $T^\Sigma_{K'}$ and by $R_\mathbf{D}$ the semisimple representation over $\mathbf{B}_\mathbf{D}$ (or any finite extension thereof) obtained from $R_{K'}$; then $\mathbf{R}_\mathbf{D}$ has trace equal to $T_\mathbf{D}$. More generally, for any component $\mathbf{J}$ of $\mathbf{h}_\mathbf{D}$, we let $T_\mathbf{J}$ denote the pseudo-representation obtained by composing the pseudo-representation $T_\mathbf{D}$ with the projection $\mathbf{h}_\mathbf{D} \twoheadrightarrow \mathbf{J}$, and we write $R_\mathbf{J}$ for the corresponding semisimple Galois representation defined over the total ring of fractions of $\mathbf{J}$.

Theorem 7.3.1. *Assume that* $(\mathbf{irred})_\mathbf{f}$ *and* $(\mathbf{dist})_\mathbf{f}$ *hold. If* $\mathbf{J}$ *is an irreducible component of* $\mathbf{T}_\mathbf{D}$*, then either (a)* $R_\mathbf{J}$ *is absolutely irreducible or (b) over some finite extension of the fraction field of* $\mathbf{J}$*,* $R_\mathbf{J} \cong R_1 \oplus R_2$ *with each* R_i *a two-dimensional representation satisfying the polarization condition* $R_i^c \cong R_i^\vee \otimes \epsilon^{-3}\boldsymbol{\omega}^{1+c}_{K'}$*.*

Proof. Assume that $R_\mathbf{J}$ is not irreducible. Since $\mathbf{J}$ is a component of $\mathbf{T}_\mathbf{D}$, the reduction $\bar{T}_\mathbf{J}$ of $T_\mathbf{J}$ modulo the maximal ideal of $\mathbf{J}$ is equal to $\bar{T}_\mathbf{D}$, which is the reduction modulo the maximal ideal of $\Lambda_\mathbf{D}$ of

$$\mathrm{tr}\ \boldsymbol{\rho}_\mathbf{D} = \sigma_{\bar{\chi}_{\mathbf{f},0}}\boldsymbol{\sigma}^c_\psi \epsilon^{-3} + \sigma_{\bar{\chi}_{\mathbf{f},0}}\boldsymbol{\sigma}^{-c}_\xi \boldsymbol{\sigma}^c_\psi \epsilon^{-2}\mathrm{tr}\ \rho_\mathbf{f} + \sigma_{\bar{\chi}_{\mathbf{f},0}} \det \rho_\mathbf{f} \boldsymbol{\sigma}^{-1}_{\xi'}\boldsymbol{\sigma}^c_\psi \epsilon^{-1}.$$

Since $\bar{\rho}_\mathbf{f}$ is irreducible and the second term in the sum satisfies the polarization condition, we are reduced to the following cases:

(1) $R_\mathbf{J}$ is a sum of two characters and an irreducible odd two dimensional Galois representation satisfying the polarization condition;
(2) $R_\mathbf{J}$ is a sum of two absolutely irreducible representations $R_1 \oplus R_2$.

Assume that $R_\mathbf{J}$ is as in case (1). Let $x \in \mathrm{Spec}\,\mathbf{J}(\overline{\mathbf{Q}}_p)$ be an arithmetic point of regular weight $\underline{k}$ for which there exist a p-ordinary cuspidal representation π_x unramified at p such that λ_{π_x} is the composite $R_{S,p} \to \mathbf{h}_\mathbf{D} \to \mathbf{J} \xrightarrow{x} \overline{\mathbf{Q}}_p$. Then $R_p(\pi_x) \cong \theta_1 \oplus \theta_2 \oplus R_3$ with θ_1 and θ_2 characters and R_3 a two-dimensional representation satisfying

$$\theta_1\theta_2^c = \sigma^{1+c}_{\chi_{\pi_x}}\epsilon^{-3} \tag{7.3.1.a}$$

$$R_3^\vee(-3)\sigma^{1+c}_{\chi_{\pi_x}} \cong R_3^c \tag{7.3.1.b}$$

$$\mathrm{tr}\ R_3 \equiv \sigma_{\bar{\chi}_\mathbf{f}\bar{\chi}_{\mathbf{f},0}}\sigma^{-c}_\xi \sigma^c_\psi \omega^{-1}\epsilon^{-2}\mathrm{tr}\ \rho_f \pmod{\mathfrak{P}_{\bar{\mathbf{Z}}_p}}, \tag{7.3.1.c}$$

where $\mathfrak{P}_{\bar{\mathbf{Z}}_p}$ is the maximal ideal of $\bar{\mathbf{Z}}_p$ and f is any specialization of $\mathbf{f}$. Using these relations, we will show that π_x is a CAP representation (that is, it has the same system of Hecke eigenvalues as a Klingen-type Eisenstein series). However, if x is chosen to have sufficiently regular weight (i.e., $k_i - k_{i+1}$ sufficiently large for all i), then π_x is not CAP by a result of Harris [Ha84, Theorem 2.5.6]. This shows that $R_{\mathbf{J}}$ is never as in case (1).

To prove that π_x is CAP, it suffices to show

(i) each θ_i is the p-adic Galois character associated with an idele class character of $\mathbf{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ of arithmetic type;
(ii) $R_3 \cong R_p(\sigma)$ for some cuspidal automorphic representation σ of $GU(1,1)(\mathbf{A})$.

As $R_p(\pi_x)$ is Hodge-Tate, so are the characters θ_1 and θ_2, and point (i) then follows from a well-known result of Serre [Se68]. It remains to prove (ii). We first show that a twist of R_3 descends to a representation of $G_{\mathbf{Q}}$ that is congruent to a twist of $\rho_{\mathbf{f}}$ modulo $\mathfrak{P}_{\bar{\mathbf{Z}}_p}$. Let $\chi := \det R_3\epsilon^3\sigma_{\chi_{\pi_x}}^{-1-c}$. Then from the polarization relation (7.3.1.b) it follows that $\chi^c = \chi^{-1}$. From (7.3.1.c) it follows that

$$\chi \equiv \sigma_{\chi_{\mathbf{f}}\chi_{\mathbf{f},0}}^{-2}\sigma_\xi^{-2c}\sigma_\psi^{2c}\omega^{-2}\epsilon^{-1}\det\rho_f\boldsymbol{\omega}_{K'}^{-1-c} \pmod{\mathfrak{P}_{\bar{\mathbf{Z}}_p}}.$$

As

$$\omega^{-2}\sigma_\psi\sigma_\psi^c \equiv \epsilon^{-1}\det\rho_f \quad \text{and} \quad \boldsymbol{\omega}_{K'} \equiv \sigma_{\chi_{\mathbf{f}}\chi_{\mathbf{f},0}}^{-1}\sigma_\psi^2\sigma_\xi^{-1}\omega^{-2},$$

we have

$$\chi \equiv \sigma_\xi^{1-c}\sigma_\psi^{c-1} \pmod{\mathfrak{P}_{\bar{\mathbf{Z}}_p}}.$$

Therefore, $\chi = \theta^{c-1}$ with θ a character of $G_{\mathcal{K}}$ congruent to $\sigma_\psi\sigma_\xi^{-1}$. Let $R := R_3 \otimes \theta$. As $R_3^c \cong R_3 \otimes \chi^{-1}$, $R \cong R^c$. Also R is congruent to $\rho_{\mathbf{f}} \otimes \omega^{-3}\sigma_\xi^{-1-c}$ modulo $\mathfrak{P}_{\bar{\mathbf{Z}}_p}$. The obstruction for R to descend to a representation of $G_{\mathbf{Q}}$ belongs to $H^2(\mathrm{Gal}(\mathcal{K}/\mathbf{Q}), \mathrm{GL}_2(\bar{\mathbf{Z}}_p))$. The image of this obstruction in $H^2(\mathrm{Gal}(\mathcal{K}/\mathbf{Q}), \mathrm{GL}_2(\bar{\mathbf{F}}_p))$ is trivial because the reduction of R descends. We deduce that the obstruction is trivial as $H^2(\mathrm{Gal}(\mathcal{K}/\mathbf{Q}), 1 + M_2(\mathfrak{P}_{\bar{\mathbf{Z}}_p})) = 1$ since p is odd. In particular, R descends to a two-dimensional p-adic representation of $G_{\mathbf{Q}}$ unramified outside finitely many places that is odd (because its reduction is odd), congruent to a twist of $\rho_{\mathbf{f}}$, and nearly ordinary at p and Hodge-Tate (since the same is true of R by Lemma 7.1.2). As $(\mathbf{dist})_{\mathbf{f}}$ is assumed to hold, it follows from the modularity results in [Wi95, TW95, Di96, SW99] that R is a Tate-twist of a representation associated with a modular form[9]. This implies (ii).

Suppose now that $R_{\mathbf{J}}$ is as in case (2). Suppose first that R_1 is one-dimensional and R_2 is three-dimensional. Let Φ_2 be the pairing on the representation space of R_2 such that $\Phi_2(R_2(g)v, w) = \boldsymbol{\omega}_{K'}^{1+c}(g)\epsilon^{-3}\Phi_2(v, R_2(cg^{-1}c)w)$ for all $v, w \in V$ and $g \in G_{\mathcal{K}}$. Since R_2 is absolutely irreducible and odd dimensional, Φ_2 has to be a symmetric bilinear form. But R_2 is residually isomorphic to the sum of a character and a twist of $\bar{\rho}_{\mathbf{f}}$ which is residually symplectic. It is easy to see that the latter is incompatible with Φ_2 being symmetric (for example, by an argument similar to that used in the proof of Lemma 4.3.3).

[9]More recent modularity results may not require $(\mathbf{dist})_{\mathbf{f}}$ at this point, but the hypothesis still figures into later arguments.

We may therefore suppose that R_1 and R_2 are two-dimensional and irreducible. If they did not satisfy the polarization property in the theorem, then necessarily $R_1^c \cong R_2^\vee \otimes \boldsymbol{\omega}_{\mathcal{K}'}^{1+c}\epsilon^{-3}$. But this would imply that $\bar{T}_{\mathbf{D}}$ is the sum of two irreducible characters of degree two or four characters of degree one, neither of which is true: $\bar{T}_{\mathbf{D}}$ is the sum of two characters of degree one and one irreducible character of degree two. ∎

Remark. Conjecturally, the components $\mathbf{J}$ such that $R_{\mathbf{J}}$ is reducible are associated with families of endoscopic representations of $U(2)\times U(2)$-type. Such families do occur when there are Eisenstein congruences for $U(1,1)$. However, if we are in a situation where such congruences do not exist, then all $\mathbf{J}$ are associated with families of stable cuspidal representations of $GU(2,2)$.

7.3.2. *Relating $\mathcal{E}_{\mathbf{D}}$ to characteristic ideals.* Let $\mathcal{E}_{\mathbf{D}} = \mathcal{E}_{\mathbf{D},K'} \subseteq \Lambda_{\mathbf{D}}$ be the Eisenstein ideal associated with $\mathbf{D}$ in 6.5.3. Assuming that $(\mathbf{irred})_{\mathbf{f}}$ and $(\mathbf{dist})_{\mathbf{f}}$ hold, we let $T_{\mathbf{f}}^+ \subseteq T_{\mathbf{f}}$ be the rank one $\mathbb{I}$-summand of $T_{\mathbf{f}}$ that is $G_{\mathbf{Q}_p}$-stable and unramified. Given a height one prime P of $\Lambda_{\mathbf{D}}$ containing $\mathcal{E}_{\mathbf{D}}$ we consider the following specific instance of the set up of 4.4.3:

- $H := G_{\mathbf{Q},\Sigma}$, $G := G_{\mathcal{K},\Sigma}$, $c =$the usual complex conjugation;
- $A_0 := \Lambda_{\mathbf{D}}$, $A := \widehat{\Lambda}_{\mathbf{D},P}$ (not to be confused with the ring A in the datum $\mathbf{D}$);
- $J_0 := \mathcal{E}_{\mathbf{D}}$, $J := \mathcal{E}_{\mathbf{D}}A$;
- $R_0 := \mathbf{T}_{\mathbf{D}}$, $I_0 := I_{\mathbf{D}}$;
- $Q \subset R_0$ is the inverse image of $P \bmod \mathcal{E}_{\mathbf{D}}$ under $\mathbf{T}_{\mathbf{D}} \to \mathbf{T}_D/I_{\mathbf{D}} = \Lambda_{\mathbf{D}}/\mathcal{E}_{\mathbf{D}}$;
- $R := \widehat{\mathbf{T}}_{\mathbf{D},Q}$, $I := I_{\mathbf{D}}R$;
- $V_0 := T_{\mathbf{f}} \otimes_{\mathbb{I}} \Lambda_{\mathbf{D}}$, $\rho_0 := \rho_{\mathbf{f}} \otimes \sigma_{\bar{\chi}_{\mathbf{f},0}} \boldsymbol{\sigma}_\xi^{-c}\boldsymbol{\sigma}_\psi^c \epsilon^{-2}$;
- $V_0^+ := T_{\mathbf{f}}^+ \otimes_{\mathbb{I}} A_0$, $V_0^- := (T_{\mathbf{f}}/T_{\mathbf{f}}^+)\otimes_{\mathbb{I}} A_0$;
- $V = V_0 \otimes_{A_0} A$, $\rho = \rho_0 \otimes_{A_0} A$, $V^\pm := V_0^\pm \otimes_{A_0} A$;
- $\chi := \sigma_{\bar{\chi}_{\mathbf{f},0}}\boldsymbol{\sigma}_\psi^c \epsilon^{-3}$ (so $\chi' = \nu\chi^{-c}$);
- $\chi' := \sigma_{\bar{\chi}_{\mathbf{f},0}}\epsilon^{-1}\det\rho_{\mathbf{f}}\boldsymbol{\sigma}_{\xi'}^{-1}\boldsymbol{\sigma}_\psi^c$;
- $\nu : \chi^c\chi' = \sigma_{\bar{\chi}_{\mathbf{f},0}}^2\epsilon^{-7}\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-2}$;
- $M := (R\otimes_A F_A)^4$, F_A the field of fractions of A;
- σ the representation on M obtained from $R_{\mathbf{D}}$.

Let $\mathcal{T} := (T_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!])(\det\rho_{\mathbf{f}}^{-1}\sigma_\xi^{-c}\boldsymbol{\varepsilon}_{\mathcal{K}}^{-c})$ and $\mathcal{T}^+ := (T_{\mathbf{f}}^+ \otimes_{\mathbb{I}} \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!])(\det\rho_{\mathbf{f}}^{-1}\sigma_\xi^{-c}\boldsymbol{\varepsilon}_{\mathcal{K}}^{-c})$, and let $Ch_{\mathcal{K}}^\Sigma(\rho_{\mathbf{f}}\otimes\sigma_\xi\boldsymbol{\varepsilon}_{\mathcal{K}}) \subset \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ be the characteristic ideal of the dual Selmer group $X_{\mathcal{K}}^\Sigma(\mathcal{T},\mathcal{T}^+)$.

Theorem 7.3.3. *Assume that* $(\mathbf{irred})_{\mathbf{f}}$ *and* $(\mathbf{dist})_{\mathbf{f}}$ *hold and that* $\mathbb{I}$ *is an integrally closed domain. Let* $P_0 \subset \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ *be a height one prime and let* $P = P_0\Lambda_{\mathbf{D}}$ *be the height one prime of* $\Lambda_{\mathbf{D}}$ *it generates. Suppose also that*

(7.3.3.a) $\quad V^+\oplus A(\chi)$ *and* $V^-\oplus A(\chi')$ *are residually disjoint modulo* P.

(i) *If* $\mathrm{ord}_P(\mathcal{E}_{\mathbf{D}}) \geq \mathrm{ord}_{P_0}(\mathcal{L}^{\Sigma}_{\bar{\chi}_{\mathbf{f}}\bar{\chi}'}) + 1$, *then*

$$\mathrm{ord}_{P_0}(Ch^{\Sigma}_{\mathcal{K}}(\rho_{\mathbf{f}} \otimes \sigma_\xi \boldsymbol{\varepsilon}_{\mathcal{K}})) \geq 1.$$

(ii) *If* $\mathcal{L}^{\Sigma}_{\bar{\chi}_{\mathbf{f}}\bar{\chi}'} \notin P_0$, *then*

$$\mathrm{ord}_{P_0}(Ch^{\Sigma}_{\mathcal{K}}(\rho_{\mathbf{f}} \otimes \sigma_\xi \boldsymbol{\varepsilon}_{\mathcal{K}})) \geq \mathrm{ord}_P(\mathcal{E}_{\mathbf{D}}).$$

Proof. We deduce this theorem from the general framework of §4. The set-up of 4.2.3 and 4.4.3 holds with the above choices of H, G, etc. That $\rho = \rho_0 \otimes_{A_0} A$ is residually disjoint from χ and χ' is a straightforward consequence of the fact that ρ is residually irreducible (since $(\mathbf{irred})_{\mathbf{f}}$ holds) and is two-dimensional. The hypothesis (7.3.3.a) ensures that χ and χ' are distinct modulo P. That ν is compatible with ρ is an immediate consequence of ρ_0 being an odd, two-dimensional representation of $G_{\mathbf{Q}}$ (and so symplectic) and ν also being odd.

Note that the (unique) extension of $\nu\chi^{-1-c}$ to an odd character of $G_{\mathbf{Q},\Sigma}$ is, in the notation of 4.4.3,

$$\nu\psi := \det \rho_{\mathbf{f}}^{-1} \sigma_{\bar{\xi}'} \boldsymbol{\varepsilon}_{\mathbf{Q}}^{-2}.$$

Note also that

$$\rho_0 \otimes \chi^{-1} = \rho_{\mathbf{f}} \otimes \det \rho_{\mathbf{f}}^{-1} \sigma_\xi^{-c} \boldsymbol{\varepsilon}_{\mathcal{K}}^{-c}.$$

Next we check that the hypotheses of 4.5.5, especially of Proposition 4.5.4, are satisfied. Recall that $T_{\mathbf{f}}^+$ is the rank one $\mathbb{I}$-summand of $T_{\mathbf{f}}$ which is $G_{\mathbf{Q},p}$-stable and unramified; it exists by virtue of the hypotheses $(\mathbf{irred})_{\mathbf{f}}$ and $(\mathbf{dist})_{\mathbf{f}}$. That hypotheses (4.5.1), (4.5.2), and (4.5.3) hold for $D = G_{\mathcal{K},\mathfrak{p}}$ is then an easy consequence of Lemma 7.2.4 and the definition of $I_{\mathbf{D}}$; we leave the details of this simple verification to the reader. That the $G_{\mathcal{K},\mathfrak{p}}$-representations $V_0^+ \oplus A_0(\chi)$ and $V_0^- \oplus A_0(\chi')$ are residually disjoint modulo the maximal ideal of A_0 (= the maximal ideal of $\Lambda_{\mathbf{D}}$) follows from (7.3.3.a). Finally, we note that (4.5.5.a) holds by Theorem 7.3.1 and Lemma 4.3.3. This establishes that the hypotheses of Propositions 4.5.6 and 4.5.8 hold.

Let L be the quotient field of A. Let $T := \Lambda_{\mathbf{D}}(\psi\nu)$ and $T^+ := 0$. Then

$$X^{\Sigma}_{\mathbf{Q}}(T, T^+) = X^{\Sigma}_{\mathbf{Q}}(\Lambda_{\mathbf{Q},A}(\sigma_{\bar{\chi}_{\mathbf{f}}\bar{\xi}'} \epsilon \boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1}), 0) \otimes_{\Lambda_{\mathbf{Q},A},\iota} \Lambda_{\mathbf{D}},$$

where $\iota : \Lambda_{\mathbf{Q},A} \to \Lambda_{\mathbf{D}}$ is given by $\iota(\gamma) = (1+W)^{-1}\gamma_+^2$. The characteristic ideal of $X^{\Sigma}_{\mathbf{Q}}(T, T^+)$ is the image under ι of the characteristic ideal of $X^{\Sigma}_{\mathbf{Q}}(\Lambda_{\mathbf{Q},A}(\sigma_{\bar{\chi}_{\mathbf{f}}\bar{\xi}'} \epsilon \boldsymbol{\varepsilon}_{\mathbf{Q}}^{-1}), 0)$. The latter equals the characteristic ideal of $X^{\Sigma}_{\mathbf{Q}_\infty,L}(\bar{\chi}_{\mathbf{f}}\bar{\xi}'\epsilon)$ by Proposition 3.2.3, which, by the Main Conjecture for $\mathbf{Q}$ (Theorem 3.5.1), is generated by $G^{\Sigma}_{\bar{\chi}_{\mathbf{f}}\bar{\xi}'}$, so the characteristic ideal of $X^{\Sigma}_{\mathbf{Q}}(T, T^+)$ is generated by $\mathcal{L}^{\Sigma}_{\bar{\chi}_{\mathbf{f}}\bar{\chi}'}$.

Let $Ch^{\Sigma}_{\mathcal{K}}(V_0(\chi^{-1}))$ be the characteristic ideal of $X^{\Sigma}_{\mathcal{K}}(V_0(\chi^{-1}), V_0^+(\chi^{-1}))$. If $\mathrm{ord}_P(\mathcal{E}_{\mathbf{D}}) \geq \mathrm{ord}_{P_0}(\mathcal{L}^{\Sigma}_{\bar{\chi}_{\mathbf{f}}\bar{\chi}'}) + 1$, then, as the characteristic ideal of $X^{\Sigma}_{\mathbf{Q}}(T, T^+)$ is generated by $\mathcal{L}^{\Sigma}_{\bar{\chi}_{\mathbf{f}}\bar{\chi}'}$, it follows from Proposition 4.5.8 that $\mathrm{ord}_P(Ch^{\Sigma}_{\mathcal{K}}(V_0(\chi^{-1})) \geq 1$. As $X^{\Sigma}_{\mathcal{K}}(V_0(\chi^{-1}), V_0^+(\chi^{-1})) = X^{\Sigma}_{\mathcal{K}}(\mathcal{T}, \mathcal{T}^+) \otimes_{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]} \Lambda_{\mathbf{D}}$, it follows that $\mathrm{ord}_{P_0}(Ch^{\Sigma}_{\mathcal{K}}(\rho_{\mathbf{f}} \otimes \sigma_\xi \boldsymbol{\varepsilon}_{\mathcal{K}})) \geq 1$. This proves part (i) of the theorem. Similarly, if $\mathrm{ord}_{P_0}(\mathcal{L}^{\Sigma}_{\bar\chi_{\mathbf{f}}\bar\chi'})=0$ then it follows from Proposition 4.5.6 that $\mathrm{ord}_P(Ch^{\Sigma}_{\mathcal{K}}(V_0(\chi^{-1})))\geq \mathrm{ord}_P(\mathcal{E}_{\mathbf{D}})$, and so $\mathrm{ord}_{P_0}(Ch^{\Sigma}_{\mathcal{K}}(\rho_{\mathbf{f}}\otimes\sigma_\xi\boldsymbol{\varepsilon}_{\mathcal{K}}))\geq \mathrm{ord}_P(\mathcal{E}_{\mathbf{D}})$. ∎

7.3.4. *Connecting characteristic ideals and p-adic L-functions.* Associated to $\mathbf{D}$ and $M_{\mathbf{D}}$ there is also a p-adic Eisenstein series $\mathbf{E}_{\mathbf{D}}$ as in Theorem 6.5.4.

Theorem 7.3.5. *Let $P\subset\Lambda_{\mathbf{D}}$ be a height one prime such that $\mathbf{E}_{\mathbf{D}}$ is non-zero modulo P and $\mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi})\geq 1$. Then*

$$\mathrm{ord}_P(Ch^{\Sigma}_{\mathcal{K}}(\rho_{\mathbf{f}}\otimes\sigma_\xi\boldsymbol{\varepsilon}_{\mathcal{K}}))\geq 1.$$

Furthermore, if $\mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\bar\chi_{\mathbf{f}}\bar\chi'})=0$ then

$$\mathrm{ord}_P(Ch^{\Sigma}_{\mathcal{K}}(\rho_{\mathbf{f}}\otimes\sigma_\xi\boldsymbol{\varepsilon}_{\mathcal{K}}))\geq \mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}).$$

Proof. By Theorem 6.5.4, $\mathrm{ord}_P(\mathcal{E}_{\mathbf{D}})\geq\mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}\mathcal{L}^{\Sigma}_{\bar\chi_{\mathbf{f}}\bar\chi'})$. So the theorem follows from Theorem 7.3.3 provided we can verify that (7.3.3.a) holds.

To prove that (7.3.3.a) holds we first note that as $G_{\mathcal{K},\mathfrak{p}}$-representations

$$V^+\oplus A(\chi)\cong\delta_{\mathbf{f}}\sigma_{\bar\chi_{\mathbf{f}},0}\boldsymbol{\sigma}_\xi^{-c}\boldsymbol{\sigma}_\psi^{-c}\epsilon^{-2}\oplus\sigma_{\bar\chi_{\mathbf{f}},0}\boldsymbol{\sigma}_\psi^{c}\epsilon^{-3}$$

and

$$V^-\oplus A(\chi')\cong\delta_{\mathbf{f}}^{-1}\sigma_{\bar\chi_{\mathbf{f}},0}\sigma_{\chi_{\mathbf{f}}}\boldsymbol{\varepsilon}_W^{-1}\boldsymbol{\sigma}_\xi^{-c}\boldsymbol{\sigma}_\psi^{c}\epsilon^{-3}\oplus\sigma_{\bar\chi_{\mathbf{f}},0}\sigma_{\chi_{\mathbf{f}}}\boldsymbol{\varepsilon}_W^{-1}\boldsymbol{\sigma}_{\xi'}^{-1}\boldsymbol{\sigma}_\psi^{c}\epsilon^{-2},$$

where $\delta_{\mathbf{f}}$ is the unramified character such that $\delta_{\mathbf{f}}(\mathrm{frob}_{\mathfrak{p}})=a(p,\mathbf{f})$. So it suffices to show that none of the four $G_{\mathcal{K},\mathfrak{p}}$-characters

$$\delta_{\mathbf{f}}^2\sigma_{\chi_{\mathbf{f}}}^{-1}\boldsymbol{\varepsilon}_W\epsilon,\ \delta_{\mathbf{f}}\sigma_{\chi_{\mathbf{f}}}^{-1}\boldsymbol{\sigma}_\xi\boldsymbol{\varepsilon}_W,\ \delta_{\mathbf{f}}\sigma_{\chi_{\mathbf{f}}}^{-1}\boldsymbol{\sigma}_\xi^{c}\boldsymbol{\varepsilon}_W,\ \sigma_{\chi_{\mathbf{f}}}\boldsymbol{\varepsilon}_W^{-1}\boldsymbol{\sigma}_{\xi'}^{-1}\epsilon$$

are congruent to 1 modulo P. That this holds for the first character (which is the quotient of the characters acting on V_0^+ and V_0^-) is immediate from $(\mathbf{dist})_{\mathbf{f}}$.

Let $P_0:=P\cap\mathbb{I}_{\mathcal{K}}$. Suppose that $\theta:=\delta_{\mathbf{f}}\sigma_{\chi_{\mathbf{f}}}^{-1}\boldsymbol{\sigma}_\xi\boldsymbol{\varepsilon}_W-1=\delta_{\mathbf{f}}\sigma_\xi\boldsymbol{\varepsilon}_{\mathcal{K}}-1$ is congruent to zero modulo P. Then $\theta\equiv 0$ modulo P_0. Let $V(P_0)\subset\mathrm{Spec}\,\mathbb{I}_{\mathcal{K}}(\overline{\mathbf{Q}}_p)$ be the closed subspace cut out by P_0. Then $V(P_0)\subset V(\theta)$ with $V(\theta)$ the subspace of $\mathrm{Spec}\,\mathbb{I}_{\mathcal{K}}(\overline{\mathbf{Q}}_p)$ defined by $\theta=0$. We will show that $V(\theta)$ has codimension two, which is a contradiction as $V(P_0)$ has codimension one. To see that $V(\theta)$ has codimension two it suffices to prove this in $(\mathrm{Spec}\,\mathbb{I}\times_{\mathrm{Spec}\,A}\mathrm{Spec}\,\Lambda_{\mathcal{K},A})(\overline{\mathbf{Q}}_p)$. By considering the restriction of θ to the inertia subgroup of $G_{\mathcal{K},\mathfrak{p}}$ it is easy to see that the projection to $\mathrm{Spec}\,\Lambda_{\mathcal{K},A}(\overline{\mathbf{Q}}_p)$ is codimension one. On the other hand, $a(p,\mathbf{f})$ is transcendental over $\mathbf{Q}_p$ so the projection of $V(\theta)$ to $\mathrm{Spec}\,\mathbb{I}(\overline{\mathbf{Q}}_p)$ has codimension one. It follows that $V(\theta)$ has codimension two. The same argument applies to $\delta_{\mathbf{f}}\sigma_{\chi_{\mathbf{f}}}^{-1}\boldsymbol{\sigma}_\xi^{c}\boldsymbol{\varepsilon}_W-1$.

Finally, suppose $\lambda:=\sigma_{\chi_{\mathbf{f}}}\boldsymbol{\varepsilon}_W^{-1}\boldsymbol{\sigma}_{\xi'}^{-1}\epsilon\equiv 1$ modulo P. Then $\lambda\equiv 1$ modulo P_0. As $\boldsymbol{\sigma}'_\xi=\sigma_{\chi_{\mathbf{f}}}^2\sigma_{\xi'}\boldsymbol{\varepsilon}_{\mathbf{Q}}^2\boldsymbol{\varepsilon}_W^{-2}$, $\lambda=\sigma_{\chi_{\mathbf{f}}\xi'}^{-1}\boldsymbol{\varepsilon}_{\mathbf{Q}}^{-2}\boldsymbol{\varepsilon}_W\epsilon$. Therefore, if $\lambda\equiv 1$ modulo P_0 then P_0 is contained in $\ker\phi$ for some arithmetic homomorphism $\phi\in\mathcal{X}^a_{\mathbb{I}_{\mathcal{K}}}$ of weight $k_\phi=3$. From the specialization property of $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}$ it follows that $\phi(\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}) \neq 0$. But $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi} \in P_0$ by hypothesis, so $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi} \in \ker\phi$. This contradiction proves that $\lambda \not\equiv 1$ modulo P. ∎

7.4. **Putting the pieces together: the proof of Theorem 3.6.1.** We now give the proof of the main theorem of this paper. Let $\mathbf{D} = (A, \mathbf{f}, 1, 1, \Sigma)$ be a p-adic Eisenstein datum with $\mathbf{f} \in \mathcal{M}^{\mathrm{ord}}(N, 1; \mathbb{I})$. Assume that

- $(\mathbf{irred})_{\mathbf{f}}$ and $(\mathbf{dist})_{\mathbf{f}}$ hold;
- A contains $\mathbf{Z}[i, D_{\mathcal{K}}^{1/2}]$;
- $\mathbb{I}$ is an integrally closed domain;
- $N = N^-N^+$ with N^- divisible only by primes that are inert in $\mathcal{K}$ and N^+ divisible only by primes that split in $\mathcal{K}$; N^- is square-free and has an odd number of prime factors and the reduction $\bar{\rho}_{\mathbf{f}}$ of $\rho_{\mathbf{f}}$ modulo the maximal ideal of $\mathbb{I}$ is ramified at all $\ell|N^-$.

By Proposition 13.4.1, after possibly replacing Σ with a larger finite set, there is an integer $M_{\mathbf{D}}$ as in 6.5.3 for which the p-adic Eisenstein series $\mathbf{E}_{\mathbf{D}}$ associated with $\mathbf{D}$ and $M_{\mathbf{D}}$ is such that if P is a height one prime of $\Lambda_{\mathbf{D}}$ dividing $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}} := \mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},1}$ then $\mathbf{E}_{\mathbf{D}}$ is non-zero modulo P. As $\mathcal{L}^{\Sigma}_1 \in \mathbb{I}[\![\Gamma^+_{\mathcal{K}}]\!]$, by Proposition 12.3.6 $\mathrm{ord}_P(\mathcal{L}^{\Sigma}_1) = 0$ if $\mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}) > 0$. Let $(\mathcal{T}, \mathcal{T}^+)$ be as in 7.3.2 and let $(\mathcal{T}^c, \mathcal{T}^{+,c})$ be the same pair with the G_K-action composed with conjugation by c. The dual Selmer groups $X^{\Sigma}_{\mathcal{K}}(T, T^+)$ and $X^{\Sigma}_{\mathcal{K}}(T^c, T^{+,c})$ are isomorphic as $\mathbb{I}_{\mathcal{K}}$-modules, and the characteristic ideal of $X^{\Sigma}_{\mathcal{K}}(T^c, T^{+,c})$ is just $Ch^{\Sigma}_{\mathcal{K}_\infty}(\mathbf{f})$. It then follow from Theorem 7.3.5 that

$$Ch^{\Sigma}_{\mathcal{K}_\infty}(\mathbf{f}) = Ch^{\Sigma}_{\mathcal{K}}(\rho_{\mathbf{f}} \otimes \boldsymbol{\varepsilon}_{\mathcal{K}}) \subseteq (\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K}}),$$

for this (possibly enlarged) Σ. That the inclusion holds for the original Σ follows from Corollary 3.2.16 and the relation (3.4.5.b). This proves Theorem 3.6.1.

8. More notation and conventions

We introduce additional notation and conventions. These are in effect for the remainder of this paper. In a (very) few instances these may involve changing the meaning of a previously defined symbol.

8.1. Characters.

8.1.1. *Local conductors.* Let ψ be a character of $\mathcal{K}_\ell^\times$. If ℓ does not split in $\mathcal{K}$ then we write $\mathrm{cond}(\psi)$ for the usual conductor of ψ. If ℓ splits in $\mathcal{K}$, then our conventions identify ψ with a pair of characters (ψ_1, ψ_2) of $\mathbf{Q}_\ell^\times$. By the conductor of ψ we mean the ideal in $\mathcal{O}_\ell = \mathbf{Z}_\ell \times \mathbf{Z}_\ell$ that is $\mathrm{cond}(\psi_1) \times \mathrm{cond}(\psi_2)$. We will write $\mathrm{cond}(\psi) = (\ell^s)$ as short hand with s taken to mean a pair $s = (s_1, s_2)$ such that $\mathrm{cond}(\psi_i) = (\ell^{s_i})$.

Our conventions regarding pairs of such exponents are that given two pairs of integers $s = (s_1, s_2)$ and $t = (t_1, t_2)$, by $\max\{s, t\}$ we mean the pair $(\max\{s_1, t_1\}, \max\{s_2, t_2\})$.

Similary, when t is an integer, by $\max\{s,t\}$ we mean $\max\{s,(t,t)\}$, and by $s \geq t$ we mean $s_1, s_2 \geq t$.

8.1.2. *Additive characters.* For a place v of $\mathbf{Q}$ we let $e_v : \mathbf{Q}_v \to \mathbf{C}$ be the standard additive character. Thus $e_\infty(x) = e(x) = e^{2\pi i x}$ and $e_\ell(1/\ell) = e(-1/\ell)$. For $x \in \mathbf{A}$ we let $e_{\mathbf{A}}(x) := \prod e_v(x_v)$, the standard additive character of $\mathbf{A}$.

8.1.3. *Gauss sums.* If ψ is a primitive Dirichlet character of conductor N then we let $G(\psi)$ be the usual Gauss sum:

$$G(\psi) := \sum_{a \in (\mathbf{Z}/N)^\times} \psi(a) e(a/N).$$

If ψ is a character of $\mathbf{Q}_\ell^\times$ and $(c_\psi) \subset \mathbf{Z}_\ell$ is the conductor of ψ, then we let

$$\mathfrak{g}(\psi, c_\psi) := \sum_{a \in (\mathbf{Z}_\ell/c_\psi)^\times} \psi(a) e_\ell(a/c_\psi).$$

We write $\mathfrak{g}_\ell(\psi)$ for $\mathfrak{g}_\ell(\psi, \ell^r)$, $r = \mathrm{ord}_\ell(c_\psi)$. If $\otimes\psi_v$ is an idele class character of $\mathbf{A}^\times$ then we set

$$\mathfrak{g}(\otimes\psi_v) := \prod_\ell \psi_\ell^{-1}(c_{\psi_\ell}) \mathfrak{g}(\psi_\ell, c_{\psi_\ell}),$$

which is independent of the choices of the c_{ψ_ℓ}'s. If $\otimes\psi_v$ is the idele class character associated with the Dirichlet character ψ then

$$\mathfrak{g}(\otimes\psi_v) = \psi(-1) G(\bar{\psi}) = \overline{G(\psi)}.$$

If ψ is a character of $\mathcal{K}_\ell$ and $c_\psi \in \mathcal{O}_\ell$ generates the conductor $\mathrm{cond}(\psi)$ and $\mathfrak{d}\mathcal{O}_\ell = (d_\ell)$, then we similarly define

$$\mathfrak{g}(\psi, c_\psi d_\ell) := \sum_{a \in (\mathcal{O}_\ell/c_\psi)^\times} \psi(a) e_\ell(\mathrm{Tr}_{\mathcal{K}/\mathbf{Q}}(a/c_\psi d_\ell)),$$

and if $\otimes\psi_v$ is an idele class character of $\mathbf{A}_\mathcal{K}^\times$ then we set

$$\mathfrak{g}(\otimes\psi_v) := \prod_\ell \psi_\ell^{-1}(c_{\psi_\ell} d_\ell) \mathfrak{g}(\psi_\ell, c_{\psi_\ell} d_\ell).$$

If ψ is a Hecke character with associated with an idele class character $\otimes\psi_v$ then we set $\mathfrak{g}(\psi) := \mathfrak{g}(\otimes\psi_v)$.

8.2. **Groups and measures.**

8.2.1. *Haar measures.* For each place v of $\mathbf{Q}$ we fix an additive Haar measure on $\mathbf{Q}_v$ so that it is the usual Lebesque measure if $v=\infty$ and so that $\mathbf{Z}_\ell$ has measure one if $v=\ell$. These give a measure on $\mathbf{A}_\mathbf{Q}$. Similarly, we define an additive Haar measure on each $\mathcal{K}_v$ such that on $\mathcal{K}_\infty=\mathbf{C}$ the measure is $2dxdy$ ($z=x+iy\in\mathbf{C}$) and such that $\mathcal{O}_\ell$ has volume $D_\ell^{-1/2}$, obtaining a measure on $\mathbf{A}_\mathcal{K}$. Note that the volume of $\mathbf{A}_\mathcal{K}/\mathcal{K}$ with respect to the induced measure is then equal to 1. Unless indicated otherwise, all integration over $\mathbf{A}_\mathbf{Q}$ or $\mathbf{A}_\mathcal{K}$ will be with respect to these measures. Multiplicative measures on $\mathbf{A}_\mathbf{Q}^\times$ and $\mathbf{A}_\mathcal{K}^\times$ are taken as the ratios of these additive measures with $|\cdot|_\mathbf{Q}$ and $|\cdot|_\mathcal{K}$, respectively.

The Haar measures on local and adelic points of algebaic groups will generally be clear from context. For example, the measures on unipotent subgroups will be taken so that the induced measures on the set of points of the obvious $\mathbf{G}_a$-subquotients are just the above measures; our conventions for tori are similar.

8.2.2. *Weyl groups.* For each prime ℓ we write $W_{G_n,\ell}$ for the Weyl group of the diagonal torus in $G_{n/\mathbf{Q}_\ell}$. For R a standard $\mathbf{Q}$-parabolic of G_n we write $W_{R,\ell}$ for the Weyl group of the diagonal torus in $R_{/\mathbf{Q}_\ell}$ (really of its standard Levi). In practice we will usually identify $W_{G_n,\ell}$, $W_{R,\ell}\backslash W_{G_n,\ell}$, etc., with sets of representatives in $G_n(\mathbf{Q}_\ell)$, which we always assume to contain 1 and, whenever possible, w_n.

8.2.3. *Lie algebras.* The Lie algebra of $G_n(\mathbf{R})$ will be denoted $\mathfrak{g}_n$; that of $\mathrm{GL}_2(\mathbf{R})$ will be denoted $\mathfrak{gl}_2$.

8.2.4. *Parabolics and compact open subgroups of G_n.* Let $P_n\subset G_n$ be the subgroup of elements g with last row $(0,...,0,*)$; it is the stabilizer of the isotropic line $0^{2n-1}\oplus\mathcal{O}\subset\mathcal{O}^{2n}$. This is a maximal parabolic. Let N_{P_n} be the unipotent radical of P_n. The inclusion

$$G_{n-1}\times Res_{\mathcal{O}/\mathbf{Z}}\mathbf{G}_m\to G_n,\quad (g,x)\mapsto\begin{pmatrix} A_g & & B_g & \\ & \mu_{n-1}(g)\bar{x}^{-1} & & \\ C_g & & D_g & \\ & & & x\end{pmatrix},$$

identifies $G_{n-1}\times Res_{\mathcal{O}/\mathbf{Z}}\mathbf{G}_m$ with a Levi subgroup M_{P_n} of P_n. Given a pair (g,x) as above, we write $m(g,x)$ for their image in M_{P_n} under the above map.

Let $Q_n\subset G_n$ be the Siegel parabolic of G_n. This is the subgroup consisting of those elements $g=\left(\begin{smallmatrix} A_g & B_g\\ C_g & D_g\end{smallmatrix}\right)\in G_n$ with $C_g=0$; it is the stabilizer of the isotropic submodule $0^n\oplus\mathcal{O}^n\subset\mathcal{O}^{2n}$. Let S_n be the group scheme defined by $S_n(R)=\{M\in\mathrm{M}_n(\mathcal{O}\otimes R)\ :\ M={}^t\bar{M}\}$. Then S_n is identified with the unipotent radical N_{Q_n} of Q_n via

$$S\mapsto r(S):=\left(\begin{smallmatrix} 1 & S\\ & 1\end{smallmatrix}\right),\quad S\in S_n(R).$$

Let $B_n\subseteq Q_n$ be the Borel defined by requiring A_g to be lower-triangular and let N_{B_n} be its unipotent radical (so $B_2=P_2\cap Q_2$ and $B_1=Q_1=P_1$). Let $T_n\subset B_n$ be the torus of diagonal matrices.

For $R = P_n$, Q_n, or B_n we let δ_R be the usual modulus character for R. However, we let δ_n be such that $\delta_n^{2n-1} = \delta_{P_n}$. (The use of the latter is more convenient for many formulas.)

For a parabolic R with unipotent radical N_R, R^{opp} is the opposite parabolic and N_R^{opp} is its unipotent radical.

We let $K_{n,\ell} = G_n(\mathbf{Z}_\ell)$. Given an element or ideal I of $\mathcal{O}$ or $\mathcal{O}_\ell$ we let $K_{Q_n,\ell}(I)$ be the stabilizer in $K_{n,\ell}$ of $I\mathcal{O}_\ell^n \oplus \mathcal{O}_\ell \subseteq \mathcal{O}_\ell^{2n}$, $K_{P_n,\ell}(I)$ the stabilizer of $\bar{I}\mathcal{O}_\ell^{2n-1} \oplus \mathcal{O}_\ell$ (where $\bar{I}$ is the conjugate element or ideal - the image of I under the action $x \mapsto \bar{x}$), and $K_{B_n,\ell}(I)$ is the subgroup of $g \in K_{Q_n,\ell}(I)$ such that A_g is lower-triangular modulo $I\mathcal{O}_\ell$. We put

$$K_\ell(I,J) := K_{Q_2,\ell}(I) \cap w_2' K_{P_2,\ell}(J) w_2', \quad w_2' := \mathrm{diag}(w_1, w_1).$$

When ℓ is understood, we frequently drop it from our notation. For non-negative integers r and s we will often write $K_{r,s}$ for $K_\ell(\lambda^r, \lambda^s)$ (ℓ will be clear from context).

8.2.5. *The group* GL_2. We denote the standard (upper-triangular) Borel of GL_2 by B' and the standard (diagonal) torus by T'. We let Z' be the center of $\mathrm{GL}_2^+(\mathbf{R})$, $K'_\infty := O_2(\mathbf{R}) \subset \mathrm{GL}_2(\mathbf{R})$ (the usual maximal compact), and $K'_{\infty,+} := SO_2(\mathbf{R})$. We let

$$K_{B'}(N) := K'(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\hat{\mathbf{Z}}) \ : \ c \in N\hat{\mathbf{Z}}\}$$

and

$$K_{B',\ell}(N) := K'_\ell(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{Z}_\ell) \ : \ c \in N\mathbf{Z}_\ell\}.$$

Additionally we let $U'(N)$ and $U'_\ell(N)$ be their respective subgroups such that $N|(d-1)$. Then $\Gamma_1(N) = U'(N) \cap \mathrm{SL}_2(\mathbf{Z})$. We also put

$$\eta := w_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).$$

8.3. **Automorphic forms and modular forms.**

8.3.1. *Automorphic forms on reductive groups.* For $G = \mathrm{GL}_2$, G_n, or U_n we will write $\mathcal{A}(G)$ for the space of automorphic forms on $G(\mathbf{A})$ and $\mathcal{A}^0(G)$ for the subspace of cusp-forms.

8.3.2. *Constant terms.* For a $\mathbf{Q}$-parabolic R of G_n, by the constant term along R of an automorphic form ϕ on $G_n(\mathbf{A})$ we mean the function

$$\phi_R(g) := \int_{N_R(\mathbf{Q})\backslash N_R(\mathbf{A})} \phi(ng)dn,$$

where N_R is the unipotent radical of R.

8.3.3. *Hermitian modular forms.* For $Z \in \mathbf{H}_n$ and $g \in G_n^+(\mathbf{R})$ we let

$$J_n(g, Z) := \det(C_g Z + D_g).$$

Let $K \subset G_n(\mathbf{A}_f)$ be an open compact subgroup and $\chi : K \to \mathbf{C}$ a finite character. Let $\kappa > 0$ be a positive integer. By a Hermitian modular form of degree n, weight κ, level K, and character χ we will mean a function $f : \mathbf{H}_n \times G_n(\mathbf{A}_f) \to \mathbf{C}$ such that $f(Z, x)$ is holomorphic as a function of Z, locally constant as a function of x, and satisfies $f(Z, xk) = \chi(k)f(Z, x)$ for all $k \in K$ and

$$f(\gamma(Z), x) = \chi(x^{-1}\gamma^{-1}x)J_n(\gamma, Z)^\kappa f(Z, x) \quad \text{for all } \gamma \in G_n^+(\mathbf{Q}) \cap xKx^{-1}$$

(plus the usual holomorphy condition at the cusps if $n = 1$). We denote the space of such functions by $M_\kappa^n(K, \chi)$. If χ is trivial then omit it from our notation. Note that if $K = \prod K_\ell$ then $M_\kappa^n(K) = \mathbf{M}_{\underline{k}}^n(K, \mathbf{C})$ with $\underline{k} = (0, ..., 0; \kappa, ..., \kappa)$, where the right-hand side is as in 5.5.3.

8.3.4. *Modular forms for* GL_2*.* We generally follow standard conventions for modular forms for GL_2, and in particular follow those given in 3.3.1 and 3.3.2.

Given a modular form $f \in M_\kappa(N, \chi)$ we write $f_{\mathbf{A}}$ for the form in $\mathcal{A}(\mathrm{GL}_2)$ defined by

$$f_{\mathbf{A}}(g) := j(g_\infty, i)^{-\kappa} \det(g_\infty)^{\kappa/2} f(g_\infty(i)), \quad g = \gamma g_\infty k \in \mathrm{GL}_2(\mathbf{Q})\mathrm{GL}_2(\mathbf{R})^+ U'(N).$$

Here $j(g, z)$ is the usual automorphy factor and $g(z)$ denotes the usual action of $g \in \mathrm{GL}_2^+(\mathbf{R})$ on the upper half-plane $\mathfrak{h} = \mathbf{H}_1$. The (unitary) central character of $f_{\mathbf{A}}$ is the idele class character associated with the Dirichlet character χ.

If $f \in S_\kappa(N, \chi)$ is an eigenform for the Hecke operators $T(m)$, $(m, N) = 1$, then we write $\pi(f)$ for the irreducible $(\mathfrak{gl}_2, K'_\infty) \times \mathrm{GL}_2(\mathbf{A}_f)$-module in $\mathcal{A}^0(\mathrm{GL}_2)$ generated by $f_{\mathbf{A}}$. With these conventions $L^S(f, s) = L^S(\pi(f), s - (\kappa - 1)/2)$ for any set of primes S containing the prime divisors of N.

Conductors and eigenvectors. Let (π, V) be an irreducible, admissible representation of $\mathrm{GL}_2(\mathbf{Q}_\ell)$ for some prime ℓ, and let χ_π be its central character. We will say that a vector $\phi \in V$ has a conductor with respect π if there is an integer $r \geq 0$ such that $\pi(k)\phi = \chi_\pi(d_k)\phi$ for all $k \in K_{B',\ell}(\ell^r)$. The conductor, denoted $\mathrm{cond}_\pi(\phi)$, is the minimal such ℓ^r.

For all integers n, the Hecke operator $U'_\ell(\ell^m)\mathrm{diag}(n, 1)U'_\ell(\ell^m)$ acts on $V^{U'_\ell(\ell^m)}$: $\phi \mapsto \sum \pi(g_i)\phi$ for $U'_\ell(\ell^m)\mathrm{diag}(n, 1)U'_\ell(\ell^m) = \sqcup g_i U'_\ell(\ell^m)$.

By an eigenvector in V we mean a vector ϕ that has a conductor, say $\mathrm{cond}_\pi(\phi) = \ell^r$, and is such that ϕ is an eigenvector for the Hecke operators $U'_\ell(\ell^m)\mathrm{diag}(n, 1)U'_\ell(\ell^m)$ for all integers n and for some (hence all) $m \geq r$; given an eigenvector ϕ, we write $a_n(\phi)$ for its eigenvalue with respect to the Hecke operator $U'_\ell(\ell^m)\mathrm{diag}(n, 1)U'_\ell(\ell^m)$.

Note that if $f \in M_\kappa(N, \chi)$ is an elliptic modular eigenform, then $f_{\mathbf{A}}$ is a pure tensor in the automorphic representation $(\pi, V) = (\otimes\pi_v, \otimes V_v)$ it generates, say $f_{\mathbf{A}} = \otimes f_v$, and each f_v is an eigenvector for π_v; if $a(1,f) = 1$ (i.e., f is a normalized eigenform) then $a_\ell(f_\ell) = \ell^{1-\kappa/2}a(\ell, f)$.

8.3.5. *A convention on sub- and superscripts.* In the case $n = 2$ we will frequently omit the sub- or superscript 2.

9. Some cuspidal Eisenstein series

In this section we define the Eisenstein series on G that we will use to study the L-values of cuspforms on GL_2. These Eisenstein series are induced from cuspforms on the Levi $M_P = G_1 \times Res_{\mathcal{K}/\mathbf{Q}}\mathbf{G}_m$ of P.

In 9.1 we recall basic facts about representations of G induced from representations of M_P, setting the stage for some explicit choices and calculations, which are detailed in 9.2. We generally work in an adelic framework. This reduces many global calculations to their local counterparts. However, for our eventual applications, the adelic results are also recast in a more classical setup. In defining the local sections, there are four situations to consider: sections at the archimedean place, at the places $\ell \neq p$ where the inducing data is unramified and no ramification is allowed, at the places $\ell \neq p$ where either there is ramification or ramification is to be allowed, and at the prime p. Each case has distinct features and is generally handled separately. For those $\ell \neq p$ where ramification is to be allowed, we take a very ramified section (supported on the 'big cell'), while at p we take the p-ordinary section which has non-zero support in the smallest cell. We identify a 'generic case' at p which - while not covering all possible inducing data - covers enough cases for the eventual application to p-adic families of Eisenstein series.

In 9.3 we explain how the local choices are combined to associate certain (good) Eisenstein series with certain global data, and then in 9.4 this is all intrepeted in a classical setup, with the Eisenstein series being seen to be holomorphic Hermitian modular forms. Of particular importance is the description of the singular terms of the Fourier expansions of these Eisenstein series as given in Lemma 9.4.1.

In 9.5 we recall the (local) Hecke algebras for the group G and calculate their actions on the sections defined in 9.2, and then in 9.6 we interpret the results in the classical setup. The key results are Propositions 9.6.1 and 9.6.2, which give the connection between the Hecke eigenvalues of the good Eisenstein series and the L-functions of the inducing data and show that when the inducing data is ordinary then so is the Eisenstein series.

Throughout this section we take $n = 2$.

9.1. **Induced representations and Eisenstein series: generalities.** In the following we establish notation for certain induced representations on $G(\mathbf{Q}_v)$ and $G(\mathbf{A})$ and recall basic facts about these representations and their connections to Eisenstein series.

9.1.1. *Induced representations: archimedean picture.* Let (π, V) be an irreducible $(\mathfrak{gl}_2, K'_\infty)$-module and suppose that π is unitary and tempered. There is an irreducible, unitary, Hilbert representation (π_1, H) of $\mathrm{GL}_2(\mathbf{R})$, unique up to isomorphism, such that (π, V) can be identified with the $(\mathfrak{gl}_2, K'_\infty)$-module comprising the K'_∞-finite, smooth vectors in H. Let χ be the central character of π_1. Let ψ and τ be unitary characters of $\mathbf{C}^\times$ such that $\psi|_{\mathbf{R}^\times} = \chi$. The representation π_1 extends to a representation ρ of $P(\mathbf{R})$ as follows. For $g = mn$, $n \in N_P(\mathbf{R})$, $m = m(bx, a) \in M_P(\mathbf{R})$ with $a, b \in \mathbf{C}^\times$, $x \in \mathrm{GL}_2(\mathbf{R})$, put

$$\rho(g)v := \tau(a)\psi(b)\pi(x)v, \quad v \in H.$$

Let $H_\infty \subseteq H$ be the smooth vectors with the usual topology (cf.[Wa92, §10.1.1]) and let $I(H_\infty)$ be the space of functions $f \in C^\infty(K_\infty, H_\infty)$ such that $f(k'k) = \rho(k')f(k)$ for $k' \in P(\mathbf{R}) \cap K_\infty$. For each $z \in \mathbf{C}$ and $f \in I(H_\infty)$ we define a function f_z on $G(\mathbf{R})$ by

$$f_z(g) := \delta(m)^{3/2+z}\rho(m)f(k), \quad g = mnk \in P(\mathbf{R})K_\infty,$$

(recall that $\delta_P = \delta^3$) and we define an action $\sigma(\rho, z)$ of $G(\mathbf{R})$ on $I(H_\infty)$ by

$$(\sigma(\rho, z)(g)f)(k) := f_z(kg).$$

The representation $(\sigma(\rho, z), I(H_\infty))$ is a smooth Fréchet representation (cf. [Wa92, §10.1.1]). Let $I(\rho)$ be the subspace of K_∞-finite vectors of $I(H_\infty)$. We obtain a $(\mathfrak{g}, K_\infty)$-module structure on $I(\rho)$ as the underlying $(\mathfrak{g}, K_\infty)$-module of the representation $(\sigma(\rho, z), I(H_\infty))$, and we denote this by $(\sigma(\rho, z),\ I(\rho))$. The representation $\sigma(\rho, z)|_{K_\infty}$ (which is independent of z) is unitary with respect to the pairing defined by

$$(f, g) := \int_{K_\infty} < f(k), g(k) >_{\pi_1} dk,$$

where $<,>_{\pi_1}$ is the Hilbert space pairing on H, and the representation $(\sigma(\rho, z), I(\rho))$ is admissible.

Let $(\pi^\vee, V)$ be the irreducible $(\mathfrak{gl}_2, K'_\infty)$-module given by $\pi^\vee(x) = \pi(\mathrm{Ad}(\eta) \cdot x)$ for x in $\mathfrak{gl}_2$ or K'_∞. This is also tempered and unitary, as is $\pi^\vee \otimes (\tau \circ \det)$ (the usual tensor product of $\pi^\vee$ with the $(\mathfrak{gl}_2, K_2)$-module associated with the character $\tau \circ \det$). We denote by $\rho^\vee, I(\rho^\vee), I^\vee(H_\infty)$, and $(\sigma(\rho^\vee, z), I(\rho^\vee))$ the representations and spaces defined as above but with π, ψ and τ replaced by $\pi^\vee \otimes (\tau \circ \det), \psi\tau\tau^c$, and $\bar{\tau}^c$, respectively (so (π_1, H) gets replaced by $(\pi_1^\vee \otimes (\tau \circ \det), H)$). Let $\tilde{\pi} := \pi^\vee \otimes \chi^{-1}$.

For each complex number z, $f \in I(H_\infty)$, and $k \in K_\infty$ consider the (Böchner) integral

$$A(\rho, z, f)(k) := \int_{N_P(\mathbf{R})} f_z(wnk)dn. \tag{9.1.1.a}$$

This integral converges absolutely and uniformly for z in compact subsets of $\{z : \mathrm{Re}(z) > 3/2\}$ (cf. [Wa92, §10.1.2]). Moreover, for such z, $A(\rho, z, f)$ is in $I^\vee(H_\infty)$ and $A(\rho, z, -) \in \mathrm{Hom}_{\mathbf{C}}(I(H_\infty), I^\vee(H_\infty))$ intertwines the actions of $\sigma(\rho, z)$ and $\sigma(\rho^\vee, -z)$. Note that if f is K_∞-finite then so is $A(\rho, z, f)$.

Let $\mathcal{F}$ be a finite collection of irreducible representations of K_∞. For any $\mu \in \mathcal{F}$ let $I(\rho)^\mu$ be the μ-isotypical subspace. Define $I(\rho)^{\mathcal{F}}$ to be the subspace $\oplus_{\mu \in \mathcal{F}} I(\rho)^\mu$.

As $I(\rho)$ is admissible the subspace $I(\rho)^{\mathcal{F}}$ is finite-dimensional. It is easily seen that if $f \in I(\rho)^{\mu}$ then $A(\rho, z, f) \in I(\rho^{\vee})^{\mu}$ (at least for $\mathrm{Re}(z) > 3/2$). Therefore the map from $\{z : \mathrm{Re}(z) > 3/2\}$ to $\mathrm{Hom}_{\mathbf{C}}(I(\rho)^{\mathcal{F}}, I(\pi^{\vee}, \tau\chi_{\pi})^{\mathcal{F}})$ given by $z \mapsto A(\rho, z, -)$ is holomorphic.

It is sometimes possible to express $A(\rho, z, f)$ in terms of well-understood functions. We do this in Section 9.2.1 for a particular choice of f in the case where π is in a (limit of) holomorphic discrete series.

9.1.2. *Induced representations: ℓ-adic picture.* Let (π, V) be an irreducible, admissible representation of $\mathrm{GL}_2(\mathbf{Q}_\ell)$ and suppose that π is unitary and tempered. Denote by χ the central character of π. Let ψ and τ be unitary characters of $\mathcal{K}_\ell^{\times}$ such that $\psi|_{\mathbf{Q}_\ell^{\times}} = \chi$. We extend π to a representation ρ of $P(\mathbf{Q}_\ell)$ on V as follows. For $g = mn$, $n \in N_P(\mathbf{Q}_\ell)$, $m = m(bx, a) \in M_P(\mathbf{Q}_\ell)$, $a, b \in \mathcal{K}_\ell^{\times}, x \in \mathrm{GL}_2(\mathbf{Q}_\ell)$, put

$$\rho(g)v := \tau(a)\psi(b)\pi(x)v, \quad v \in V.$$

Let $I(\rho)$ be the space of functions $f : K_\ell \to V$ such that (i) there exists an open subgroup $U \subseteq K_\ell$ such that $f(gu) = f(g)$ for all $u \in U$ and (ii) $f(k'k) = \rho(k')f(k)$ for $k' \in P(\mathbf{Z}_\ell)$. For each $f \in I(\rho)$ and each $z \in \mathbf{C}$ we define a function f_z on $G(\mathbf{Q}_\ell)$ by

$$f_z(g) := \delta(m)^{3/2+z}\rho(m)f(k), \quad g = mnk \in P(\mathbf{Q}_\ell)K_\ell$$

We define a representation $\sigma(\rho, z)$ of $G(\mathbf{Q}_\ell)$ on $I(\rho)$ by

$$(\sigma(\rho, z)(g)f)(k) := f_z(kg).$$

The representation $\sigma(\rho, z)|_{K_\ell}$ (which is independent of z) is unitary with respect to the pairing

$$(f, g) := \int_{K_\ell} < f(k), g(k) >_{\pi} dk,$$

where $<, >_{\pi}$ is the pairing implicit in our hypotheses on π, and $(\sigma(\rho, z), I(\rho))$ is admissible. Moreover, if π, ψ, and τ are unramified then

$$\dim_{\mathbf{C}} I(\rho)^{K_\ell} = 1. \tag{9.1.2.a}$$

In particular if $\phi \in V$ is a newvector for π and F_ρ is defined by $F_\rho(mk) = \rho(m)\phi$, $mk \in P(\mathbf{Z}_\ell)K_\ell$, then $I(\rho)^{K_\ell}$ is spanned by F_ρ.

Let $(\pi^{\vee}, V)$ be given by $\pi^{\vee}(g) = \pi(\eta^{-1}g\eta)$. This representation is also tempered and unitary. We denote by $\rho^{\vee}, I(\rho^{\vee})$, and $(\sigma(\rho^{\vee}, z), I(\rho^{\vee}))$ the representations and spaces defined as above but with π, ψ, and τ replaced by $\pi^{\vee} \otimes (\tau \circ \det)$, $\psi\tau\tau^c$, and $\overline{\tau}^c$, respectively. Let $\tilde{\pi} := \pi^{\vee} \otimes \chi^{-1}$.

For $f \in I(\rho)$, $k \in K_\ell$, and $z \in \mathbf{C}$ consider the integral

$$A(\rho, z, f)(k) := \int_{N_P(\mathbf{Q}_\ell)} f_z(wnk)dn. \tag{9.1.2.b}$$

As a consequence of our hypotheses on π this integral converges absolutely and uniformly for z and k in compact subsets of $\{z : \mathrm{Re}(z) > 3/2\} \times K_\ell$; the proof is the same as in the real case. Moreover, for such z, $A(\rho, z, f) \in I(\rho^\vee)$ and the operator $A(\rho, z, -) \in \mathrm{Hom}_{\mathbf{C}}(I(\rho), I(\rho^\vee))$ intertwines the actions of $\sigma(\rho, z)$ and $\sigma(\rho^\vee, -z)$.

For any open subgroup $U \subseteq K_\ell$ let $I(\rho)^U \subseteq I(\rho)$ be the finite-dimensional subspace consisting of functions satisfying $f(ku) = f(k)$ for all $u \in U$. Then the function $\{z \in \mathbf{C} : \mathrm{Re}(z) > 3/2\} \to \mathrm{Hom}_{\mathbf{C}}(I(\rho)^U, I(\rho^\vee)^U)$, $z \mapsto A(\rho, z, -)$, is holomorphic. It is well-known that this map has a meromorphic continuation to all of $\mathbf{C}$. For example, if τ, ψ and π are unramified then letting $F_{\rho^\vee} := F_\rho$, which is in $I(\rho^\vee)$ as well, we have by [La71, (4)]

$$A(\rho, z, F_\rho) = D_\ell^{-1} \frac{L(\tilde{\pi} \otimes \xi, z) L(\bar{\tau}', 2z)}{L(\tilde{\pi} \otimes \xi, z+1) L(\bar{\tau}', 2z+1)} F_{\rho^\vee}, \tag{9.1.2.c}$$

where

$$\xi := \psi/\tau, \quad \tau' := \tau|_{\mathbf{Q}_\ell^\times}. \tag{9.1.2.d}$$

More generally, let q be the order of the residue field of a prime of $\mathcal{K}$ over ℓ, and for $L(\tilde{\pi} \otimes \xi, z) L(\bar{\tau}', 2z) = \prod_{i=1}^t (1 - \alpha_i q^{-z})^{-1}$, $t \leq 3$, put

$$a(\rho, z) := \prod_{i=1}^t (1 - \alpha_i^2 q^{-2z}).$$

Lemma 9.1.3. *For any open subgroup $U \subseteq K_\ell$ the map $\{z \in \mathbf{C} : \mathrm{Re}(z) > 3/2\} \to \mathrm{Hom}_{\mathbf{C}}(I(\rho)^U, I(\rho^\vee)^U)$ given by $z \mapsto a(\rho, z) A(\rho, z, -)$ has a holomorphic continuation to all of $\mathbf{C}$. In particular, $A(\rho, z, -)$ has an analytic continuation to $\{z \in \mathbf{C} : \mathrm{Re}(z) \neq 0, \pm 1/2\}$.*

This is a special case of [Sha81, Thm. 2.2.2].

9.1.4. *Induced representations: global picture.* The space of cuspforms $\mathcal{A}^0(\mathrm{GL}_2)$ is an admissible $(\mathfrak{gl}_2, K'_\infty) \times \mathrm{GL}_2(\mathbf{A}_f)$-module. Let V be an irreducible submodule and write π for the action of $(\mathfrak{gl}_2, K') \times \mathrm{GL}_2(\mathbf{A}_f)$ on V. Then (π, V) can be identified with a restricted tensor product of local irreducible admissible representations. More precisely, there is an irreducible admissible $(\mathfrak{gl}_2, K'_\infty)$-module (π_∞, V_∞), an irreducible admissible $\mathrm{GL}_2(\mathbf{Q}_\ell)$-representation (π_ℓ, V_ℓ) for each prime ℓ, and an isomorphism $V \simeq \otimes V_w$ (restricted tensor product with w running over all the places of $\mathbf{Q}$) intertwining the actions of π and $\otimes \pi_w$. The representations π_ℓ are almost always unramified, and implicit in the definition of $\otimes V_w$ is a choice of a newvector ϕ_ℓ in V_ℓ for the primes ℓ for which π_ℓ is unramified. We fix once-and-for-all such identifications $V = \otimes V_w$ and $\pi = \otimes \pi_w$. We will assume that for each place w, (π_w, V_w) satisfies the hypotheses of sections 9.1.1 and 9.1.2. Let χ_π be the central character of π.

Let $\tau, \psi : \mathbf{A}_{\mathcal{K}}^\times \to \mathbf{C}^\times$ be Hecke characters such that $\psi|_{\mathbf{A}_{\mathbf{Q}}^\times} = \chi_\pi$ and let $\tau = \otimes \tau_w$ and $\psi = \otimes \psi_w$ be their local decompositions, w running over places[10] of $\mathbf{Q}$. We associate

[10]Here and throughout we will often view ψ as a function on the $\mathbf{A}$-points of $\mathrm{Res}_{\mathcal{K}/\mathbf{Q}} \mathbf{G}_m$, so for a place w of $\mathbf{Q}$, $\psi_w = \otimes_{v|w} \psi_v$.

with the triple (τ, ψ, π) a representation of $(P(\mathbf{R}) \cap K_\infty) \times P(\mathbf{A}_f)$ on V as follows. For $m \in (P(\mathbf{R}) \cap K_\infty) \times P(\mathbf{A}_f)$ and $v = \otimes v_w \in V$ put

$$\rho(m)v := \otimes(\rho_w(m_w)v_w),$$

where ρ_w is the representation associated with the triple (τ_w, ψ_w, π_w) as in 9.1.1 or 9.1.2.

Let $K_f := \prod_{w\nmid\infty} K_w$ and $K_\mathbf{A} := K_\infty \times K_f$. Let $I(\rho)$ be the space of functions $f : K_\mathbf{A} \to V$ such that (i) $f(k'k) = \rho(k')f(k)$ for $k' \in P(\mathbf{A}) \cap K_\mathbf{A}$, $k \in K_\mathbf{A}$, and (ii) f factors through $K_\infty \times K_f/K'$ for some open subgroup $K' \subseteq K_f$ and f is K_∞-finite and smooth as a function on $K_\infty \times K_f/K'$. Let $\otimes I(\rho_w)$ be the restricted tensor product with respect to the F_{ρ_w}'s at those w at which τ_w, ψ_w, and π_w are unramified. We assume that the F_{ρ_w}'s are defined using the fixed newvectors ϕ_w. The $\mathbf{C}$-linear map

$$\otimes I(\rho_w) \to I(\rho), \quad \otimes f_w \mapsto (k \mapsto \otimes f_w(k_w))$$

is easily seen to be an isomorphism of $K_\mathbf{A}$-representations. Henceforth we identify the spaces $I(\rho)$ and $\otimes I(\rho_w)$ in this way.

For each $z \in \mathbf{C}$ and $f \in I(\rho)$ we define a function f_z on $G(\mathbf{A})$ as follows. For $f = \otimes f_w$ we put

$$f_z(g) := \otimes f_{w,z}(g_w)$$

where $f_{w,z}$ is defined as in §9.1.1, 9.1.2. We define an action $\sigma(\rho, z)$ of $(\mathfrak{g}, K_\infty) \times G(\mathbf{A}_f)$ on $I(\rho)$ by

$$\sigma(\rho, z) := \otimes\sigma(\rho_w, z).$$

It follows from the admissibility of each $(\sigma(\rho_w, z), I(\rho_w))$ that $(\sigma(\rho, z), I(\rho))$ is an admissible $(\mathfrak{g}, K_\infty) \times G(\mathbf{A}_f)$-module.

We define $\rho^\vee$, $I(\rho^\vee)$, and $\sigma(\rho^\vee, z)$ in the same way but with each ρ_w replaced by $\rho_w^\vee$, and we make a similar identification of $I(\rho^\vee)$ with $\otimes I(\rho_w^\vee)$ (where the restricted tensor product is defined with respect to the $F_{\rho_w^\vee}$'s; if τ_w, ψ_w, and π_w are all unramified then $F_{\rho_w^\vee} = F_{\rho_w}$ is in $I(\rho_w^\vee)$).

For each $z \in \mathbf{C}$ there are maps

$$I(\rho), I(\rho^\vee) \hookrightarrow \mathcal{A}^0(M_P(\mathbf{Q})N_P(\mathbf{A})\backslash P(\mathbf{A})),$$

both given by

$$f \mapsto (g \mapsto f_z(g)(1)).$$

So for f in $I(\rho)$ or $I(\rho^\vee)$, in the context of automorphic forms we will often write f_z to mean the cuspform in $\mathcal{A}^0(M_P(\mathbf{Q})N_P(\mathbf{A})\backslash P(\mathbf{A}))$ given by this recipe (the space denotes the space of cuspforms on $P(\mathbf{A})$ as in [MoWa95]).

9.1.5. *Klingen-type Eisenstein series on G.* Let π, ψ, and τ be as in 9.1.4. For $f \in I(\rho)$, $z \in \mathbf{C}$, and $g \in G(\mathbf{A})$ the series

$$E(f, z, g) := \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} f_z(\gamma g) \tag{9.1.5.a}$$

is known to converge absolutely and uniformly for (z, g) in compact subsets of $\{z \in \mathbf{C} \;:\; \mathrm{Re}(z) > 3/2\} \times G(\mathbf{A})$ and to define an automorphic form on G. (cf. [MoWa95, II.1.5],[La76, Lemma 4.1]). The map $f \mapsto E(f, z, -)$ intertwines the action of $\sigma(\rho, z)$ and the usual action of $(\mathfrak{g}, K_\infty) \times G(\mathbf{A}_f)$ on $\mathcal{A}(G)$.

To describe the constant terms of $E(f, z, g)$ we consider the the global analog of the integrals (9.1.1.a) and (9.1.2.b). For $f \in I(\rho)$, $z \in \mathbf{C}$, and $k \in K_{\mathbf{A}}$ put

$$A(\rho, z, f)(k) := \int_{N_P(\mathbf{A})} f_z(wnk)dn. \tag{9.1.5.b}$$

It is known that this integral converges absolutely and uniformly for z in compact subsets of $\{z \in \mathbf{C} \;:\; \mathrm{Re}(z) > 3/2\}$. For such z, $A(\rho, z, f) \in I(\rho^\vee)$, and the operator $A(\rho, z, -) \in \mathrm{Hom}_{\mathbf{C}}(I(\rho), I(\rho^\vee))$ intertwines the actions of $\sigma(\rho, z)$ and $\sigma(\rho^\vee, -z)$. Moreover, for any pair $\mathcal{F} = (\mathcal{F}_\infty, U)$ consisting of a finite set $\mathcal{F}_\infty$ of irreducible representations of K_∞ and a compact open subgroup $U \subseteq G(\mathbf{A}_f)$, the function $\{z \in \mathbf{C} \;:\; \mathrm{Re}(z) > 3/2\} \to \mathrm{Hom}_{\mathbf{C}}(I(\rho)^{\mathcal{F}}, I(\rho^\vee)^{\mathcal{F}})$, $z \mapsto A(\rho, z, -)$, is holomorphic (the superscript '$\mathcal{F}$' denotes the sum of the μ-isotypical pieces, $\mu \in \mathcal{F}_\infty$, of the space of vectors fixed by U). If $f = \otimes f_w$, then, for $\mathrm{Re}(z) > 3/2$ at least,

$$A(\rho, z, f)(k) = \otimes A(\rho_w, z, f_w)(k_w), \tag{9.1.5.c}$$

where $A(\rho_w, z, f_w)(k_w)$ is the integral (9.1.1.a) or (9.1.2.b).

The convergence properties of the series (9.1.5.a) and the integral (9.1.5.b) imply the following lemma about the constant terms of the $E(f, z, g)$'s (cf. [MoWa95, II.1.7]).

Lemma 9.1.6. *Let R be a standard $\mathbf{Q}$-parabolic of G (i.e., $R \supseteq B$). Suppose* $\mathrm{Re}(z) > 3/2$.

(i) *If $R \neq P$ then $E(f, z, g)_R = 0$;*
(ii) *$E(f, z, -)_P = f_z + A(\rho, f, z)_{-z}$.*

Let $\mathcal{U} := \{z \in \mathbf{C} \;:\; \mathrm{Re}(z) > 3/2\}$. Let $\mathcal{F} = (\mathcal{F}_\kappa, U)$ be as above and and let $\varphi : \mathbf{C} \to I(\rho)^{\mathcal{F}}$ be a meromorphic function. Let $\mathcal{U}_\varphi \subseteq \mathcal{U}$ be the subregion on which φ is holomorphic. The functions

$$E(\varphi, -) : \mathcal{U} \to \mathcal{A}(G)^{\mathcal{F}}, \quad E(\varphi, z)(g) = E(\varphi(z), z, g),$$

$$A(\varphi, -) : \mathcal{U} \to I(\rho^\vee)^{\mathcal{F}}, \quad A(\varphi, z) = A(\varphi(z), z, f),$$

are meromorphic on $\mathcal{U}$ and holomorphic on $\mathcal{U}_\varphi$. The general theory of Eisenstein series provides a meromorphic continuation of these functions to all of $\mathbf{C}$, but this is not needed for our purposes and so not recalled here.

9.2. **Induced representations again: good sections.**

9.2.1. *Archimedean sections.* Returning to the set-up of 9.1.1 we further assume that

$$\pi \cong \pi(\mu_1, \mu_2), \quad \mu_1 = \mathrm{sgn}^{a_1}|\cdot|^{\kappa-1/2}, \quad \mu_2 = \mathrm{sgn}^{a_2}|\cdot|^{1-\kappa/2}, \\ a_1 + a_2 \equiv \kappa \pmod 2, \quad \kappa \geq 1, \tag{9.2.1.a}$$

and that

$$\psi(x) = \tau(x) = (x/|x|)^{-\kappa}. \tag{9.2.1.b}$$

Let $V_\kappa := \mathbf{C}v^+ \oplus \mathbf{C}v^-$. We define an action ξ_κ of K_∞ on V_κ by $\mathrm{diag}(1_2, -1_2)v^\pm = v^\mp$ and

$$k \cdot v^+ = J(k, \mathbf{i})^{-\kappa} v^+ \quad k \cdot v^- = \det(k)^{-\kappa} J(k, \mathbf{i})^\kappa v^-$$

for all $k \in K_\infty^+$. A straightforward application of Frobenius reciprocity yields

$$\dim_{\mathbf{C}} \mathrm{Hom}_{K_\infty}(V_\kappa, I(\rho)) = 1 = \dim_{\mathbf{C}} \mathrm{Hom}_{K_\infty}(V_\kappa, I(\rho^\vee)). \tag{9.2.1.c}$$

If $\Psi \in \mathrm{Hom}_{K_\infty}(V_\kappa, I(\rho))$ then for any $k \in K_\infty^+$,

$$\Psi(v^+)(k) = \Psi(k \cdot v^+)(1) = J(k, \mathbf{i})^{-\kappa} \Psi(v^+)(1).$$

Thus $\Psi(v^+)(1)$ is a multiple of the unique (up to scalar) vector $x^+ \in V$ such that $k \cdot x^+ = j(k, i)^{-\kappa} x^+$ for all $k \in K'_{\infty,+}$. The same holds for any $\Psi \in \mathrm{Hom}_{K_\infty}(V_\kappa, I(\rho^\vee))$.

Fix $0 \neq \Psi \in \mathrm{Hom}_{K_\infty}(V_\kappa, I(\rho))$ and $0 \neq \Psi^\vee \in \mathrm{Hom}_{K_\infty}(V_\kappa, I(\rho^\vee))$ so that

$$\Psi(v^+)(1) = \Psi^\vee(v^+)(w). \tag{9.2.1.d}$$

(This is possible by the preceding observation that these values are *a priori* scalar multiples of each other.) Note that Ψ and $\Psi^\vee$ are uniquely determined up to the same non-zero scalar multiple by (9.2.1.d). It is then clear from (9.2.1.c) that there exists a constant $c(\rho, z)$ independent of the choice of the pair $\Psi, \Psi^\vee$ satisfying (9.2.1.d) such that

$$A(\rho, z, \Psi(v)) = c(\rho, z)\Psi^\vee(v) \quad \text{for all } v \in V_\kappa. \tag{9.2.1.e}$$

Lemma 9.2.2. *Under the hypotheses* (9.2.1.a), (9.2.1.b) *and with the above notation,*

$$c(\rho, z) = i^{-\kappa}\pi^3 2^{3-2z} \frac{\Gamma(2z)\Gamma(z + (\kappa-1)/2)}{\Gamma(z + (\kappa+1)/2)^2 \Gamma(z + (3-\kappa)/2)}.$$

Proof. The proof of this lemma is a straightforward calculation. Let $V(\mu_1, \mu_2)$ be the usual realization of $\pi(\mu_1, \mu_2)$ as a space of smooth, K'_∞-finite functions f on $\mathrm{GL}_2(\mathbf{R})$ such that

$$f(\left(\begin{smallmatrix} a & * \\ & d \end{smallmatrix}\right) g) = \mu_1(a)\mu_2(d)|a/d|^{1/2} f(g).$$

Let $W(z)$ be the space of smooth, K_∞-finite functions $f : G(\mathbf{R}) \to \mathbf{C}$ such that for $x, y \in \mathbf{R}^\times, a, b \in \mathbf{C}^\times$,

$$f(\begin{pmatrix} ax & & * & * \\ * & * & * & * \\ & & ay & * \\ & & & b \end{pmatrix} g) = \psi(ab)\mu_1(x)\mu_2(y)|x/y|^{\frac{1}{2}} |a^2/b^2xy|^{\frac{3}{2}+z} f(g).$$

Then $W(z)$ is an admissible $(\mathfrak{g}, K_\infty)$-module in the standard way (K_∞ acts by right translation).

Fix an identification of (π, V) with $(\pi(\mu_1,\mu_2), V(\mu_1,\mu_2))$ and let $\iota : V \xrightarrow{\sim} V(\mu_1,\mu_2)$ be the corresponding identification. This also identifies $(\pi^\vee, V)$ with $(\pi(\mu_1,\mu_2), V(\mu_1,\mu_2))$, the corresponding identification $\iota^\vee : V \xrightarrow{\sim} V(\mu_1,\mu_2)$ being given by $\iota^\vee(v) := (g \mapsto \iota(v)(w_1{}^t g))$. Then the maps $\Phi : I(\rho) \to W(z)$ and $\Phi^\vee : I(\rho^\vee) \to W(z)$ given by

$$f \mapsto (g \mapsto \iota(f_z(g))(1)) \quad \text{and} \quad f \mapsto (g \mapsto \iota^\vee(f_z(g))(1)),$$

respectively, identify $(\sigma(\rho, z), I(\rho))$ and $(\sigma(\rho^\vee, z), I(\rho^\vee))$ with $(\mathfrak{g}, K_\infty)$-submodules of $W(z)$.

We define an intertwining operator $A(z,-) : W(z) \to W(-z)$ by

$$A(z,f)(g) := \int_{N_P(\mathbf{R})} f(wng)dn.$$

(The convergence of this operator can be proven just as is done for (9.1.1.a), so we omit doing so here.) From our identifications we obtain

$$A(z,\Phi(f)) = \Phi^\vee(A(\rho,z,f)), \quad f \in I(\rho).$$

Putting $f^+ = \Psi(v^+)$, it then follows from (9.2.1.d) and (9.2.1.e) that

$$\begin{aligned} A(z,\Phi(f^+))(w) &= \Phi^\vee(A(\rho,z,f^+))(w) \\ &= \Phi^\vee(c(\rho,z)\Psi^\vee(v^+))(w) \\ &= c(\rho,z)\iota^\vee(\Psi^\vee(v^+)(w))(1) \\ &= c(\rho,z)\iota^\vee(f^+(1))(1) \\ &= c(\rho,z)\iota(f^+(1))(w_1) \\ &= c(\rho,z)i^\kappa\iota(f^+(1))(1) \\ &= c(\rho,z)i^\kappa\Phi(f^+)(1). \end{aligned} \tag{9.2.2.a}$$

Therefore, to prove the lemma it suffices to compute the left-hand side of (9.2.2.a). To do this, note that for $\mathrm{Re}(z)$ sufficiently large we have

$$A(z,\Phi(f^+))(w) = \int_{\mathbf{C}}\int_{\mathbf{R}}\int_{\mathbf{C}} \Phi(f^+)(\begin{pmatrix} 1 & n_3 & & \\ & 1 & & \\ & n_1 & 1 & \\ \bar{n}_1 & n_2-\bar{n}_1\bar{n}_3 & -\bar{n}_3 & 1 \end{pmatrix})dn_1dn_2dn_3.$$

This integral can be evaluated by standard techniques: integrating the variables in order, treating each as coming from a unipotent subgroup of a rank one group (essentially as is done in the computation of the Gindikin-Karpelevich formula; c.f. [GK62] and [La71]). The computation yields

$$A(z,\Phi(f^+))(w) = \pi^3 2^{3-2z}\frac{\Gamma(2z)\Gamma(z+(\kappa-1)/2)}{\Gamma(z+(\kappa+1)/2)^2\Gamma(z+(3-\kappa)/2)}\Phi(f^+)(1).$$

The lemma follows upon comparing this with (9.2.2.a). ∎

9.2.3. *ℓ-adic sections.* There are three possibilities for $\mathcal{K}_\ell$: (1) $\mathcal{K}_\ell$ is an unramified field extension of $\mathbf{Q}_\ell$, (2) $\mathcal{K}_\ell$ is a totally ramified field extension of $\mathbf{Q}_\ell$, and (3) $\mathcal{K}_\ell$ is a split extension of $\mathbf{Q}_\ell$, in which case we have fixed an identification $\mathcal{K}_\ell = \mathbf{Q}_\ell \times \mathbf{Q}_\ell$. In cases (1) and (2) we let λ be a uniformizer of $\mathcal{O}_\ell$ with the additional restriction that $\lambda = \ell$ in case (1). In both cases we let q be the order of the residue field of $\mathcal{O}_\ell$. In the third case we let $\lambda := q := \ell$.

Returning to the set-up of 9.1.2, we let (λ^{r_ψ}) and (λ^s) be the conductors of ψ and ξ, respectively. For $K_{r,s}$ with $r \geq \max\{r_\psi, s\}$ we define a character ν of $K_{r,s}$ by

$$\nu(\begin{pmatrix} \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & * \\ * & * \end{pmatrix}) := \psi(ad-bc)\bar{\xi}(d).$$

For $K \subseteq K_{r,s}$ let

$$I(\rho, K) := \{f \in I(\rho) \ : \ \rho(k)f = \nu(k)f, \ k \in K\}.$$

Let $\phi \in V$ be any vector having a conductor with respect to $\tilde{\pi}$ (see 8.3.4) and let $(\lambda^{r_\phi}) := \mathrm{cond}_{\tilde{\pi}}(\phi)$. For any $K_{r,t}$ with $r \geq \max(r_\psi, r_\phi, s)$ and $t \geq s$ we define $F_{\phi,r,t} \in I(\rho, K_{r,t})$ by

$$F_{\phi,r,t}(g) := \begin{cases} \nu(k)\rho(p)\phi & g = pwk \in P(\mathbf{Z}_p)wK_{r,t} \\ 0 & \text{otherwise.} \end{cases}$$

Note that since $P(\mathbf{Z}_\ell)wK_Q(\lambda) = P(\mathbf{Z}_\ell)wQ(\mathbf{Z}_\ell)$, if $r, r' \geq 1$ then $F_{\phi,r,t} = F_{\phi,r',t}$.

Lemma 9.2.4. *Let $K = K_{r,t}$ as above. Suppose $r > 0$.*

(i) *Suppose $t = s$. If $F \in I(\rho, K)$ is supported on $P(\mathbf{Z}_p)wK_Q(\lambda)$ then F is supported on $P(\mathbf{Z}_p)wK$ and so is determined by its value on w.*
(ii) *If $t > 0$ then for $\mathrm{Re}(z) > 3/2$, $A(\rho, z, F_{\phi,r,t})(1) = D_\ell|\lambda^t|_{\mathcal{K}}\phi$.*

Proof. For part (i) we note that it is enough to show that the function

$$f(A) := \bar{\psi}(\det A)F(w\mathrm{diag}(A, {}^t\bar{A}^{-1})), \quad A \in \mathrm{GL}_2(\mathcal{O}_\ell),$$

is supported on $B'(\mathcal{O}_\ell)K_{B',\ell}(\lambda^s)$. To see that f satisfies this we observe that for $k \in K_{B',\ell}(\lambda^s)$,

$$\begin{aligned} f(\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) Ak) &= \xi(dd_k)\pi(\begin{pmatrix} {\scriptstyle (a\bar{a})^{-1}} & \\ & {\scriptstyle 1} \end{pmatrix})f(A) \\ &= \xi(dd_k)F(\begin{pmatrix} {\scriptstyle (a\bar{a})^{-1}1_2} & \\ & {\scriptstyle 1_2} \end{pmatrix} w \left(\begin{smallmatrix} A & \\ & {}^t\bar{A}^{-1} \end{smallmatrix}\right)) \\ &= \xi(dd_k)f(A). \end{aligned}$$

Then a well-known argument (such as that used to characterize newvectors for principal series representations - cf. the proof on [Ca73, Thm. 1]) shows f has the desired support.

For part (ii) we note that $P(\mathbf{Q}_\ell)wK_{r,t} = P(\mathbf{Q}_\ell)w(N_P(\mathbf{Z}_p) \cap K_{r,t})$, and so

$$wN_P(\mathbf{Q}_\ell) \cap P(\mathbf{Q}_\ell)wK_{r,t} = w(N_P(\mathbf{Q}_\ell) \cap w^{-1}P(\mathbf{Q}_\ell)w)(N_P(\mathbf{Z}_\ell) \cap K_{r,t}) = w(N_P(\mathbf{Z}_\ell) \cap K_{r,t}).$$

It then follows that for $F = F_{\phi,r,t}$

$$A(\rho, z, F)(1) = \int_{N_P(\mathbf{Z}_\ell)\cap K_{r,t}} F_z(wn)dn = D_\ell|\lambda^t|_{\mathcal{K}}F(w) = D_\ell|\lambda^t|_{\mathcal{K}}\phi.$$

■

9.2.5. *p-adic sections.* We consider the situation of the previous section for the prime p, recalling that p is split in $\mathcal{K}$ (i.e., possibility (3) holds).

Let ρ_1 be the representation of $P(\mathbf{Q}_p)$ associated with $(\pi_1, \psi_1, \tau_1) := (\tilde{\pi}, \bar{\psi}, \psi^c\bar{\psi}\bar{\tau}^c)$. Note that $\xi_1 = \psi_1/\tau_1$ satisfies $\bar{\xi}_1^c = \xi$. Let (p^{r_ψ}) be the conductor of ψ (so also the conductor of ψ_1) and let (p^s) and (p^{s_1}) be the respective conductors of ξ and ξ_1 (if $s = (s', s'')$ then $s_1 = (s'', s')$). Let $\nu_1 : K_{r_1,s_1} \to \mathbf{C}$, $r_1 := \max\{r_\psi, s_1\}$, be the character associated with ρ_1 (denoted ν in the preceding section). Let ν_0 be the character of K_{r_1,s_1} defined by

$$\nu_0(\begin{pmatrix} * & * \\ * & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \end{pmatrix}) := \psi(ad - bc)\bar{\xi}(d).$$

Note that $\nu_0(k) = \nu_1(k)\psi(\det k)\xi_1(\mu(k))$. For any subgroup $K \subseteq K_{r_1,s_1}$ we let

$$I(\rho, K)^0 := \{f \in I(\rho) \ : \ f(gk) = \nu_0(k)f(g), k \in K\}.$$

Let $\phi \in V$ be an eigenvector for π such that $p|\mathrm{cond}_\pi(\phi)$ and let (p^{r_ϕ}) be its conductor. Let $r := \max\{r_\phi, r_1\}$ and let $t := \max\{s_1, 1\}$. Put $K_\phi := K_{r,t}$. Let $F_\phi^1 = F_{\phi,r,t} \in I(\rho_1, K_\phi)$ be as in the preceding section. For $\mathrm{Re}(z) < -3/2$ we define $F^0_{\phi,z} \in I(\rho, K_\phi)^0$ by

$$F^0_{\phi,z}(g) := \psi(\det g)\bar{\xi}(\mu(g))A(\rho_1, -z, F^1_\phi)(g).$$

For general $z \in \mathbf{C}$ we define $F^0_{\phi,z}$ to be the value at z of the meromorphic continuation to $\mathbf{C}$ of the function $z \mapsto \psi(\det g)\bar{\xi}(\mu(g))A(\rho^\vee, -z, F^1_\phi)(g) \in I(\rho)^U$, where $U \subset K$ is any open subgroup fixing F^1_ϕ. The existence of this meromorphic continuation is a consequence of Lemma 9.1.3 as is the fact that $F^0_{\phi,z}$ is defined if $\mathrm{Re}(z) \neq 0, -1/2$. We explicitly determine $F^0_{\phi,z}$ in one useful case.

<u>*The Generic Case*</u>. We say that π, ϕ, ψ, and τ are in the Generic Case if

- $\pi_1 \cong \pi(\mu_1, \mu_2)$ with μ_1 unramified and μ_2 ramified,
- ϕ is a newvector for π,
- $\mathrm{cond}(\psi_1) =: (p^{n_1})$ and $\mathrm{cond}(\tau_1) =: (p^{m_1})$, $n_1 = (n_1', n_1'')$ and $m_1 = (m_1', m_1'')$, satisfy $n_1' > m_1' > m_1'' > n_1'' > 0$.

In this case the characters ν_1 and ν_0 can be extended to a larger group. Recall that we have identified $G(\mathbf{Q}_p) = \mathrm{GL}_4(\mathbf{Q}_p) \times \mathbf{Q}_p^\times$. We let $K_\phi' \subset G(\mathbf{Z}_p)$ be the subgroup identified with

$$\{(g, x) \in \mathrm{GL}_4(\mathbf{Z}_p) \times \mathbf{Z}_p^\times \ : \ g = \begin{pmatrix} * & * & * & * \\ p^{n_1}* & * & * & * \\ p^{n_1'}* & p^{m_1'}* & * & p^{m_1''}* \\ p^{n_1'}* & p^{m_1'}* & * & * \end{pmatrix}\}.$$

Note that $K_\phi \subseteq K'_\phi$ and that the formulas defining ν_1 and ν_0 extend to K'_ϕ. We define $I(\rho_1, K'_\phi)$ and $I(\rho, K'_\phi)^0$ as we did $I(\rho_1, K_\phi)$ and $I(\rho, K_\phi)^0$. Then $F^1_\phi \in I(\rho_1, K'_\phi)$ and $F^0_\phi \in I(\rho, K'_\phi)$.

Proposition 9.2.6. $\dim_{\mathbf{C}} I(\rho_1, K'_\phi) = 1$.

Proof. We can view $I(\rho_1)$ as a space of V-valued functions on $\mathrm{GL}_4(\mathbf{Z}_p) \times \mathbf{Z}_p^\times$. Since $1 \times \mathbf{Z}_p^\times \subset K'_\phi$, any $f \in I(\rho, K'_\phi)$ is determined by its restriction to $\mathrm{GL}_4(\mathbf{Z}_p)$. Let $I'(\rho_1)$ be the restrictions of the functions in $I(\rho_1)$ to $\mathrm{GL}_4(\mathbf{Z}_p)$ and let $K_1 := K'_\phi \cap \mathrm{GL}_4(\mathbf{Z}_p)$. To prove the proposition it therefore suffices to show that $I'(\rho_1, K_1) := \{f \in I'(\rho_1) : f(gk) = \nu_1(k)f(g), k \in K_1\}$ is one-dimensional. To do this we first identify $I'(\rho_1)$ with a principal series representation of GL_4 induced from its standard upper-triangular Borel R.

Let $\tau_1 = (\tau', \tau'')$ and $\psi_1 = (\psi', \psi'')$ be the identifications of τ_1 and ψ_1 with pairs of characters of $\mathbf{Q}_p^\times$. By hypothesis, (π_1, V) can be identified with a principal series representation: $\iota : \pi_1 \simeq \pi(\mu_1, \mu_2)$. So if D denotes the diagonal torus of GL_4 and $\lambda : D(\mathbf{Q}_p) \to \mathbf{C}$ is the character

$$\lambda(\mathrm{diag}(a,b,c,d)) = \bar{\tau}''(a)\bar{\psi}''(b)\psi'(c)\tau'(d),$$

which we extend to $R(\mathbf{Q}_p)$ in the usual way, then

$$f \mapsto (g \mapsto \iota(f(w'gw'')(1))),$$

where $w' := \mathrm{diag}(\eta, 1)$ and $w'' := \mathrm{diag}(1, \eta^{-1})$, identifies $I'(\rho_1)$ with the space $W := \{f : \mathrm{GL}_4(\mathbf{Z}_p) \to \mathbf{C} \ : \ f \text{ smooth}, \ f(rg) = \lambda(r)f(g), r \in R(\mathbf{Z}_p)\}$. Let δ_R be the modulus character of R. We extend each function $f \in W$ to $\mathrm{GL}_4(\mathbf{Q}_p)$ by $f(g) = \delta_R(r)^{1/2}f(k)$, $g = rk \in R(\mathbf{Q}_p)\mathrm{GL}_4(\mathbf{Z}_p)$; this identifies W with the principal series representation $U(\lambda)$ of $\mathrm{GL}_4(\mathbf{Q}_p)$ induced from the character λ of $R(\mathbf{Q}_p)$.

Let $K' := w''K_1(w'')^{-1}$. Then

$$K' = \{ \begin{pmatrix} * & * & * & * \\ p^{n_1}* & * & * & * \\ p^{n'_1}* & p^{m'_1}* & * & * \\ p^{n'_1}* & p^{m'_1}* & p^{m''_1}* & * \end{pmatrix} \in \mathrm{GL}_4(\mathbf{Z}_p)\}.$$

Let $\nu' : D(\mathbf{Q}_p) \to \mathbf{C}$ be the character

$$\nu'(\mathrm{diag}(a,b,c,d)) = \psi'(a)\tau'(b)\bar{\tau}''(c)\bar{\psi}''(d).$$

Then ν' also defines a character of K' via

$$\nu'(\begin{pmatrix} a & * & * & * \\ * & b & * & * \\ * & * & c & * \\ * & * & * & d \end{pmatrix}) := \nu'(\mathrm{diag}(a,b,c,d)).$$

Let $W(K') := \{f \in W \ : f(gk) = \nu'(k)f(g), k \in K'\}$. Then $I'(\rho_1, K_1)$ is identified with $W(K')$, so we want to prove that $W(K')$ is one-dimensional. Equivalently, we want to prove that $\dim_{\mathbf{C}} \mathrm{Hom}_{K'}(U(\lambda), \nu') = 1$. As λ is in the Weyl orbit of ν' and since ν' is unitary and regular, the principal series $U(\lambda)$ and $U(\nu')$ are irreducible and equivalent as $\mathrm{GL}_4(\mathbf{Q}_p)$-representations [Ca, Thms. 6.3.11, 6.6.1], and so it suffices to show that $\dim_{\mathbf{C}} \mathrm{Hom}_{K'}(U(\nu'),\nu') = 1$. But $\mathrm{Hom}_{K'}(U(\nu'),\nu') = \mathrm{Hom}_{\mathrm{GL}_4(\mathbf{Z}_p)}(U(\nu'),V(\nu'))$, where $V(\nu')$ is the representation of $\mathrm{GL}_4(\mathbf{Z}_p)$ induced from the character ν' of K'. That $\dim_{\mathbf{C}} \mathrm{Hom}_{\mathrm{GL}_4(\mathbf{Z}_p)}(U(\nu'),V(\nu')) = 1$ then follows from [Ho73, Thm. 1]. ∎

Lemma 9.2.7. *In the Generic Case, $F^0_{\phi,z}$ is defined for all z, supported on $P(\mathbf{Z}_p)K'_\phi$, and satisfies $F^0_{\phi,z}(1) = |p^s|_{\mathcal{K}}\phi$. Moreover, $I(\rho,K'_\phi)^0$ is one-dimensional and spanned by $F^0_{\phi,z}$.*

Proof. Our hypotheses on the conductors of the various characters ensures that if $\mathrm{Re}(z) = 0$ then both $\rho_1\otimes\delta^z$ and $\rho_1^\vee\otimes\delta^{-z}$ are irreducible, regular, unitary representations of $M_P(\mathbf{Q}_p)$. It then follows that $(\sigma(\rho_1,z),I(\rho_1))$ and $(\sigma(\rho_1^\vee,-z),I(\rho_1^\vee))$ are irreducible (see [Ca, Thm. 6.6.1]), hence so is $(\sigma(\rho,-z),I(\rho))$. Our hypotheses also imply that $a(\rho_1,z) = 1$, where $a(\rho_1,z)$ is as in Lemma 9.1.3. Then by this same lemma, $A(\rho_1,z,-)$ is defined for all z. Since it follows from part (ii) of Lemma 9.2.4 that $A(\rho_1,z,F^1_\phi)$ is non-zero for all z, if $\mathrm{Re}(z) = 0$ then $A(\rho_1,z,F^1_\phi)$ exists and is non-zero. Therefore, for $\mathrm{Re}(z) = 0$, $f\mapsto A(\rho_1,z,f)$ determines a $G(\mathbf{Z}_p)$-equivariant isomorphism $I(\rho_1)\xrightarrow{\sim} I(\rho_1^\vee)$. It then follows from Proposition 9.2.6 that $I(\rho_1^\vee,K'_\phi) := \{f\in I(\rho_1^\vee) \;:\; f(gk) = \nu_1(k)f(g), k\in K'_\phi\}$ is one-dimensional. Hence $I(\rho,K'_\phi)^0$ is also one-dimensional. As the function

$$F'(g) := \begin{cases} \nu_0(k)\rho(m)\phi & g = mnk\in P(\mathbf{Z}_p)K'_\phi \\ 0 & \text{otherwise}\end{cases}$$

is in $I(\rho,K'_\phi)^0$, it spans $I(\rho,K'_\phi)^0$. Thus $F^0_{\phi,z}(g) = \psi(\det g)\xi_1(\mu(g))A(\rho_1,-z,F^1_\phi)(g)$ is a constant multiple of $F'(g)$. Appealing to part (ii) of Lemma 9.2.4 shows that $F^0_{\phi,z}(1) = |p^s|_{\mathcal{K}}\phi$, proving the lemma. ∎

9.3. **Good Eisenstein series.**

9.3.1. *Eisenstein data.* Let (π,V) be an irreducible $(\mathfrak{gl}_2,K'_\infty)\times\mathrm{GL}_2(\mathbf{A}_f)$-subrepresentation of $\mathcal{A}^0(\mathrm{GL}_2)$ and let $V = \otimes V_\pi$ and $\pi = \otimes\pi_w$ be identifications as in 9.1.4. We assume that each (π_w,V_w) satisfies the hypotheses of either 9.1.1 or 9.1.2. In addition, we assume that π_∞ is as in (9.2.1.a). Let χ be the central character of π.

By an Eisenstein datum for π (and $\mathcal{K}$) we will mean a 4-tuple $\mathcal{D} = \{\Sigma,\varphi,\psi,\tau\}$ consisting of a finite set of primes Σ, a cuspform $\varphi\in V$ that is completely reducible (that is, φ is identified with a pure tensor $\otimes\phi_w\in\otimes V_w$), and unitary Hecke characters $\psi = \otimes\psi_w$ and $\tau = \otimes\tau_w$ of $\mathbf{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$, all such that

- Σ contains p, all primes that ramify in $\mathcal{K}$, and all primes ℓ such that π_ℓ, ψ_ℓ, or τ_ℓ is ramified;
- for all $k\in K'_{\infty,+}$, $\pi_\infty(k)\phi_\infty = j(k,i)^{-\kappa}\phi_\infty$;
- if $\ell\notin\Sigma$ then ϕ_ℓ is the newvector implicit in the identification $V = \otimes V_w$;
- if $\ell\in\Sigma$, $\ell\neq p$, then ϕ_ℓ has a conductor with respect to $\tilde{\pi}_\ell$;

- if $\ell = p$, then ϕ_p is an eigenvector for π_p but not a newvector if π_p is unramified (i.e., the conductor of ϕ relative to π_p is not (1));
- $\psi|_{\mathbf{A}_{\mathbf{Q}}^\times} = \chi$;
- $\tau_\infty(x) = (x/|x|)^{-\kappa} = \psi_\infty(x)$.

Here κ is as in (9.2.1.a). The inclusion in Σ of the primes that ramify in $\mathcal{K}$ simplifies later formulas. Note that since p is unramified in $\mathcal{K}$, Σ contains a prime other than p.

Let $\mathcal{D} = (\Sigma, \varphi, \psi, \tau)$ be an Eisenstein datum, fixed throughout the rest of this section. Let ρ_w, ρ, $I(\rho_w)$, $I(\rho)$, etc., to be the objects associated with the triples (π_w, ψ_w, τ_w) and (π, ψ, τ) in 9.1.1-9.1.4.

Let $\xi = \otimes \xi_w := \psi/\tau$. For $\ell \in \Sigma$, $\ell \neq p$, let $t_\ell > 0$ be the smallest integer such that ℓ^{t_ℓ} is contained in each of $\mathrm{cond}_{\tilde{\pi}_\ell}(\phi_\ell)$, $\mathrm{cond}(\psi_\ell)$, and $\mathrm{cond}(\xi_\ell)$. Put

$$K_{\mathcal{D}} := K_{\phi_p} \times \prod_{\ell \in \Sigma, \ell \neq p} K_\ell(\ell^{t_\ell}, \ell^{t_\ell}) \times \prod_{\ell \notin \Sigma} G(\mathbf{Z}_\ell)$$

where K_{ϕ_p} is as in 9.2.5. Let $\nu_\ell : K_\ell(\ell^{t_\ell}, \ell^{t_\ell}) \to \mathbf{C}$ be the character denoted by ν in 9.2.3 (defined using ψ_ℓ and ξ_ℓ for ψ and ξ in 9.2.3). Similary, let $\nu_0 : K_{\phi_p} \to \mathbf{C}^\times$ be as in 9.2.5 (defined using ψ_p and τ_p for ψ and τ in 9.2.5). Let $\nu_{\mathcal{D}} : K_{\mathcal{D}} \to \mathbf{C}^\times$ be the character defined by

$$\nu_{\mathcal{D}}((k_w)) := \nu_0(k_p) \times \prod_{\ell \in \Sigma, \ell \neq p} \nu_\ell(k_\ell).$$

Let $U_{\mathcal{D}} := \ker(\nu_{\mathcal{D}})$.

Let (ξ_κ, V_κ) be the representation of K_∞ in 9.2.1. Let $F_\kappa \in I(\rho_\infty)^{\xi_\kappa}$ be the unique vector such that $F_\kappa(1) = \phi_\infty$ (the uniqueness follows from (9.2.1.c)). Let $\mathcal{F}_{\mathcal{D}} := (\xi_\kappa, U_{\mathcal{D}})$ and let

$$\varphi_{\mathcal{D}} : \mathcal{U} \to I(\rho)^{\mathcal{F}_{\mathcal{D}}}, \quad \varphi_{\mathcal{D}}(z) = F_\kappa \otimes F^0_{\phi_p, z} \otimes_{\ell \in \Sigma, \ell \neq p} F_{\phi_\ell, t_\ell, t_\ell} \otimes_{\ell \neq \Sigma} F_{\rho_\ell},$$

where $\mathcal{U} := \{z \in \mathbf{C} \ : \ \mathrm{Re}(z) > 3/2\}$. The analytic properties of $\varphi_{\mathcal{D}}$ are inherited from $F^0_{\phi_p, z}$, so to see that $\varphi_{\mathcal{D}}$ is holomorphic it suffices to observe that $z \mapsto F^0_{\phi_p, z}$ is, and holomorphy of the latter follows from Lemma 9.1.3. Let

$$E_{\mathcal{D}}(z, g) := E(\varphi_{\mathcal{D}}, z)(g) \quad \text{and} \quad A_{\mathcal{D}}(z, g) := A(\varphi_{\mathcal{D}}, z)(g).$$

These then are also holomorphic functions on $\mathcal{U}$ with values in $\mathcal{A}(G)^{\mathcal{F}_{\mathcal{D}}}$ and $I(\rho^\vee)^{\mathcal{F}_{\mathcal{D}}}$, respectively.

In what follows we let $\rho_f := \otimes_{w \neq \infty} \rho_w$ and $I(\rho_f) := \otimes_{w \neq \infty} I(\rho_w)$ (the restricted tensor product with respect to the F_{ρ_ℓ}'s for almost all ℓ), and define $A(\rho_f, z, -)$ as we did $A(\rho, z, -)$. Then $I(\rho)$ is identified with $I(\rho_\infty) \otimes I(\rho_f)$ and if $\mathrm{Re}(z) > 3/2$ then $A(\rho, z, \otimes f_w) = A(\rho, z, f_\infty) \otimes A(\rho_f, z, \otimes_{w \nmid \infty} f_w)$.

Lemma 9.3.2. *Suppose $\kappa > 6$ and let $z_\kappa := (\kappa - 3)/2$. Let $F = F_\kappa \otimes F_f \in I(\rho) = I(\rho_\infty) \otimes I(\rho_f)$.*

(i) $A(\rho, z_\kappa, F) = 0$.

(ii) $E(F, z_\kappa, g)_P = F_{z_\kappa}(g)$.

In particular, $A_{\mathcal{D}}(z_\kappa, g) = 0$ *and* $E_{\mathcal{D}}(z_\kappa, -)_P = \varphi_{\mathcal{D}}(z_\kappa)$.

Proof. Since $z_\kappa > 3/2$, $A(\rho, z_\kappa, F) = A(\rho_\infty, z_\kappa, F_\kappa) \otimes A(\rho_f, z_\kappa, F_f)$ by (9.1.5.c). By (9.2.1.e), $A(\rho_\infty, z_\kappa, F_\kappa) = c(\rho_\infty, z_\kappa)F_\kappa^\vee$, where $F_\kappa^\vee \in I(\rho_\infty^\vee)$ is the unique vector such that $F_\kappa^\vee(w) = \phi_\infty$. It follows easily from Lemma 9.2.2 that $c(\rho_\infty, z_\kappa) = 0$. This proves part (i). Part (ii) then follows from Lemma 9.1.6. ∎

If $\kappa > 6$, then for any $F = F_\kappa \otimes F_f \in I(\rho)$ we define a function of $(Z, x) \in \mathbf{H} \times G(\mathbf{A}_f)$:

$$E(Z, x; F) := J(g, i)^\kappa \mu(g)^{-\kappa} E(F, z_\kappa, gx), \quad g \in G^+(\mathbf{R}),\ g(i) = Z.$$

We write $E_{\mathcal{D}}(Z, x)$ for $E(Z, x; \varphi_{\mathcal{D}}(z_\kappa))$.

Proposition 9.3.3. *Suppose* $\kappa > 6$ *and* $F = F_\kappa \otimes F_f$. *Then* $E(Z, x; F)$ *is a Hermitian modular form of weight* κ. *In particular,* $E_{\mathcal{D}} \in M_\kappa(K_{\mathcal{D}}, \nu_{\mathcal{D}})$.

Proof. It is enough to prove that $E(Z, x; F)$ is a holomorphic function on $\mathbf{H}$, and to prove this it is enough to prove that $F_x(Z) := J(g, i)^\kappa \mu(g)^{-\kappa} F_{z_\kappa}(gx)$, $g \in G^+(\mathbf{R})$, $g(i) = Z$, is holomorphic for any x. For, by (9.1.5.a),

$$E(Z, x; F) = \sum_{\gamma \in P(\mathbf{Q}) \backslash G(\mathbf{Q})} J(\gamma, Z)^{-\kappa} F_{\gamma x}(\gamma(Z)),$$

with the series converging uniformly for Z in any compact subset of $\mathbf{H}$ and so defining a holomorphic function on $\mathbf{H}$ if each $F_{\gamma x}(Z)$ is holomorphic .

Let $\varphi_x = \phi_\infty \otimes F_{f, z_\kappa}(x) \in V$. By the hypotheses on π_∞ and the choice of ϕ_∞,

$$f_x(z) := j(g, i)^\kappa \det(g)^{-\kappa/2} \varphi_x(g), \quad g \in \mathrm{GL}_2(\mathbf{R}),\ g(i) = z \in \mathfrak{h}, \tag{9.3.3.a}$$

is a holomorphic weight κ modular form on the upper half-plane $\mathfrak{h}$. Suppose now that $Z \in \mathbf{H}$ and $g \in G^+(\mathbf{R})$ is such that $g(i) = Z$. Without loss of generality we may assume

$$g = \begin{pmatrix} t &&& \\ & y && \\ && t^{-1} & \\ &&& y^{-1} \end{pmatrix} \begin{pmatrix} 1 &&& \\ n & 1 && \\ && 1 & -\bar{n} \\ &&& 1 \end{pmatrix} \begin{pmatrix} 1 && a & b \\ & 1 & \bar{b} & c \\ && 1 & \\ &&& 1 \end{pmatrix}, \quad t, y \in \mathbf{R}^\times,\ n, b \in \mathbf{C},\ a, c \in \mathbf{R}.$$

Then

$$Z = \left(\begin{smallmatrix} z_1 & z_2 \\ \bar{z}_2 & z_3 \end{smallmatrix}\right), \quad z_1 = t^2 i + t^2 a \in \mathfrak{h},$$

and

$$\begin{aligned} F_x(Z) &= (ty)^{-\kappa} F_{\kappa, z_\kappa}(g) \otimes F_{f, z_\kappa}(x) \\ &= t^{-\kappa}(\pi_\infty(\left(\begin{smallmatrix} t & ta \\ & t^{-1} \end{smallmatrix}\right))\phi_\infty \otimes F_{f, z_\kappa}(x))(1) \\ &= t^{-\kappa}\varphi_x(\left(\begin{smallmatrix} t & ta \\ & t^{-1} \end{smallmatrix}\right)) \\ &= f_x(z_1), \end{aligned} \tag{9.3.3.b}$$

and so $F_x(Z)$ is holomorphic. ∎

9.3.4. *Fourier expansions.* Let $\mathcal{D}$ be an Eisenstein datum. For $f \in I(\rho)$ (ρ being associated with $\mathcal{D}$ as in the previous section) and for any z at which $E_{\mathcal{D}}(f, z, g)$ is holomorphic (so in particular for $\mathrm{Re}(z) > 3/2$)

$$E(f, z, g) = \sum_{\beta \in S(\mathbf{Q})} \mu(\beta, f, z, g),$$

where

$$\mu(\beta, f, z, g) := \int_{S(\mathbf{Q})\backslash S(\mathbf{A})} E(f, z, \left(\begin{smallmatrix} 1 & m \\ & 1 \end{smallmatrix}\right) g) e_{\mathbf{A}}(-\mathrm{Tr}\,\beta m) dm.$$

We note that

$$\text{(9.3.4.a)} \qquad \begin{aligned} \mu(\beta, f, z, \left(\begin{smallmatrix} 1 & m \\ & 1 \end{smallmatrix}\right) g) &= e_{\mathbf{A}}(\mathrm{Tr}\,\beta m)\mu(\beta, f, z, g), \quad m \in S(\mathbf{A}) \\ \mu(\beta, f, z, \mathrm{diag}(u, {}^t\bar{u}^{-1})g) &= \mu({}^t\bar{u}\beta u, f, z, g), \quad u \in \mathrm{GL}_2(\mathcal{K}). \end{aligned}$$

If $\kappa > 6$ and $F = F_\kappa \otimes F_f$, then $E(Z, x; F)$ is a holomorphic Hermitian modular form of weight κ by Proposition 9.3.3 and so has a Fourier expansion

$$\text{(9.3.4.b)} \qquad E(Z, x; F) = \sum_{\beta \in S(\mathbf{Q}), \beta \geq 0} c(\beta, F, x) e(\mathrm{Tr}\,\beta Z), \quad c(\beta, F, x) \in \mathbf{C}.$$

Comparing (9.3.4.a) and (9.3.4.b) yields

$$\text{(9.3.4.c)} \qquad \begin{aligned} c(\beta, F, x) \det(\bar{u})^\kappa e(i\mathrm{Tr}\,(\beta u {}^t\bar{u})) &= \mu(\beta, F, z_\kappa, \mathrm{diag}(u, {}^t\bar{u}^{-1})x), \quad u \in \mathrm{GL}_2(\mathbf{C}) \\ c(\beta, F, \mathrm{diag}(\zeta, {}^t\bar{\zeta}^{-1})x) &= \det {}^t\bar{\zeta}^{-\kappa} c({}^t\bar{\zeta}\beta\zeta, F, x), \quad \zeta \in \mathrm{GL}_2(\mathcal{K}). \end{aligned}$$

Comparing the first of the preceding equations with part (ii) of Lemma 9.3.2 and (9.3.3.b) yields

$$\text{(9.3.4.d)} \qquad c(\left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right), F, x) = a(n, f_x), \quad n \in \mathbf{Q},$$

where f_x is as in (9.3.3.a).

Specializing to the case $F = \varphi_{\mathcal{D}}(z)$, we write $\mu_{\mathcal{D}}(\beta, z, g)$ for $\mu(\beta, F, z, g)$, and if $z = z_\kappa$ then we write $c_{\mathcal{D}}(\beta, x)$ for $c(\beta, F, x)$.

Lemma 9.3.5. *Suppose $\kappa > 6$. Let $\beta = \left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right) \in S(\mathbf{Q})$.*

(i) *If $x_\ell \notin P(\mathbf{Q}_\ell) w K_{t_\ell, t_\ell}$ for some $\ell \in \Sigma$, $\ell \neq p$, then $c_{\mathcal{D}}(\beta, x) = 0$.*
(ii) *If π_p, ϕ_p, ψ_p, and τ_p are in the Generic Case of 9.2.5 and if $x_p \notin P(\mathbf{Q}_p) K'_{\phi_p}$, then $c_{\mathcal{D}}(\beta, x) = 0$.*

This follows easily from (9.3.4.d) and the definitions of $F_{\phi_\ell, t_\ell, t_\ell}$, $F^0_{\phi_p, z}$, and f_x (with an appeal to Lemma 9.2.7 for part (ii)).

9.4. **The classical picture I.** By a classical datum we will mean a 4-tuple $\mathfrak{D} = (f, \psi, \xi, \Sigma)$ consisting of

- an eigenform $f \in S_\kappa(N, \chi)$ of level $N = Mp^r$, $p \nmid M$;
- an idele class character ψ of $\mathcal{K}$ such that $\psi_\infty(z) = z^{-\kappa}$ and $\psi|_{\mathbf{A}_\mathbf{Q}^\times} = \chi|\cdot|_\mathbf{Q}^{-\kappa}$;
- an idele class character ξ of $\mathcal{K}$ of finite order;
- a finite set of primes Σ containing all primes that divide $\mathrm{Nm}(\mathfrak{f}_\psi \mathfrak{f}_\xi)N$ and all primes that ramify in $\mathcal{K}$.

If $\kappa > 6$ we associate with each classical datum $\mathfrak{D}$ a holomorphic Hermitian Eisenstein series $E_\mathfrak{D}$ of weight κ. We do this by first associating to $\mathfrak{D}$ an Eisenstein datum $\mathcal{D}$ and then setting $E_\mathfrak{D}(Z, x) := E_\mathcal{D}(Z, x)$.

Let $\mathfrak{D} = (f, \psi, \xi, \Sigma)$ be a classical datum as above. Let χ_0 be the unique unitary idele class character of $\mathbf{A}^\times$ such that $\chi_0|_{\hat{\mathbf{Z}}^\times} = \prod_{\ell|M} \chi_\ell$. Let $w^{(M)} \in \mathrm{GL}_2(\mathbf{A}_f)$ be defined by $w_\ell^{(M)} = \eta$ if $\ell|M$ and $w_\ell^{(M)} = 1$ otherwise, and let φ be the automorphic form on $\mathrm{GL}_2(\mathbf{A})$ defined by

$$\varphi(g) := f_\mathbf{A}(gw^{(M)})\chi_0^{-1}(\det g).$$

Note that $\varphi(gk) = \chi_0^{-1}(d)\chi_p(d_p)\varphi(g)$ for all $k \in U''(N)$ where

$$U''(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\hat{\mathbf{Z}}) \ : \ b \in M\hat{\mathbf{Z}}, c \in p^r\hat{\mathbf{Z}}\}.$$

If $\sigma_N \in \mathrm{SL}_2(\mathbf{Z})$ is such that $\sigma_N^{-1}U'(N) = w^{(M)}U'(N)$ and we let

$$f_\mathfrak{D} := f|_\kappa \sigma_N,$$

then

$$\varphi(g) = j(g_\infty)^{-\kappa}\det(g_\infty)^{\kappa/2} f_\mathfrak{D}(g_\infty(i)),$$

$$g = \gamma g_\infty u \in \mathrm{GL}_2(\mathbf{Q})\mathrm{GL}_2^+(\mathbf{R})U_1''(N), \quad U_1''(N) = \{k \in U''(N) \ : \ d_k - 1 \in N\widehat{\mathbf{Z}}\}.$$

More generally, for $x \in \mathrm{GL}_2(\mathbf{A}_f)$ we put $f_\mathfrak{D}(z, x) := |\det x|_\mathbf{Q}^{-\kappa/2} j(g, i)^\kappa \det(g)^{-\kappa/2}\varphi(gx)$, $g \in \mathrm{GL}_2^+(\mathbf{R})$ such that $g(i) = z$ (so $f_\mathfrak{D}(z) = f_\mathfrak{D}(z, u)$ for all $u \in U_1''(N)$).

Let (π, V) be the unitary cuspidal automorphic representation in $\mathcal{A}^0(\mathrm{GL}_2)$ generated by φ. This has central character $\chi_\pi = \chi\chi_0^{-2}$. In particular, $\chi_\pi|_{\widehat{\mathbf{Z}}^\times} = \chi_p \prod_{\ell|M} \chi_\ell^{-1}$. Since $f_\mathbf{A}$ is an eigenform and hence a pure tensor, say $f_\mathbf{A} = \otimes f_v$, in the representation it generates, it follows from the definition of φ that φ is also a pure tensor, say $\varphi = \otimes \phi_v$. If $\ell \nmid M$, then ϕ_ℓ is an eigenform relative to π_ℓ with eigenvalue $a_{\ell^n}(\varphi_\ell) = \chi_{0,\ell}^{-1}(\ell^n)a_{\ell^n}(f_\ell) = \chi_{0,\ell}^{-1}(\ell^n)\ell^{1-\kappa/2}a(\ell^n, f)$. If $\ell \mid M$ then φ_ℓ is an eigenform relative to $\tilde{\pi}_\ell$ and so has a conductor with respect to $\tilde{\pi}_\ell$.

Let $\psi_0 := \psi\chi_0^{-1}|\cdot|_\mathcal{K}^{\kappa/2}$ and let $\tau_0 := \psi\bar{\xi}|\cdot|_\mathcal{K}^{\kappa/2}$. Note that $\psi_0|_{\mathbf{A}^\times} = \chi_\pi$. Then $\mathcal{D} := (\Sigma, \varphi, \psi_0, \tau_0)$ is an Eisenstein datum for π, which we refer to as the Eisenstein datum associated with $\mathfrak{D}$. We will often write $K_\mathfrak{D}$ for $K_\mathcal{D}$ and $K_{\mathfrak{D},\ell}$ for the ℓ-component of $K_\mathcal{D}$. Similarly, we will write $\nu_\mathfrak{D}$ for $\nu_\mathcal{D}$.

If $\kappa > 6$ then $E_{\mathfrak{D}}(Z,x) := E_{\mathcal{D}}(Z,x)$ is defined and is a holomorphic Hermitian modular form of weight κ; in particular, $E_{\mathfrak{D}} \in M_\kappa(K_{\mathfrak{D}}, \nu_{\mathfrak{D}})$. Writing $c_{\mathfrak{D}}(\beta,x)$ for $c_{\mathcal{D}}(\beta,x)$ we then have

$$E_{\mathfrak{D}}(Z,x) = \sum_{\beta\in S(\mathbf{Q}),\beta\geq 0} c_{\mathfrak{D}}(\beta,x)e(\mathrm{Tr}\,(\beta Z)). \tag{9.4.0.a}$$

Lemma 9.4.1. *Let* $x \in G(\widehat{\mathbf{Z}})$. *Let* $\beta = \left(\begin{smallmatrix} n & 0 \\ 0 & 0\end{smallmatrix}\right) \in S(\mathbf{Q})$, $n \geq 0$.

(i) *If* $x_\ell \notin P(\mathbf{Z}_\ell)wK_{\mathfrak{D},\ell}$ *for some* $\ell \in \Sigma, \ell \neq p$, *then* $c_{\mathfrak{D}}(\beta,x) = 0$.
(ii) *If* $\pi_p, \phi_p, \psi_{0,p}$, *and* $\tau_{0,p}$ *are in the Generic Case of 9.2.5 and* $x_p \notin P(\mathbf{Q}_p)K'_{\phi_p}$ *then* $c_{\mathfrak{D}}(\beta,x) = 0$.
(iii) *If* $x = mnw^{(M)}k \in M_P(\widehat{\mathbf{Z}})N_P(\widehat{\mathbf{Z}})w^{(M)}K_{\mathfrak{D}}$, $m = m(aA,b)$ *with* $a,b \in (\mathcal{O}\otimes\widehat{\mathbf{Z}})^\times$ *and* $A \in \mathrm{GL}_2(\widehat{\mathbf{Z}})$, *then*
$$c_{\mathfrak{D}}(\beta,x) = p^{-r}\nu_{\mathfrak{D}}(k)\psi\bar{\xi}(b)\chi_0^{-1}\psi(a)|a\bar{a}\det A|_{\mathbf{Q}}^\kappa a(n, f_{\mathfrak{D}}(-,A)), \quad p^r||\mathrm{Nm}(\mathfrak{f}_\xi).$$

Proof. Parts (i) and (ii) are immediate consequences of Lemma 9.3.5. To prove part (iii), let $F_{\mathcal{D}} := \varphi_{\mathcal{D}}(z_\kappa) \in I(\rho)$, where $\mathcal{D}$ is the Eisenstein datum associated with the classical datum $\mathfrak{D}$ and $\varphi_{\mathcal{D}}$ is as in 9.3.1, and write $F_{\mathcal{D}} = F_\kappa \otimes F_{\mathcal{D},f} \in I(\rho) = I(\rho_\infty)\otimes I(\rho_f)$. Then $F_{\mathcal{D}}(x) = \phi_\infty \otimes F_{\mathcal{D},f}(w^{(M)})$. Following the notation preceding (9.3.3.a), we let $\varphi_x = F_{\mathcal{D}}(x)$. Then

$$\varphi_x = \nu_{\mathcal{D}}(k)\tau_0(b)\psi_0(b)|a\bar{a}\det A/b\bar{b}|_{\mathbf{Q}}^{\kappa/2}\pi(A)\phi_\infty \otimes F^0_{\phi_p,z_\kappa}(1)\otimes_{\ell\neq p}\phi_\ell.$$

By Corollary 9.2.7, $F^0_{\phi_p,z_\kappa}(1) = p^{-r}\phi_p$, and therefore

$$\varphi_x = p^{-r}\nu_{\mathcal{D}}(k)\tau_0(b)\psi_0(b)|a\bar{a}\det A/b\bar{b}|_{\mathbf{Q}}^{\kappa/2}\pi(A)\varphi,$$

and so

$$f_x(z) = p^{-r}\nu_{\mathcal{D}}(k)\tau_0(b)\psi_0(b)|a\bar{a}\det A/b\bar{b}|_{\mathbf{Q}}^{\kappa/2} j(g,i)^\kappa \det g^{-\kappa/2}\varphi(gA)$$

for $g \in \mathrm{GL}_2^+(\mathbf{R})$ such that $g(i) = z$. Part (iii) now follows from (9.3.4.d) and the definitions of ψ_0, τ_0, and $f_{\mathfrak{D}}(z,A)$. ∎

Let $\pi(f) = \otimes\pi_v(f)$ be the representation generated by $f_{\mathbf{A}}$ and let

$$W'(f) := \prod_{\ell\neq p}\epsilon(\pi_\ell(f), 1/2),$$

where the epsilon factors are defined with respect to the additive characters e_ℓ. Let

$$f'_{\mathfrak{D}}(z) := W'(f)^{-1}M^{-\kappa/2}f_{\mathfrak{D}}(Mz) \in S_\kappa(\Gamma_1(N)).$$

If f is primitive at each $\ell|M$ then $a(1,f'_{\mathfrak{D}}) = 1$ and

$$a(n,f'_{\mathfrak{D}}) = \prod_{\ell^r||n,\ell\nmid M} a(\ell^r,f)\bar{\chi}_{0,\ell}(\ell^r) \prod_{\ell^r||n,\ell|M,a(\ell,f)\neq 0} a(\ell^r,f)^{-1}\ell^{r(\kappa-1)}\bar{\chi}_{0,\ell}(\ell^r). \tag{9.4.1.a}$$

9.5. **Hecke operators and L-functions.**

9.5.1. *Local Hecke algebras.* Let ℓ be a prime. For ease of notation, we will assume throughout this section that ℓ is unramified in $\mathcal{K}$. We will also write K for $K_{r,s}$ if r and s are understood or if their exact values are unimportant.

If $r = s = 0$ (i.e., $K = K_\ell$) then we let $\tilde{\mathcal{H}}_K$ be the free abelian group on the set $\{KgK \ : \ g \in G(\mathbf{Q}_\ell)\}$ of double-cosets. This is a ring under the usual double-coset multiplication. It is isomorphic to the ring $C_c^\infty(K\backslash G(\mathbf{Q}_\ell)/K, \mathbf{Z})$ of locally-constant, $\mathbf{Z}$-valued, K-bi-invariant functions on $G(\mathbf{Q}_\ell)$ (this latter ring is denoted R_ℓ in 5.5.11); the isomorphism comes by identifying the double coset KgK with its characteristic function.

For any $s > 0$ we let

$$M_2(\mathcal{O}_\ell; s) := \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in M_2(\mathcal{O}_\ell) \cap \mathrm{GL}_2(\mathcal{K}_\ell) \ : \ c \in \lambda^s\mathcal{O}_\ell, a \in \mathcal{O}^\times\}.$$

We let $M_2(\mathcal{O}_\ell; 0) := M_2(\mathcal{O}_\ell)$. If $r, s > 0$ we let $\tilde{\mathcal{H}}_K$ be the free group on the set of double-cosets $\{K\mathrm{diag}(A, {}^t\bar{A}^{-1})K \ : \ A \in M_2(\mathcal{O}_\ell; s), {}^t\bar{A}S(\mathbf{Z}_\ell)A \subseteq S(\mathbf{Z}_\ell)\}$. Then $\tilde{\mathcal{H}}_K$ is a commutative ring under the usual double-coset multiplication; parahoric decomposition of K with respect to the parabolic Q shows that the map

$$K\mathrm{diag}(A, {}^t\bar{A}^{-1})K \mapsto |\det A\bar{A}|_\ell^{-2}\, K'_\ell(\ell^s)AK'_\ell(\ell^s)$$

determines an isomorphism of $\tilde{\mathcal{H}}_K$ with a commutative GL_2-Hecke algebra that is compatible with double-coset multiplication.

We set $\mathcal{H}_K := \tilde{\mathcal{H}}_K \otimes \mathbf{Z}[q^{1/2}, q^{-1/2}]$.

9.5.2. *The Satake map.* Recall that $T \subset G$ is the standard diagonal torus. For a prime ℓ let $X_\ell := T(\mathbf{Q}_\ell)/T(\mathbf{Z}_\ell)$. Given $t \in T(\mathbf{Q}_\ell)$ we write $[t]$ for its image in X_ℓ. We single out some elements of $T(\mathbf{Q}_\ell)$ (and hence of X_ℓ) for future use. If ℓ does not split in $\mathcal{K}$ let

$$t_1 := \mathrm{diag}(1, \ell, 1, \ell^{-1}) \quad t_2 := \mathrm{diag}(\ell, 1, \ell^{-1}, 1).$$

If ℓ splits in $\mathcal{K}$ let

$$t_1^{(1)} := \mathrm{diag}(1, (\ell, 1), 1, (1, \ell^{-1})), \quad t_2^{(1)} := \mathrm{diag}((\ell, 1), 1, (1, \ell^{-1}), 1),$$

$$t_i^{(2)} := \bar{t}_i^{(1)}, \quad t_i = t_i^{(1)}t_i^{(2)}, \quad i = 1, 2.$$

Let $t_0 := \mathrm{diag}(\ell, \ell, 1, 1)$. Let also $z_0 := \mathrm{diag}(\ell, \ell, \ell, \ell)$. If ℓ splits in $\mathcal{K}$ let $z_0^{(1)} = (\mathrm{diag}(\ell, \ell, \ell, \ell), 1)$ and $z_0^{(2)} = (1, \mathrm{diag}(\ell, \ell, \ell, \ell))$. Put

$$\mathcal{R}_\ell := \mathbf{Z}[X_\ell, q^{1/2}, q^{-1/2}] = \begin{cases} \mathbf{Z}[\{[t_i], [t_i]^{-1}\}_{i=1,2}, [t_0], [t_0]^{-1}, q^{1/2}, q^{-1/2}] & \ell \text{ non-split} \\ \mathbf{Z}[\{[t_i^{(j)}], [t_i^{(j)}]^{-1}\}_{1\le i,j\le 2}, [t_0], [t_0]^{-1}, q^{1/2}, q^{-1/2}] & \ell \text{ split.}\end{cases}$$

That is, $\mathcal{R}_\ell$ is the group ring of X_ℓ over $\mathbf{Z}[q^{1/2}, q^{-1/2}]$. The local Weyl group $W_{G,\ell}$ acts on $\mathcal{R}_\ell$ through its action on X_ℓ: $w \cdot [t] = [wtw^{-1}]$.

Any element of $KgK \in \tilde{\mathcal{H}}_K$ has a decomposition $KgK = \sqcup t_i n_i K$, $t_i \in T(\mathbf{Q}_\ell)$, $n_i \in N_B(\mathbf{Q}_\ell)$. We define $\mathcal{S}_K(KgK) \in \mathcal{R}_\ell$ by $\mathcal{S}_K(KgK) = \sum \delta_B^{1/2}(t_i)[t_i]$. This extends linearly to a map $\mathcal{S}_K : \mathcal{H}_K \to \mathcal{R}_\ell$.

Proposition 9.5.3. *Suppose $r = s = 0$ or $r, s > 0$. The map $\mathcal{S}_K : \mathcal{H}_K \to \mathcal{R}_\ell$ is an injection of rings. Moreover, when $r = s = 0$ the map $\mathcal{S}_K$ identifies $\mathcal{H}_K$ with the $\mathcal{R}_\ell^{W_{G,\ell}}$.*

When $r = s = 0$ this proposition follows from the usual Satake isomorphism. When $r, s > 0$ the previously described isomorphism of $\tilde{\mathcal{H}}_K$ with a GL_2-Hecke algebra reduces this to a well-known situation for GL_2.

Suppose $r = s = 0$. As a consequence of this proposition we may define elements in $\mathcal{H}_K$ by specifying their images in $R_\ell^{W_{G,\ell}}$. When ℓ is inert in $\mathcal{K}$ we let $T_i \in \mathcal{H}_K$, $i = 1, ..., 4$, be determined by

$$1 + \sum_{i=1}^{4} \mathcal{S}_K(T_i)X^i = \prod_{i=1}^{2}(1 - q^{3/2}[t_i]X)(1 - q^{3/2}[t_i]^{-1}X). \tag{9.5.3.a}$$

Similarly, when ℓ splits in $\mathcal{K}$ we let $T_i^{(j)} \in \mathcal{H}_K$, $i = 1, .., 4, j = 1, 2$, be determined by

$$1 + \sum_{i=1}^{4} \mathcal{S}_K(T_i^{(j)})X^i = \prod_{i=1}^{2}(1 - q^{3/2}[t_i^{(j)}]X)(1 - q^{3/2}[t_i^{(j')}]^{-1}X), \quad j' = 3 - j. \tag{9.5.3.b}$$

The T_i and $T_i^{(j)}$ actually belong to $\tilde{\mathcal{H}}_{\mathcal{K}}$. This is well-known in the split case (it can be deduced, for example from the discussion on pp.228-229 of [G98]). In the inert case we just note that $T_1 = Kt_1K$, $T_2 = Kt_1t_2K + q^6$, $T_3 = q^3T_1$, and $T_4 = 1$. Let Z_0 be defined by $\mathcal{S}_K(Z_0) = [z_0]$, and if ℓ splits let $Z_0^{(j)}$ be defined by $\mathcal{S}_K(Z_0^{(j)}) = [z_0^j]$. We let $\mathcal{H}'_K \subset \tilde{\mathcal{H}}_K$ be the subring generated by $\{T_1, .., T_4, Z_0\}$ if ℓ is inert and by $\{T_1^{(j)}, ..., T_4^{(j)}, Z_0^{(j)} \ : \ j = 1, 2\}$ if ℓ splits.

9.5.4. *Actions on induced representations.* We return to the set-up of 9.1.2, freely using the notation from there and from 9.2.3.

Let KgK be such that $KgK = \sqcup b_iK$ with $b_i \in B(\mathbf{Q}_\ell)$ or $b_i \in \alpha B(\mathbf{Q}_\ell)\alpha^{-1}$, $\alpha = \mathrm{diag}(\eta, \eta^{-1})$. For each $z \in \mathbf{C}$ we define an action of KgK on $I(\rho, K)$ by

$$[KgK]_z f := \begin{cases} \sum \psi^{-1}(a_i)\tau^{-1}(d_i)\sigma(\rho, z)(b_i)f & r, s > 0 \\ \sum \sigma(\rho, z)(b_i)f & r = s = 0, \end{cases} \qquad b_i = \begin{pmatrix} a_i & * & * & * \\ * & d_i & * & * \\ & & * & * \\ & & * & * \end{pmatrix}.$$

This defines an action of $\mathcal{H}_K$ on $I(\rho, K)$. When $r = s = 0$ this action is the same as the usual convolution action of the characteristic function of KgK on the space of functions $\{f_z \ : \ f \in I(\rho, K)\}$.

9.5.5. *The unramified case.* Continuing with the conventions of the previous paragraph, suppose $r = s = 0$ (so in particular, π, ψ, and τ are unramified). Then $K = K_\ell$ and $I(\rho, K)$ is one-dimensional. It follows that F_ρ is an eigenvector for the action of $\mathcal{H}_K$ associated with each $z \in \mathbf{C}$. The eigenvalues can be determined as follows.

Given a character $\alpha : T(\mathbf{Q}_\ell) \to \mathbf{C}$ let $I(\alpha)$ be the set of locally-constant functions $f : G(\mathbf{Q}_\ell) \to \mathbf{C}$ such that $f(tng) = \alpha\delta_B^{1/2}(t)f(g)$, $t \in T(\mathbf{Q}_\ell)$, $b \in N_B(\mathbf{Q}_\ell)$; $G(\mathbf{Q}_\ell)$ acts on $I(\alpha)$ via right translation. So $I(\alpha)$ is the usual induction of α from B to G. The space $I(\alpha)^K$ is non-zero (in which case it is one-dimensional) if and only if α is trivial on $T(\mathbf{Z}_\ell)$, in which case α defines a character of X_ℓ and hence a homomorphism $\mathcal{R}_\ell \to \mathbf{C}$. Let $\lambda_\alpha : \mathcal{H}_K \to \mathbf{C}$ be the homomorphism obtained by composition with $\mathcal{S}_K$; this extends to a homorphism $\mathcal{H}_K[X] \to \mathbf{C}[X]$. There is an action of $\mathcal{H}_K$ on $I(\alpha)^K$ defined by

$$[KgK]f(x) := \sum f(xb_i), \quad KgK = \sqcup b_iK, b_i \in B(\mathbf{Q}_\ell),$$

and we have

$$[h]f = \lambda_\alpha(h)f, \quad h \in \mathcal{H}_K, f \in I(\alpha)^K.$$

Each $(I(\rho), \sigma(\rho, z))$ is isomorphic to some $I(\alpha)$, with α trivial on $T(\mathbf{Z}_\ell)$, as representations of $G(\mathbf{Q}_\ell)$. Such an isomorphism is $\mathcal{H}_K$-invariant. Suppose $\pi \simeq \pi(\mu_1, \mu_2)$. Then $(I(\rho), \sigma(\rho, z))$ is isomorphic to $I(\alpha)$ with

$$\alpha(\mathrm{diag}(\lambda\bar{x}^{-1}, \lambda\bar{y}^{-1}, x, y)) = \psi(x)\tau(y)\mu_1(\lambda/x\bar{x})|\lambda/y\bar{y}|_\ell^z.$$

Let $\lambda_{\rho,z} := \lambda_\alpha$; this extends to a homomorphism $\tilde{\mathcal{H}}_K[\![X]\!] \to \mathbf{C}[\![X]\!]$.

If ℓ does not split in $\mathcal{K}$ we set

$$Z_K(X) := 1 + \sum_{i=1}^{4} T_i X^i \in \mathcal{H}'_K[X]. \tag{9.5.5.a}$$

If ℓ splits in $\mathcal{K}$ then we define $Z_K^{(j)}(X)$ by the same formula but with T_i replaced by $T_i^{(j)}$. If ℓ does not split in $\mathcal{K}$, then it follows from (9.5.3.a) and (9.5.5.a) that

$$\begin{aligned}\lambda_{\rho,z}(Z_K) = (1-&q^{3/2}\psi(\ell)\mu_1(q)^{-1}X)(1 - q^{3/2}\bar{\psi}^c(\ell)\mu_1(q)X) \\ &\times (1 - \tau(\ell)q^{z+3/2}X)(1 - \bar{\tau}^c(\ell)q^{-z+3/2}X).\end{aligned} \tag{9.5.5.b}$$

Similarly, if ℓ splits in $\mathcal{K}$ then it follows from (9.5.3.b) that

$$\begin{aligned}\lambda_{\rho,z}(Z_K^{(j)}) = (1-&q^{3/2}\psi_j(\ell)\mu_1(q)^{-1}X_j)(1 - q^{3/2}\bar{\psi}_{j'}(\ell)\mu_1(q)X_j) \\ &\times (1 - \tau_j(\ell)q^{z+3/2}X)(1 - \bar{\tau}_{j'}(\ell)q^{-z+3/2}X),\end{aligned} \tag{9.5.5.c}$$

where $j' = 3 - j$.

9.5.6. *Ramified cases.* Suppose now we are in the situation of 9.2.3 but that $K = K_{r,t}$ with $r, t > 0$ and with λ^r contained in $\mathrm{cond}_{\tilde{\pi}}(\phi)$ and λ^t contained in $\mathrm{cond}(\xi)$.

Lemma 9.5.7. *Suppose ℓ splits in $\mathcal{K}$. If ϕ is an eigenform for $\tilde{\pi}$ such that $\ell|\mathrm{cond}_{\tilde{\pi}}(\phi)$ and if $\mathrm{cond}(\xi) = (\ell^t)$, then for any $z \in \mathbf{C}$, $F_{\phi,r,t}$ is an eigenform for the action of each $[KdK]_z$, $d = (\mathrm{diag}(\ell^{a_1}, \ell^{a_2}, \ell^{a_4}, \ell^{a_3}), 1)$ with $a_1 \geq a_2 \geq a_3 \geq a_4$. In particular,*

$$[KdK]_z F_{\phi,r,t} = \ell^{(a_2-a_3)(3/2+z)}\ell^{a_1-a_2+a_3-a_4}a_{\ell^{a_1-a_4}}(\phi)F_{\phi,r,t},$$

where $a_{\ell^{a_1-a_4}}(\phi)$ is the eigenvalue with respect to $\tilde{\pi}$.

Proof. Let d be as in the lemma. Then $KdK = \sqcup n_i dK$ with n_i running over the elements

$$\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{x} & 1 \end{pmatrix} \begin{pmatrix} 1 & & \alpha & \beta \\ & 1 & \bar{\beta} & \gamma \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$x \in \mathcal{O}_\ell/(\ell^{a_1-a_2}, \ell^{a_3-a_4}),\ \alpha \in \mathbf{Z}_\ell/\ell^{a_1-a_4},\ \beta \in \mathcal{O}_\ell/(\ell^{a_1-a_3}, \ell^{a_2-a_4}),\ \gamma \in \mathbf{Z}_\ell/\ell^{a_2-a_3}\},$$

and so the operator $[KdK]_z$ is defined. Let $F := F_{\phi,r,t}$ and $F' := [KdK]_z F$.

Let $W' \subset W_{G,\ell}$ be such that $G(\mathbf{Z}_\ell) = \sqcup_{x\in W'} P(\mathbf{Z}_\ell)xK_Q(\lambda)$. Suppose $n \in K_Q(\lambda)$. Then $F_z(xnn_id) \neq 0$ only if $xnn_i \in P(\mathbf{Q}_\ell)w(K \cap N_B(\mathbf{Z}_\ell))d^{-1} \subseteq P(\mathbf{Q}_\ell)wN_B(\mathbf{Z}_\ell)$. Since $n_i \in Q(\mathbf{Z}_\ell)$ it follows that $xn \in P(\mathbf{Q}_\ell)wQ(\mathbf{Z}_\ell) = P(\mathbf{Q}_\ell)wK_Q(\lambda)$. Therefore it must be that $x \in wW_{Q,\ell}$ and hence F' is supported on $P(\mathbf{Z}_\ell)wQ(\mathbf{Z}_\ell)$. Then by part (i) of Lemma 9.2.4 it follows that F' is supported on $P(\mathbf{Z}_\ell)wK$ and that $F' = cF$ if and only if $F'(w) = c\phi$.

Since $F_z(wn_id) \neq 0$ only if $n_i \in w^{-1}P(\mathbf{Z}_\ell)wd(K \cap N_B(\mathbf{Z}_\ell))d^{-1}$, it is easily seen from the description of n_i that this happens only if $\beta = 0$,$\gamma = 0$. And so

$$\begin{aligned} F'(w) &= \psi((\ell^{-a_1}, \ell^{a_4}))\tau((\ell^{-a_2}, \ell^{a_3})) \sum_{\{n_i\ :\ \beta=\gamma=0\}} F_z(wn_id) \\ &= \ell^{a_1-a_2+a_3-a_4}\psi((\ell^{-a_1}, \ell^{a_4}))\tau((\ell^{-a_2}, \ell^{a_3})) \sum_{\{n_i\ :\ x=\beta=\gamma=0\}} F_z(wn_id) \\ &= \ell^{a_1-a_2+a_3-a_4}\ell^{(a_2-a_3)(3/2+z)} \sum_{\alpha\in\mathbf{Z}/\ell^{a_1-a_4}} \tilde{\pi}\left(\left(\begin{smallmatrix} 1 & \alpha \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \ell^{a_1-a_4} & \\ & 1 \end{smallmatrix}\right)\right)\phi \\ &= \ell^{(a_2-a_3)(3/2+z)}\ell^{a_1-a_2+a_3-a_4} a_{\ell^{a_1-a_4}}(\phi)\phi. \end{aligned}$$

■

9.5.8. *Actions on p-adic sections.* Supposing we are in the setting of 9.2.5, we modify the actions in 9.5.4 to get actions on $I(\rho, K_\phi)^0$: for $z \in \mathbf{C}$ and $K_\phi gK_\phi \in \mathcal{H}_{K_\phi}$

$$[K_\phi gK_\phi]_z^0 f := \sum \psi^{-1}(a_i)\tau^{-1}(d_i)\sigma(\rho, z)(b_i)f, \quad K_\phi tK_\phi = \sqcup b_iK_\phi, b_i = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & a_i & * \\ & & * & d_i \end{pmatrix}.$$

This is related to the action on $I(\rho_1, K_\phi)$ as follows. Let $f \in I(\rho_1, K_\phi)$. If $f' \in I(\rho, K_\phi)^0$ is given by $f'(x) = \psi(\det x)\bar{\xi}(\mu(x))A(\rho_1, -z, f)(x)$, then

$$[K_\phi gK_\phi]_{-z}f = cf \Longrightarrow [K_\phi gK_\phi]_z^0 f' = cf'. \tag{9.5.8.a}$$

Lemma 9.5.9. *For $z \in \mathbf{C}$, $F_{\phi,z}^0$ is an eigenvector for the action of $[K_\phi dK_\phi]_z$, $d = (\mathrm{diag}(p^{a_1}, p^{a_2}, p^{a_4}, p^{a_3}), 1)$ with $a_1 \geq a_2 \geq a_3 \geq a_4$. In particular,*

$$[K_\phi dK_\phi]_z^0 F_{\phi,z}^0 = p^{(a_2-a_3)(3/2-z)}p^{a_1-a_2+a_3-a_4}a_{p^{a_1-a_4}}(\phi).$$

This follows immediately from (9.5.8.a) and Lemma 9.5.7.

For an Eisenstein datum $\mathcal{D}$, let $\mathcal{U}_{\mathcal{D},p} \subseteq \mathcal{H}_{K_{\mathcal{D},p}}$ be the subalgebra generated by the double cosets $K_{\mathcal{D},p}dK_{\mathcal{D},p}$ with d as in the preceding lemma.

9.5.10. *Consequences for Eisenstein series.* Let $\mathcal{D} = (\Sigma, \varphi, \psi, \tau)$ be an Eisenstein datum for a unitary cuspidal representation (π, V) of $\mathrm{GL}_{2/\mathbf{Q}}$ as in 9.3.1, the notation from which we freely use. Let

$$\mathcal{H}^\Sigma := \otimes_{\ell \notin \Sigma} \mathcal{H}'_{K_\ell},$$

the restricted tensor product being with respect to the identity elements. Then $\mathcal{H}^\Sigma$ acts on $I(\rho)^\Sigma := \otimes_{\ell\notin\Sigma} I(\rho_\ell, K_\ell)$, and hence on $I(\rho)_\mathcal{D} := I(\rho_\infty) \otimes I(\rho_p, K_{\mathcal{D},p})^0 \otimes_{\ell\in\Sigma\setminus\{p\}} I(\rho_\ell) \otimes I(\rho)^\Sigma$, in the obvious way: if $h = \otimes h_\ell \in \mathcal{H}^\Sigma$ and $f = \otimes f_\ell \in I(\rho)^\Sigma$ then $[h]_z f = \otimes [h_\ell]_z f_\ell$. The action of $\mathcal{U}_{\mathcal{D},p}$ on $I(\rho, K_{\mathcal{D},p})^0$ also gives an action on $I(\rho)_\mathcal{D}$ that clearly commutes with the action of $\mathcal{H}^\Sigma$.

Let $\mathcal{A}(G)_\mathcal{D} := \{f \in \mathcal{A}(G) \ : \ f(gk) = \nu_\mathcal{D}(k) f(g), k \in K_{\mathcal{D},p} \otimes \prod_{\ell\notin\Sigma} K_\ell\}$. Then $\mathcal{H}^\Sigma$ acts on $\mathcal{A}(G)_\mathcal{D}$ in the obvious way; this is the usual double-coset action. We also define an action of $\mathcal{U}_{\mathcal{D},p}$ on $\mathcal{A}(G)_\mathcal{D}$ in analogy with the action on $I(\rho_p, K_{\mathcal{D},p})^0$. In particular, for $h \in \mathcal{H}_\mathcal{D} := \mathcal{U}_{\mathcal{D},p} \otimes \mathcal{H}^\Sigma$,

$$h \cdot E(f, z, g) = E([h]_z f, z, g), \quad f \in I(\rho)_\mathcal{D},$$

whenever $E(f, z, g)$ is defined. Hence $E_\mathcal{D}(z, g)$ is an eigenform for the action of $\mathcal{H}_\mathcal{D}$ with the same eigenvalues as $\varphi_\mathcal{D}(z) = F_\kappa \otimes F^0_{\phi_p,z} \otimes_{\ell\neq p} F_{\phi_\ell,t_\ell,t_\ell}$ (with respect to z). We define a homomorphism $\lambda_\mathcal{D} : \mathcal{U}_{\mathcal{D},p} \otimes \mathcal{H}^\Sigma \to \mathbf{C}$ by

$$h \cdot E_\mathcal{D}(z_\kappa, g) = \lambda_\mathcal{D}(h) E_\mathcal{D}(z_\kappa, g).$$

Note that if $\ell \notin \Sigma$ then the restriction of $\lambda_\mathcal{D}$ to $\mathcal{H}'_{K_\ell} \subset \mathcal{H}^\Sigma$ is just λ_{ρ,z_κ}. We let $Z_{\mathcal{D},\ell} = \lambda_\mathcal{D}(Z_{K_{\mathcal{D},\ell}})$, by which we mean the polynomial obtained from applying $\lambda_\mathcal{D}$ to the coefficients of $Z_{K_{\mathcal{D},\ell}}$.

9.6. **The classical picture II.** Let $\mathfrak{D} = (f, \psi, \xi, \Sigma)$ be a classical datum, and let $\mathcal{D} = (\Sigma, \varphi, \psi_0, \tau_0)$ be its associated Eisenstein datum. Recall that if $\ell \notin \Sigma$ then there is an action of $\mathcal{H}_{K_\ell}$ on $M_\kappa(K_\mathfrak{D}, \nu_\mathfrak{D})$ defined via correspondences (cf. 5.5.11). For $h = K_\ell g K_\ell = \sqcup b_i K_\ell$, $b_i \in B(\mathbf{Q}_\ell)$, this action is just

$$(h \cdot f)(Z, x) = |\mu(g)|_\ell^{-\kappa} \sum f(Z, x b_i), \quad f \in M_\kappa(K_\mathfrak{D}, \nu_\mathfrak{D}).$$

We also define an action of $\mathcal{U}_{\mathcal{D},p}$ on $M_\kappa(K_\mathfrak{D}, \nu_\mathfrak{D})$ as we did on $\mathcal{A}(G)_\mathcal{D}$ in 9.5.10, but modified by the factor $|\mu(g)|_p^{-\kappa}$ as above.

Suppose $\kappa > 6$. We define $\lambda_\mathfrak{D} : \mathcal{H}^\Sigma \to \mathbf{C}$ by $h \cdot E_\mathfrak{D} = \lambda_\mathfrak{D}(h) E_\mathfrak{D}$. If $h = K_\ell g K_\ell \in \mathcal{H}'_{K_\ell}$ then $\lambda_\mathfrak{D}(h) = |\mu(g)|_\mathbf{A}^{-\kappa} \lambda_\mathcal{D}(h)$. We continue to denote by $\lambda_\mathfrak{D}$ its extension to a homomorphism $\mathcal{H}^\Sigma[X] \to \mathbf{C}[X]$.

Let v be a finite place of $\mathcal{K}$ lying over a rational prime $\ell \notin \Sigma$. If ℓ splits in $\mathcal{K}$ let $i_v = 1$ or 2, according to whether the valuation associated to v comes from the projection onto the first or second factor of $K_\ell = \mathbf{Q}_\ell \times \mathbf{Q}_\ell$. Let $Q_v(X) \in \mathcal{H}'_{K_\ell}[X]$ be defined by

(9.6.0.a) $$Q_v(X) := \begin{cases} Z_{K_\ell}(Z_0 X) & \text{if } \ell \text{ does not split} \\ Z^{(i_v)}_{K_\ell}(Z_0^{(3-i_v)} X) & \text{if } \ell \text{ splits.} \end{cases}$$

Proposition 9.6.1. *Suppose $\kappa > 6$. Let v be a finite place of $\mathcal{K}$ lying over a rational prime $\ell \notin \Sigma$, and let q_v be the order of the residue field of v. Then $\lambda_{\mathfrak{D}}(Q_v)(q_v^{-s})$ is the Euler factor at v of the Dirichlet series*

$$L_{\mathcal{K}}^{\Sigma}(f, \bar{\chi}_0\bar{\xi}^c\psi^c, s-2)L^{\Sigma}(\bar{\chi}_0\psi^c, s-3)L^{\Sigma}(\chi\bar{\chi}_0\bar{\xi}'\psi^c, s-\kappa).$$

In this proposition, χ_0 is the idele class character so denoted in 9.4.

Recall that f is ordinary at p if the eigenvalue of the Hecke operator U_p acting on f is a p-adic unit (with respect to the fixed embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$). This is equivalent to $a(p,f)$ being a p-adic unit and hence to $\chi_0(p)p^{\kappa/2-1}a_p(\phi_p) = a(p,f)$ being a p-adic unit, where $\varphi = \otimes\phi_v$ is the form appearing in the Eisenstein datum $\mathcal{D}$ associated with $\mathfrak{D}$.

Proposition 9.6.2. *Suppose $\kappa > 6$ and $p|\mathfrak{f}_\xi$. Let $t := (\mathrm{diag}(p^{a_1}, p^{a_2}, p^{a_4}, p^{a_3}), 1) \in T(\mathbf{Q}_p)$ with $a_1 \le a_2 \le a_3 \le a_4$. Let u_t be the operator defined in 6.2. Then*

$$u_t \cdot E_{\mathfrak{D}} = \prod_{i=1}^{4} \beta_i^{a_i},$$

where

$$(\beta_1, ..., \beta_4) = (a(p,f)^{-1}\psi_{p,2}(p), \chi_{0,p}^{-1}\psi_{p,2}\xi_{p,2}(p), p^{\kappa}\chi_{0,p}\psi_{p,1}^{-1}\xi_{p,1}^{-1}(p), a(p,f)p^{\kappa}\psi_{p,1}^{-1}(p)).$$

In particular, if f is ordinary, then so is $E_{\mathfrak{D}}$.

Proof. The operator u_t acts on $E_{\mathfrak{D}}$ as

$$p^{(\kappa-2)(a_3+a_4)+2(a_1+a_2)}\tau_{0,p}((p^{-a_3}, p^{a_2}))\psi_{0,p}((p^{-a_4}, p^{a_1})K_{\mathfrak{D},p}t^{-1}K_{\mathfrak{D},p}.$$

Since $p|\mathfrak{f}_\xi$ (so $p|\mathrm{cond}(\xi_{0,p})$), and since $p|N$ (so $p|\mathrm{cond}_{\pi_{0,p}}(\phi_p)$), it follows that $\pi_{0,p}$, $\psi_{0,p}$, $\tau_{0,p}$, and ϕ_p satisfy the hypotheses of Lemma 9.5.9. In particular, $E_{\mathfrak{D}}$ is an eigenform for the action of u_t. That the eigenvalues are as stated then also follows from Lemma 9.5.9. That $E_{\mathfrak{D}}$ is ordinary if f is ordinary follows from the simple observation that in this case each of the β_i's is a p-adic unit. For this last point, the crucial observation is that the values of χ_0 and ξ are p-adic units (these are finite characters) while the valuation of $\psi_p((p^a, p^b)) = \psi_{p,1}(p^a)\psi_{p,2}(p^b)$ is the same as $p^{a\kappa}$. ∎

10. Hermitian theta functions

In this section we recall the Weil representations and theta functions associated with certain definite Hermitian matrices and define some specific Schwartz functions that enter into our later expressions for Fourier coefficients of the Eisenstein series $E_{\mathfrak{D}}$. As in §9, we adopt an adelic point of view for the most part, first defining and analyzing various local Schwartz functions and then combining them into global objects. The motivation for the specific Schwartz functions included here come from the calculations in §11. This section can be safely omitted from a first reading and only referred to as need arises in subsequent calculations (see especially 11.8 and 11.9).

10.1. **Generalities.** Let V be the two-dimensional $\mathcal{K}$-space of column vectors.

The local set-up. Let v be a place of $\mathbf{Q}$. Let $h \in S_2(\mathbf{Q}_v)$, $\det h \neq 0$. Then $< x, y >_h := {}^t\bar{x}hy$ defines a non-degenerate Hermitian pairing on $V_v := V \otimes \mathbf{Q}_v$. Let U_h be the unitary group of this pairing and let GU_h be its similitude group (algebraic groups over $\mathbf{Q}_v$) with similitude character $\mu_h : GU_h \to \mathbf{G}_m$. Let $V_1 := \mathcal{K}^2$ and $< -, - >_1$ be the pairing on V_1 defined by $< x, y >_1 = xw_1{}^t\bar{y}$. The unitary group of this pairing is just $U_{1/\mathbf{Q}}$. Let $W := V_v \otimes_{\mathcal{K}_v} V_{1,v}$, where $V_{1,v} := V_1 \otimes \mathbf{Q}_v$. Then $(-,-) := \mathrm{Tr}_{\mathcal{K}_v/\mathbf{Q}_v}(< -, - >_h \otimes_{\mathcal{K}_v} < -, - >_1)$ is a $\mathbf{Q}_v$-linear pairing on W that makes W into an 8-dimensional symplectic space over $\mathbf{Q}_v$. The canonical embedding of $U_h \times U_1$ into $Sp(W)$ realizes the pair (U_h, U_1) as a dual pair in $Sp(W)$. Let λ_v be a character of $\mathcal{K}_v^\times$ such that $\lambda_v|_{\mathbf{Q}_v^\times} = 1$. In [Ku94], a splitting $U_h(\mathbf{Q}_v) \times U_1(\mathbf{Q}_v) \hookrightarrow Mp(W, \mathbf{Q}_v)$ of the metaplectic cover $Mp(W, \mathbf{Q}_v) \to Sp(W, \mathbf{Q}_v)$ is associated with the character λ_v; we use this splitting to identify $U_h(\mathbf{Q}_v) \times U_1(\mathbf{Q}_v)$ with a subgroup of $Mp(W, \mathbf{Q}_v)$.

We let $\omega_{h,v}$ be the corresponding Weil representation of $U_h(\mathbf{Q}_v) \times U_1(\mathbf{Q}_v)$ (associated with λ_v and e_v) on the Schwartz space $\mathcal{S}(V_v)$: the action of (u, g) on $\Phi \in \mathcal{S}(V_v)$ is written $\omega_{h,v}(u, g)\Phi$. If $u = 1$ we often omit u, writing $\omega_{h,v}(g)$ to mean $\omega_{h,v}(1, g)$. Then $\omega_{h,v}$ satisfies:

- $\omega_{h,v}(u, g)\Phi(x) = \omega_{h,v}(1, g)\Phi(u^{-1}x)$;
- $\omega_{h,v}(\mathrm{diag}(a, {}^t\bar{a}^{-1}))\Phi(x) = \lambda(a)|a|_{\mathcal{K}}\Phi(xa)$, $a \in \mathcal{K}_v^\times$;
- $\omega_{h,v}(r(S))\Phi(x) = \Phi(x)e_v(< x, x >_h S)$, $S \in \mathbf{Q}_v$;
- $\omega_{h,v}(\eta)\Phi(x) = |\det h|_v \int_{V_v} \Phi(y)e_v(\mathrm{Tr}_{\mathcal{K}/\mathbf{Q}} < y, x >_h)dy$.

We often drop the subscript v from $\omega_{h,v}$.

The global set-up. Let $h \in S_2(\mathbf{Q})$, $h > 0$. We define global versions of U_h, GU_h, W, and $(-,-)$ analogously to the above. Fixing an idele class character $\lambda = \otimes\lambda_v$ of $\mathbf{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ such that $\lambda|_{\mathbf{A}_{\mathbf{Q}}^\times} = 1$, the associated local splittings described above then determine a global splitting $U_h(\mathbf{A}) \times U_1(\mathbf{A}) \hookrightarrow Mp(W, \mathbf{A})$ and hence an action $\omega_h := \otimes\omega_{h,v}$ of $U_h(\mathbf{A}) \times U_1(\mathbf{A})$ on the Schwartz space $\mathcal{S}(V \otimes \mathbf{A})$.

10.1.1. *Theta functions.* Given $\Phi \in \mathcal{S}(V^n \otimes \mathbf{A})$ we let

$$\Theta_h(u, g; \Phi) := \sum_{x \in V} \omega_h(u, g)\Phi(x).$$

This is an automorphic form on $U_h(\mathbf{A}) \times U_1(\mathbf{A})$.

10.2. **Some useful Schwartz functions.** We now define various Schwartz functions that show up in later formulas.

10.2.1. *Archimedean Schwartz functions.* Let $\Phi_{h,\infty} \in \mathcal{S}(V \otimes \mathbf{R})$ be

$$\Phi_{h,\infty}(x) = e^{-2\pi < x,x >_h}.$$

Henceforth we assume that

(10.2.1.a) $$\lambda_\infty(z) = (z/|z|)^{-2}.$$

Lemma 10.2.2. *Given $z \in \mathfrak{h}$, let $\Phi_{h,z}(x) := e(< x, x >_h z)$ (so $\Phi_{h,i} = \Phi_{h,\infty}$). For any $g \in U_1(\mathbf{R})$,*

$$\omega_h(g)\Phi_{h,z} = J_1(g,z)^{-2}\Phi_{h,g(z)}.$$

In particular, if $k \in K^+_{\infty,1}$ then $\omega_h(k)\Phi_{h,\infty} = J_1(k,i)^{-2}\Phi_{h,\infty}$.

Proof. Since $U_1(\mathbf{R})$ is generated by η and the elements of $Q_1(\mathbf{R}) \cap U_1(\mathbf{R})$ it suffices to check the asserted formula for these, and in these cases the formula is a simple consequence of the formulas for the actions of the Weil representation ω_h. ■

10.2.3. *ℓ-adic Schwartz functions.* Let $\Phi_0 \in S(V_\ell)$ be the characteristic function of the set of column vectors with entries in $\mathcal{O}_\ell$. For $y \in \mathrm{GL}_2(\mathcal{K}_\ell)$ let $\Phi_{0,y}(x) := \Phi_0(y^{-1}x)$.

Lemma 10.2.4. *Let $h \in S_2(\mathbf{Q}_\ell)$, $\det h \neq 0$. Let $y \in \mathrm{GL}_2(\mathcal{K}_\ell)$. Suppose ${}^t\bar{y}hy \in S_2(\mathbf{Z}_\ell)^*$.*

(i) *If λ is unramified, ℓ is unramified in $\mathcal{K}$, and $h, y \in \mathrm{GL}_2(\mathcal{O}_\ell)$, then*

$$\omega_h(U_1(\mathbf{Z}_\ell))\Phi_{0,y} = \Phi_{0,y}.$$

(ii) *If $D_\ell \det {}^t\bar{y}hy | \ell^r$, $r > 0$, then*

$$\omega_h(k)\Phi_{0,y} = \lambda(a_k)\Phi_{0,y}, \quad k \in \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_1(\mathbf{Z}_\ell) \ : \ \ell^r | c\}.$$

Proof. We first note that for $b \in \mathbf{Z}_\ell$

$$\begin{aligned}\omega_h(\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right))\Phi_{0,y}(x) &= \Phi_{0,y}(x)e({}^t\bar{x}hxb) \\ &= \Phi_{0,y}(x)e({}^t\bar{x}{}^t\bar{y}^{-1t}\bar{y}hyy^{-1}xb) = \Phi_{0,y}(x),\end{aligned}$$

the last equality following since $\Phi_{0,y}(x) = 0$ unless the entries of $y^{-1}x$ are in $\mathcal{O}_\ell$, in which case ${}^t\bar{x}{}^t\bar{y}^{-1t}\bar{y}hyy^{-1}x \in \mathbf{Z}_\ell$ as ${}^t\bar{y}hy \in S_2(\mathbf{Z}_\ell)^*$ and so the exponential term equals 1. For $a \in \mathcal{O}_\ell^\times$ we also have $\omega_h(\mathrm{diag}(a, \bar{a}^{-1}))\Phi_{0,y} = \lambda(a)\Phi_{0,y}$. Therefore $\omega_h(g)\Phi_{0,y} = \lambda(a_g)\Phi_{0,y}$ for all $g \in B_1(\mathbf{Z}_\ell) \cap U_1(\mathbf{Z}_\ell)$.

Let $\Phi' := \omega_h(\eta)\Phi_{0,y}$. By definition

$$\Phi'(x) = |\det h|_\ell \int_{yM_{2\times 1}(\mathcal{O}_\ell)} e_\ell(\mathrm{Tr}_{\mathcal{K}/\mathbf{Q}}{}^t\bar{u}hx)du.$$

Then $\Phi'(x) = 0$ unless $hx \in {}^t\bar{y}^{-1}M_{2\times 1}(\mathcal{O}_\ell)\delta_{\mathcal{K}}^{-1}$, in which case $\Phi'(x) = |\det h/y\bar{y}|_\ell D_\ell^{-1}$.

Suppose λ is unramified. If $D_\ell = 1$ (so $\delta_{\mathcal{K}} \in \mathcal{O}_\ell^\times$) and $h, y \in \mathrm{GL}_2(\mathcal{O}_\ell)$, then it follows that $\Phi' = \Phi_0$. Since $U_1(\mathbf{Z}_\ell)$ is generated by η and $B_1(\mathbf{Z}_\ell) \cap U_1(\mathbf{Z}_\ell)$, it then follows that in this case $\omega_h(U_1(\mathbf{Z}_\ell))\Phi_0 = \Phi_0$, proving (i).

To prove part (ii) we note that any k as in the statement can be written as

$$k = \left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix}\right) g, \quad \ell^r | c, \ g \in B_1(\mathbf{Z}_\ell) \cap U_1(\mathbf{Z}_\ell).$$

Then

$$\omega_h(k)\Phi_{0,y} = \lambda(a_k)\omega_h((\begin{smallmatrix} 1 & \\ c & 1\end{smallmatrix}))\Phi_{0,y} = \lambda(a_k)\omega_h(\eta^{-1}\left(\begin{smallmatrix} 1 & -c \\ & 1\end{smallmatrix}\right))\Phi'.$$

Let $\Phi'' := \omega_h((\begin{smallmatrix} 1 & -c \\ & 1\end{smallmatrix}))\Phi'$. Then $\Phi''(x) = \Phi'(x)e(-{}^t\bar{x}hxc)$, so $\Phi''(x)$ is zero unless $x = h^{-1}{}^t\bar{y}^{-1}v$ with $v \in M_{2\times 1}(\mathcal{O}_\ell)\delta^{-1}$, in which case ${}^t\bar{x}hxc = {}^t\bar{v}y^{-1}h^{-1}{}^t\bar{y}^{-1}vc \in \mathbf{Z}_\ell$ as $y^{-1}h^{-1}{}^t\bar{y}^{-1}c \in D_\ell S_2(\mathbf{Z}_\ell)^*$. Therefore $\Phi'' = \Phi'$. From this it follows that $\omega_h(k)\Phi_{0,y} = \lambda(a_k)\omega_h(\eta^{-1})\Phi'' = \lambda(a_k)\omega_h(\eta^{-1})\Phi' = \lambda(a_k)\Phi_{0,y}$, proving (ii). ∎

Let θ be a character of $\mathcal{K}_\ell^\times$ and let $x \in \mathrm{cond}(\theta)$ be invertible (i.e., $x \in \mathcal{K}_\ell^\times$). Let

$$\Phi_{\theta,x}(u) := \sum_{a\in(\mathcal{O}_\ell/x)^\times} \theta(a)\Phi_0({}^t(u_1 + a/x, u_2)), \quad u = {}^t(u_1, u_2).$$

For $y \in \mathrm{GL}_2(\mathcal{K}_\ell)$ we let $\Phi_{\theta,x,y}(u) := \Phi_{\theta,x}(y^{-1}u)$. We let $\Phi_{h,\theta,x} := \omega_h(\eta^{-1})\Phi_{\theta,x}$ and $\Phi_{h,\theta,x,y} := \omega_h(\eta^{-1})\Phi_{\theta,x,y}$.

Lemma 10.2.5. *Let $h \in S_2(\mathbf{Q}_\ell)$, $\det h \neq 0$. Let $y \in \mathrm{GL}_2(\mathcal{K}_\ell)$. Suppose ${}^t\bar{y}hy \in S_2(\mathbf{Z}_\ell)^*$. Let θ be a character of $\mathcal{K}_\ell^\times$ and let $0 \neq x \in \mathrm{cond}(\theta)$ be such that $\ell|x$. Let $(c) := \mathrm{cond}(\theta)\cap(\varpi_\ell)$, where $\varpi_\ell = \ell$ if ℓ is split in $\mathcal{K}$ and otherwise ϖ_ℓ is a uniformizer at ℓ.*

(i) *If $cD_\ell \det {}^t\bar{y}hy|x$ and $y^{-1}hy \in \mathrm{GL}_2(\mathcal{O}_\ell)$ and $D_\ell = 1$ or $y^{-1}h^{-1}{}^t\bar{y}^{-1} = (\begin{smallmatrix} * & * \\ * & d\end{smallmatrix})$ with $d \in \mathbf{Z}_\ell$, then*

$$\omega_h(k)\Phi_{\theta,x,y} = \lambda\theta(a_k)\Phi_{\theta,x,y}, \quad k \in U_1(\mathbf{Z}_\ell), D_\ell|c_k, x\bar{x}|b_k.$$

(ii) *The support of $\Phi_{h,\theta,x,y}$ is in $h^{-1}{}^t\bar{y}^{-1}L^*_{\theta,x}$, where if ℓ is non-split in $\mathcal{K}$ then*

$$L^*_{\theta,x} := \{{}^t(u_1, u_2)\ : u_2 \in \delta_{\mathcal{K}}^{-1}\mathcal{O}_\ell, \bar{u}_1 \in \frac{x}{c\delta_{\mathcal{K}}}\begin{cases}\mathcal{O}_\ell & \mathrm{cond}(\theta) = \mathcal{O}_\ell \\ \mathcal{O}_\ell^\times & \mathrm{cond}(\theta) \neq \mathcal{O}_\ell\end{cases}\},$$

and if ℓ splits in K then

$$L^*_{\theta,x} := \{{}^t(u_1, u_2)\ :\ u_2 \in \delta_{\mathcal{K}}^{-1}\mathcal{O}_\ell, \bar{u}_{1,i} \in \frac{x_i}{c_i\delta_{\mathcal{K}}}\begin{cases}\mathbf{Z}_\ell & \mathrm{cond}(\theta_i) = \mathbf{Z}_\ell \\ \mathbf{Z}_\ell^\times & \mathrm{cond}(\theta_i) \neq \mathbf{Z}_\ell\end{cases}\},$$

with $\bar{u}_1 = (\bar{u}_{1,1}, \bar{u}_{1,2})$, $x = (x_1, x_2)$, $c = (c_1, c_2) \in \mathcal{K}_\ell = \mathbf{Q}_\ell \times \mathbf{Q}_\ell$ and $\theta = (\theta_1, \theta_2)$. Furthermore, for $v = h^{-1}{}^t\bar{y}^{-1}u$ with $u \in L^*_{\theta,x}$

$$\Phi_{h,\theta,x,y}(v) = |\det hy\bar{y}|_\ell D_\ell^{-1} \sum_{a\in(\mathcal{O}_\ell/x)^\times} \theta(a)e_\ell(\mathrm{Tr}_{\mathcal{K}/\mathbf{Q}}a\bar{u}_1/x).$$

Proof. We first note that $\Phi_{\theta,x,y}$ is supported on the lattice yL_x, $L_x := \{u = {}^t(u_1, u_2)\ :\ u_1 \in \frac{1}{x}\mathcal{O}_\ell^\times, u_2 \in \mathcal{O}_\ell\}$ and that for $v = yu \in yL_x$, $\Phi_{\theta,x,y}(v) = \theta(-xu_1)$.

Let $\Phi' := \omega_h(\eta)\Phi_{\theta,x,y}$. Then

$$\Phi'(v) = |\det hy\bar{y}|_\ell \int_{L_x} \theta(-xu_1)e_\ell(\mathrm{Tr}_{\mathcal{K}/\mathbf{Q}}{}^t\bar{u}{}^t\bar{y}hv)du.$$

It follows that $\Phi'(v) = 0$ unless $v \in h^{-1t}\bar{y}^{-1}L^*_{\theta,x}$ with

$$L^*_{\theta,x} := \{{}^t(w_1, w_2) \ : w_2 \in \delta^{-1}\mathcal{O}_\ell, \bar{w}_1 \in \frac{x}{c\delta_{\mathcal{K}}}\begin{cases}\mathcal{O}_\ell & \mathrm{cond}(\theta) = \mathcal{O}_\ell \\ \mathcal{O}_\ell^\times & \mathrm{cond}(\theta) \neq \mathcal{O}_\ell\end{cases}\}$$

if ℓ non-split in $\mathcal{K}$, and with

$$L^*_{\theta,x} := \{{}^t(w_1, w_2) \ : \ w_2 \in \delta_{\mathcal{K}}^{-1}\mathcal{O}_\ell, \bar{w}_{1,i} \in \frac{x_i}{c_i\delta_{\mathcal{K}}}\begin{cases}\mathbf{Z}_\ell & \mathrm{cond}(\theta_i) = \mathbf{Z}_\ell \\ \mathbf{Z}_\ell^\times & \mathrm{cond}(\theta_i) \neq \mathbf{Z}_\ell\end{cases}\}$$

if ℓ splits in $\mathcal{K}$, where $\bar{w}_1 = (\bar{w}_{1,1}, \bar{w}_{1,2})$, $x = (x_1, x_2)$, $c = (c_1, c_2) \in \mathcal{K}_\ell = \mathbf{Q}_\ell \times \mathbf{Q}_\ell$, and $\theta = (\theta_1, \theta_2)$. It then follows that for $v = h^{-1t}\bar{y}^{-1}w \in h^{-1t}\bar{y}^{-1}L^*_{\theta,x}$,

$$\Phi'(v) = |\det hy\bar{y}|_\ell D_\ell^{-1}\theta(-1) \sum_{a \in (\mathcal{O}_\ell/x)^\times} \theta(a)e_\ell(\mathrm{Tr}_{\mathcal{K}/\mathbf{Q}} a\bar{w}_1/x).$$

Part (i) can now be proved by a simple modification of the arguments proving Lemma 10.2.4. Part (ii) is just the formula for $\omega_\beta(-1)\Phi'(v) = \Phi'(-v)$. ∎

Lemma 10.2.6. *Suppose ℓ splits in $\mathcal{K}$. Let $(c) := \mathrm{cond}(\theta)$ and suppose $c = (\ell^r, \ell^s)$ with $r, s > 0$. Let $\gamma = (1, \eta) \in \mathrm{SL}_2(\mathcal{O}_\ell) = \mathrm{SL}_2(\mathbf{Z}_\ell) \times \mathrm{SL}_2(\mathbf{Z}_\ell)$. Suppose $h = \mathrm{diag}(\alpha, \beta)$ with $\alpha, \beta \in \mathbf{Z}_\ell^\times$.*

(i) *$\Phi_{h,\theta,c,\gamma}$ is supported on*

$$L' := \{u = {}^t(a, b) \ : \ a \in \mathbf{Z}_\ell \times \mathbf{Z}_\ell^\times, b \in \mathbf{Z}_\ell^\times \times \mathbf{Z}_\ell\},$$

and for $u \in L'$

$$\Phi_{h,\theta,c,\gamma}(u) = \theta_1^{-1}(\alpha a_2)\mathfrak{g}(\theta_1)\theta_2^{-1}(\beta b_1)\mathfrak{g}(\theta_2),$$

where $a = (a_1, a_2), b = (b_1, b_2) \in \mathbf{Z}_\ell \times \mathbf{Z}_\ell$, and $\theta = (\theta_1, \theta_2)$.

(ii) *$\omega_h(u, k)\Phi_{h,\theta,c} = \theta_1^{-1}(a_g)\theta_2(d_g)\lambda\theta(d_k)\Phi_{h,\theta,c}$ for $u = (g, g') \in U_h(\mathbf{Z}_\ell)$ with $p^{\max(r,s)}|c_g$ and for $k \in U_1(\mathbf{Z}_\ell)$ such that $p^{\max(r,s)}|c_k$.*

Proof. Part (i) follows immediately from part (ii) of Lemma 10.2.5. The claim of part (ii) for the action of u follows immediately from part (i), while the claim for the action of k can be proved by a simple modification of the arguments proving Lemma 10.2.4 (with an appeal to part (i) of Lemma 10.2.5). ∎

10.3. **Connections with classical theta functions.** Assume (10.2.1.a) holds. If $\Phi \in \mathcal{S}(V \otimes \mathbf{A})$ is such that $\Phi(u) = \Phi_{h,\infty}(u_\infty)\Phi_f(u_f)$ for some $\Phi_f \in \mathcal{S}(V \otimes \mathbf{A}_f)$, then we let

$$\theta_h(g, z; \Phi) := J_1(g_\infty, i)^2\theta_h(g, g_\infty; \Phi), \quad z \in \mathfrak{h}, g_\infty \in U_1(\mathbf{R}), g_\infty(i) = z, g \in U_h(\mathbf{A}).$$

By Lemma 10.2.2

$$\begin{aligned}\theta_h(g,z;\Phi) &= \sum_{x\in V} J_1(g_\infty,i)^2\omega_h(g,g_\infty)\Phi(x)\\ &= \sum_{x\in V}\Phi_f(g^{-1}x)J_1(g_\infty,i)^2\omega_h(g_\infty)\Phi_{h,i}(x)\\ &= \sum_{x\in V}\Phi_f(g^{-1}x)\Phi_{h,g_\infty(i)}(x)\\ &= \sum_{x\in V}\Phi_f(g^{-1}x)e(<x,x>_h z).\end{aligned} \tag{10.3.0.a}$$

This last series is clearly holomorphic in z, so $\theta_h(g,z;\Phi)$ is a weight 2 modular form.

11. Siegel Eisenstein series and their pull-backs

In this section we recall the pull-back formulas of Garrett and Shimura that, among other things, realize the Klingen-type Eisenstein series from 9.1.5 as inner-products of cuspforms on $\mathfrak{h}$ with the pull-back to $\mathbf{H}_1\times\mathfrak{h}$ of Siegel Eisenstein series on $\mathbf{H}_3$. The Fourier-Jacobi coefficients of these latter series are more amenable to calculation, and we combine explicit formulas for these coefficients with the pull-back formulas to express the Fourier coefficients of the Klingen-type Eisenstein as Peterson inner-products of cuspsforms and theta and Eisenstein series on $\mathfrak{h}$. These last formulas are crucial ingredients in the analysis in §13 of the p-adic interpolations of §12

The organization of this section is similar to §9. We begin in 11.1, 11.2, and 11.3 with generalities on Siegel Eisenstein series on G_n, their pullbacks to embedded $G_m\times G_{m'}$ ($m'=m$ or $m+1$ and $n=m+m'$), and their Fourier-Jacobi coefficients. Then in 11.4 we make explicit choices of local Siegel sections of representations of $G_n(\mathbf{Q}_v)$ induced from characters of the Siegel parabolic $Q(\mathbf{Q}_v)$. As before, we separately consider the sections at the archimedean place, at the $\ell\neq p$ (unramified and ramified) places, and at p. In case, for the sections defined, we compute the local Fourier coefficient, the pullback section (that is the local section of the representation of $G_{m'}(\mathbf{Q}_v)$ obtained by pulling back the Siegel section and integrating against a specific vector in a representation of $G_m(\mathbf{Q}_v)$), and the local Fourier-Jacobi coefficient; for the last two calculations we specialize to $m=1$. Among the key results are proofs that the local sections defined in §9 arise as pullbacks of these Siegel sections. In particular, Proposition 11.4.13 is crucial to the identification of the sections at p in 9.2.5 as pullbacks of Siegel sections. In 11.5, 11.6, and 11.7, we explain how the Eisenstein series $E_{\mathcal{D}}$ and $E_{\mathfrak{D}}$ are obtained by pulling back certain Siegel Eisenstein series and how the Fourier coefficients of $E_{\mathfrak{D}}$ can be expressed in terms of the Fourier-Jacobi coefficients the Siegel Eisenstein series. The rest of this section is taken up with further analysis of the resulting formulas for the Fourier coefficients of $E_{\mathfrak{D}}$. The most important results are Proposition 11.8.2 and Lemmas 11.8.2 and 11.9.4.

11.1. **Siegel Eisenstein series on G_n: the general set-up.** We recall the definition of the Siegel Eisenstein series on G_n and some of their well-known properties.

For a place v of $\mathbf{Q}$ and a character χ of $\mathcal{K}_v^\times$ we let $I_n(\chi)$ be the space of smooth $K_{n,v}$-finite functions $f : K_{n,v} \to \mathbf{C}$ such that $f(qk) = \chi(\det D_q)f(k)$ for all $q \in Q_n(\mathbf{Q}_v) \cap K_{n,v}$. Given $z \in \mathbf{C}$ and $f \in I(\chi)$ we define a function $f(z, -) : G_n(\mathbf{Q}_v) \to \mathbf{C}$ by $f(z, qk) := \chi(\det(D_q))|\det A_q D_q^{-1}|_v^{z+n/2} f(k)$, $q = \begin{pmatrix} A_q & B_q \\ & D_q \end{pmatrix} \in Q_n(\mathbf{Q}_v)$ and $k \in K_{n,v}$.

For an idele class character $\chi = \otimes \chi_v$ of $\mathbf{A}_\mathcal{K}^\times$ we similarly define a space $I_n(\chi)$ of smooth $K_{n,\mathbf{A}}$-finite functions on $K_{n,\mathbf{A}}$. We also similarly define $f(z, -)$ given $f \in I_n(\chi)$ and $z \in \mathbf{C}$. There is an identification $\otimes I_n(\chi_v) = I_n(\chi)$, the former being the restricted tensor product defined using the spherical vectors $f_v^{sph} \in I_n(\chi_v)$, $f_v^{sph}(K_{n,v}) = 1$, at the finite places v where χ_v is unramified: $\otimes f_v$ is identified with $k \mapsto \prod_v f_v(k_v)$.

Let $\mathcal{U} \subseteq \mathbf{C}$ be an open set. By a meromorphic (resp. holomorphic) section of $I_n(\chi)$ on $\mathcal{U}$ we mean a function $\varphi : \mathcal{U} \to I_n(\chi)$ taking values in a finite-dimensional subspace $V \subset I_n(\chi)$ and such that $\varphi : \mathcal{U} \to V$ is meromorphic (resp. holomorphic).

The functions $f(z, -)$ are sections of the induced representations obtained by parabolic induction from the one-dimensional representation $q \mapsto \chi(\det D_q)\delta_{Q_n}(q)^{z/n}$ of the Siegel parabolic Q_n (so the local representations are degenerate principal series).

Let $\chi = \otimes \chi_v$ be a unitary idele class character of $\mathbf{A}_\mathcal{K}^\times$. For $f \in I_n(\chi)$ we consider the Eisenstein series

$$E(f; z, g) := \sum_{\gamma \in Q_n(\mathbf{Q}) \backslash G_n(\mathbf{Q})} f(z, \gamma g), \tag{11.1.0.b}$$

often referred to as a Siegel Eisenstein series. This series converges absolutely and uniformly for (z, g) in compact subsets of $\{\mathrm{Re}(z) > n/2\} \times G_n(\mathbf{A})$ and defines an automorphic form on G_n and a holomorphic function on $\{\mathrm{Re}(z) > n/2\}$. The convergence of this series was essentially shown by Godement (cf. [Bo66, Thm. 11.2]); it also follows from [Sh97, Prop. A3.7] (see also the paragraph preceeding Lemma 18.8 of [Sh97]).

The Eisenstein series $E(f; z, g)$ has a meromorphic continuation in z to all of $\mathbf{C}$. If $\varphi : \mathcal{U} \to I_n(\chi)$ is a meromorphic section, then we put $E(\varphi; z, g) := E(\varphi(z); z, g)$. This is clearly a meromorphic function of $z \in \mathcal{U}$ and an automorphic form on G_n for those z where it is holomorphic. Both the meromorphic continuation and the functional equation given in (11.1.1.d) below are well-known (compare [LR05, p.335]) and follow from the general theory of Eisenstein series. This is explained for symplectic and orthogonal groups in [GPSR, Part A, §5]; the unitary case is similar.

Remark. Our conventions here depart somewhat from some of the literature; here we have essentially identified the Levi of $U_n \cap Q_n$ with $\mathrm{GL}_{n/\mathcal{K}}$ via D_g whereas A_g (or even ${}^tA_g$) is common. Our conventions seem better suited for some of the later calculations related to objects in §9. The effect of our choices is that in appeals to the literature χ must frequently be replaced with $(\chi^c)^{-1}$. For example the space $\{f(z, g) : f \in I_n(\chi)\}$ is the space $I((\chi^c)^{-1}, z)$ in [LR05].

11.1.1. *Intertwining operators and functional equations.* Let χ be a unitary character of $\mathcal{K}_v^\times$, v a place of $\mathbf{Q}$. For $f \in I_n(\chi)$, $z \in \mathbf{C}$, and $k \in K_{n,v}$, we consider the integral

$$M(z,f)(k) := \bar{\chi}^n(\mu_n(k)) \int_{N_{Q_n}(\mathbf{Q}_v)} f(z, w_n r k) dr. \tag{11.1.1.a}$$

For z in compact subsets of $\{\mathrm{Re}(z) > n/2\}$ this integral converges absolutely and uniformly, with the convergence being uniform in k. Clearly, $M(z,f) \in I_n(\bar{\chi}^c)$. It thus defines a holomorphic section $z \mapsto M(z,f)$ on $\{\mathrm{Re}(z) > 3/2\}$. This has a continuation to a meromorphic section on all of $\mathbf{C}$. In particular, if v is finite and χ is unramified then

$$M(z, f_v^{sph})(k) = D_v^{-n(n-1)/4} \prod_{i=0}^{n-1} \frac{L(2z+i-n+1, \bar{\chi}'\chi_{\mathcal{K}}^i)}{L(2z+n-i, \bar{\chi}'\chi_{\mathcal{K}}^i)}, \tag{11.1.1.b}$$

where $\chi' = \chi|_{\mathbf{Q}_v^\times}$.

Let $\chi = \otimes \chi_v$ be a unitary idele class character. For $f \in I_n(\chi)$, $z \in \mathbf{C}$, and $k \in K_{n,\mathbf{A}}$ we consider the integral $M(z,f)(k)$ as in (11.1.1.a) but with the integration being over $N_{Q_n}(\mathbf{A})$. This again converges absolutely and uniformly for z in compact subsets of $\{\mathrm{Re}(z) > n/2\}$, with the convergence being uniform in k. Thus $z \mapsto M(z,f)$ defines a holomorphic section $\{\mathrm{Re}(z) > n/2\} \to I_n(\bar{\chi}^c)$. This, too, has a continuation to a meromorphic section on $\mathbf{C}$. For $\mathrm{Re}(z) > n/2$ at least, we have

$$M(z,f) = \otimes_v M(z, f_v), \quad f = \otimes f_v. \tag{11.1.1.c}$$

The functional equation of the Siegel Eisenstein series is the identity

$$E(f; z, g) = \chi^n(\mu_n(g)) E(M(z,f); -z, g). \tag{11.1.1.d}$$

This should be viewed as an identification of meromorphic functions on $\mathbf{C}$. If $\varphi : \mathcal{U} \to I_n(\chi)$ is a meromorphic section, then we set $M(\varphi)(z) = M(z, \varphi(z))$. This is clearly a meromorphic section on $\mathcal{U}$. From (11.1.1.d) it follows that

$$E(\varphi; z, g) = \chi^n(\mu_n(g)) E(M(\varphi), -z, g). \tag{11.1.1.e}$$

11.2. **Pull-backs of Siegel Eisenstein series.** We recall the pull-back formulas of Garrett and Shimura, which expresses Klingen-type Eisenstein series in terms of restrictions (pull-backs) of Siegel Eisenstein series to subgroups. But first we define various maps between groups that intervene in the statement of the general formula as well as in the particular instances used in subsequent sections.

11.2.1. *Some isomorphisms and embeddings.* Let $V_n := \mathcal{K}^{2n}$. Then w_n defines a skew-Hermitian pairing $< -, - >_n$ on V_n: $< x, y >_n := x w_n {}^t\bar{y}$. The group $G_{n/\mathbf{Q}}$ is the unitary similitude group $GU(V_n)$ of the Hermitian space $(V_n, < -, - >_n)$.

We write an element $v \in V_n$ as $v = (v_1, v_2)$ with $v_i \in \mathcal{K}^n$. We write an element $v \in V_{n+1}$ as $v = (v_1, x, v_2, y)$ with $(v_1, v_2) \in V_n$ and $x, y \in \mathcal{K}$. Let $W_n := V_{n+1} \oplus V_n$ and $W_n' := V_n \oplus V_n$. The matrices $w_{n+1} \oplus -w_n$ and $w_n \oplus -w_n$ define Hermitian pairings on W_n and W_n', respectively, and we write $GU(W_n)$ and $GU(W_n')$ for their respective similitude groups (algebraic groups over $\mathbf{Q}$). Let $X_n := \{(v_1, 0, v_2, y) \oplus (v_1, v_2) \in W_n\}$ and $X'_n := \{(v_1, v_2) \oplus (v_1, v_2) \in W'_n\}$. These are maximal isotropic subspaces of W_n and W'_n, respectively, and we let $P_{X_n} \subset GU(W_n)$ and $P_{X'_n} \subset GU(W'_m)$ be their respective stabilizers.

The isomorphisms $W_n \xrightarrow{\sim} V_{2n+1}$, $(v_1, x, v_2, y) \oplus (u_1, u_2) \mapsto (v_1, x, u_2, v_2, y, u_1)$, and $W'_n \xrightarrow{\sim} V_{2n}$, $(v_1, v_2) \oplus (u_1, u_2) \mapsto (v_1, u_2, v_2, u_1)$ are given by the matrices (written in block form conforming to how we have written the elements of W_n, W'_n, etc.)

$$R := \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{pmatrix} \quad \text{and} \quad R' := \begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0 \end{pmatrix},$$

respectively, and the maps $g \mapsto R^{-1}gR$ and $g \mapsto R'^{-1}gR'$ determine $\mathbf{Q}$-isomorphisms $\alpha_n : GU(W_n) \xrightarrow{\sim} G_{2n+1}$ and $\alpha'_n : GU(W'_n) \xrightarrow{\sim} G_{2n}$. The maps $(v_1, x, u_2, v_2, y, u_1) \mapsto (v_1 - u_1, x, u_2 - v_2, v_2, y, u_1)$ and $(v_1, u_2, v_2, u_1) \mapsto (v_1 - u_1, u_2 - v_2, v_2, u_1)$ are given respectively by the matrices

$$S := \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&-1&1&0&0\\ 0&0&0&0&1&0\\ -1&0&0&0&0&1 \end{pmatrix} \in G_n \quad \text{and} \quad S' := \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&-1&1&0\\ -1&0&0&1 \end{pmatrix},$$

and $g \mapsto S^{-1}gS$ and $g \mapsto (S')^{-1}gS'$ define isomorphisms $\beta_n : G_{2n+1} \xrightarrow{\sim} G_{2n+1}$ and $\beta'_n : G_{2n} \xrightarrow{\sim} G_{2n}$, respectively. We let $\gamma_n := \beta_n \circ \alpha_n : GU(W_n) \xrightarrow{\sim} G_{2n+1}$ and $\gamma'_n := \beta'_n \circ \alpha'_n : GU(W'_n) \xrightarrow{\sim} G_{2n}$. Clearly, $X_n \cdot RS = \{(0, 0, 0, *, *, *)\}$, so $\gamma_n(P_{X_n}) = Q_{2n+1}$, the Siegel parabolic of G_{2n+1}. Similarly, $\gamma'_n(P_{X'_n}) = Q_{2n}$.

Let $G_{n+1,n} := \{(g, g') \in G_{n+1} \times G_n \ : \ \mu_{n+1}(g) = \mu_n(g')\}$ and let $G_{n,n} := \{(g, g') \in G_n \times G_n \ : \ \mu_n(g) = \mu_n(g')\}$. These are clearly subgroups of $GU(W_n)$ and $GU(W'_n)$, respectively. It is easy to see that

$$\gamma_n^{-1}(Q_{2n+1}) \cap G_{n+1,n} = \{(m(g, x)n, g) \ : \ g \in G_n, x \in Res_{\mathcal{K}/\mathbf{Q}}\mathbf{G}_m, n \in N_{P_{n+1}}\}$$

and

$$\gamma_n'^{-1}(Q_{2n}) \cap G_{n,n} = \{(g, g) \ : \ g \in G_n\}.$$

The obvious inclusion $W'_n \hookrightarrow W_n$ induces an embedding $\sigma_n : GU(W'_n) \hookrightarrow GU(W_n)$ given by

$$\sigma_n(g) := \begin{pmatrix} a_1 & 0 & a_2 & 0 & b_1 & b_2\\ 0 & \mu_n(g) & 0 & 0 & 0 & 0\\ a_3 & 0 & a_4 & 0 & b_3 & b_4\\ 0 & 0 & 0 & 1 & 0 & 0\\ c_1 & 0 & c_2 & 0 & d_1 & d_2\\ c_3 & 0 & c_4 & 0 & d_3 & d_4 \end{pmatrix}, \quad g = \begin{pmatrix} A & B\\ C & D \end{pmatrix}, A = \begin{pmatrix} a_1 & a_2\\ a_3 & a_4 \end{pmatrix}, a_i \in M_{n/\mathcal{K}}, \text{ etc.},$$

and, by restriction, an embedding $G_{n,n} \hookrightarrow G_{n+1,n}$; the latter is given by $(g, g') \mapsto (m(g,1), g')$. These are compatible with the embedding $\iota_n : G_{2n} \to G_{2n+1}$ defined by

$$\iota_n(g) := \begin{pmatrix} a_1 & 0 & a_2 & b_1 & 0 & b_2 \\ 0 & \mu_n(g) & 0 & 0 & 0 & 0 \\ a_3 & 0 & a_4 & b_3 & 0 & b_4 \\ c_1 & 0 & c_2 & d_1 & 0 & d_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ c_3 & 0 & c_4 & d_3 & 0 & d_4 \end{pmatrix}, \quad A_g = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right), a_i \in M_{n/\mathcal{K}}, \text{ etc.},$$

in the sense that $\iota_n(\gamma_n'(g)) = \gamma_n(\sigma_n(g))$. Note that $Q_{2n+1} \cap \iota_n(G_{2n}) = \iota_n(Q_{2n})$.

11.2.2. *The pull-back formulas.* Let χ be a unitary idele class character of $\mathbf{A}_{\mathcal{K}}^\times$. Given a cuspform ϕ on G_n we consider

$$\text{(11.2.2.a)} \qquad \begin{gathered} F_\phi(f; z, g) := \int_{U_n(\mathbf{A})} f(z, \gamma(g, g_1 h))\bar{\chi}(\det g_1 h)\phi(g_1 h) dg_1, \\ f \in I_{m+n}(\chi), \;\; g \in G_m(\mathbf{A}), h \in G_n(\mathbf{A}), \;\; \mu_m(g) = \mu_n(h), \quad m = n+1 \text{ or } n, \end{gathered}$$

with $\gamma = \gamma_n$ or γ_n' depending on whether $m = n+1$ or $m = n$. This is clearly independent of h. The pull-back formulas are the identities in the following proposition.

Proposition 11.2.3. *Let χ be a unitary idele class character of $\mathbf{A}_{\mathcal{K}}^\times$.*

(i) *If $f \in I_{2n}(\chi)$, then $F_\phi(f; z, g)$ converges absolutely and uniformly for (z, g) in compact sets of $\{\mathrm{Re}(z) > n\} \times G_n(\mathbf{A})$, and for any $h \in G_n(\mathbf{A})$ such that $\mu_n(h) = \mu_n(g)$*

$$\text{(11.2.3.a)} \qquad \int_{U_n(\mathbf{Q})\backslash U_n(\mathbf{A})} E(f; z, \gamma_n'(g, g_1 h))\bar{\chi}(\det g_1 h)\phi(g_1 h) dg_1 = F_\phi(f; z, g).$$

(ii) *If $f \in I_{2n+1}(\chi)$, then $F_\phi(f; z, g)$ converges absolutely and uniformly for (z, g) in compact sets of $\{\mathrm{Re}(z) > n + 1/2\} \times G_{n+1}(\mathbf{A})$, and for any $h \in G_n(\mathbf{A})$ such that $\mu_n(h) = \mu_{n+1}(g)$*

$$\text{(11.2.3.b)} \qquad \begin{aligned} &\int_{U_n(\mathbf{Q})\backslash U_n(\mathbf{A})} E(f; z, \gamma_n(g, g_1 h))\bar{\chi}(\det g_1 h)\phi(g_1 h) dg_1 \\ &\qquad\qquad = \sum_{\gamma \in P_{n+1}(\mathbf{Q})\backslash G_{n+1}(\mathbf{Q})} F_\phi(f; z, \gamma g), \end{aligned}$$

with the series converging absolutely and uniformly for (z, g) in compact subsets of $\{\mathrm{Re}(z) > n + 1/2\} \times G_{n+1}(\mathbf{A})$.

Part (i) is the well-known doubling formula of Piatetski-Shapiro and Rallis [GPSR, Part A]. Part (ii) is a straightforward generalization. Both formulas follow from a description of representatives of $Q_{n+m}\backslash G_{n+m}/\gamma(G_{m,n})$ (for such a description see [Sh97, Props. 2.4,2.7]) and the cuspidality of ϕ. We will be interested only in the case $n = 1$. In this case the right-hand side of (11.2.3.b) is an Eisenstein series of the type considered in 9.1.5.

Let (π, V), ψ, τ, ρ, and $I(\rho)$ be as in 9.1.4. We extend $\phi \in V$ to a cuspform on $G_1(\mathbf{A})$ by setting $\phi((a, g)) := \psi(a)\phi(g)$. If $f \in I_3(\tau)$, then, at least for $\mathrm{Re}(z) > 3/2$,

$F_\phi(f;z,g)$ converges and its restriction to $K_{\mathbf{A}}$ belongs to $I(\rho)$. The map $z \mapsto F_\phi(f;z,-)$ is a holomorphic map, which we also denote $F_\phi(f)$, from $\{\mathrm{Re}(z) > 3/2\}$ to a finite-dimensional subspace of $I(\rho)$. In this instance (11.2.3.a) becomes

$$\int_{U_1(\mathbf{Q})\backslash U_1(\mathbf{A})} E(f;z,\gamma_1(g,g_1h))\bar{\tau}(\det g_1h)\phi(g_1h)dg_1 = E(F_\phi(f),z,g), \tag{11.2.3.c}$$

where $E(F_\phi(f),z,g)$ is the Klingen-type Eisenstein series from 9.1.5. This is an instance of the general pull-back formula of Garrett and Shimura. Note that the meromorphic continuation of $E(f;z,g)$ yields, via the left-hand side of (11.2.3.c), a meromorphic continuation of $E(F_\phi(f),z,g)$.

Let $f \in I_{2n+1}(\chi)$ and let $f' \in I_{2n}(\chi)$ be defined by $f'(k) = f(\iota_n(k))$. Then

$$F_\phi(f;z,m(g,1)) = F_\phi(f';z+1/2,g).$$

It then follows from (11.2.3.a) that, at least for $\mathrm{Re}(z) > n-1/2$,

$$\begin{aligned} F_\phi(f;z,m(g,1)) = \int_{U_n(\mathbf{Q})\backslash U_n(\mathbf{A})} & E(f';z+1/2,\gamma_n'(g,g_1h)) \\ & \times \bar{\chi}(\det g_1h)\phi(g_1h)dg_1, \end{aligned} \tag{11.2.3.d}$$

where $h \in G_n(\mathbf{A})$ satisfies $\mu_n(h) = \mu_n(g)$.

Remark. In 11.6 below we use the identity (11.2.3.c) to express $E_{\mathcal{D}}$ and $E_{\mathfrak{D}}$ as inner-products of pull-backs of explicit Siegel Eisenstein series with cuspforms. The resulting inner-product identities are crucial ingredients in the p-adic interpolations of §12.

11.3. **Fourier-Jacobi expansions: generalities.** Let $0 \le r < n$ be an integer. Each Eisenstein series $E(f;z,g)$ has a Fourier-Jacobi expansion

$$E(f;z,g) = \sum_{\beta\in S_{n-r}(\mathbf{Q})} E_\beta(f;z,g) \tag{11.3.0.e}$$

where

$$E_\beta(f;z,g) := \int_{S_{n-r}(\mathbf{Q})\backslash S_{n-r}(\mathbf{A})} E(f;z,\begin{pmatrix} 1_n & \begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix} \\ & 1_n \end{pmatrix} g)e_{\mathbf{A}}(-\mathrm{Tr}\,(\beta S))dS. \tag{11.3.0.f}$$

Lemma 11.3.1. *Let $f = \otimes_v f_v \in I_n(\chi)$ be such that for some prime ℓ the support of f_ℓ is in $Q_n(\mathbf{Q}_\ell)w_nQ_n(\mathbf{Q}_\ell)$. Let $\beta \in S_n(\mathbf{Q})$ and $q \in Q_n(\mathbf{A})$. If $\mathrm{Re}(z) > n/2$ then*

$$E_\beta(f;z,q) = \prod_v \int_{S_n(\mathbf{Q}_v)} f_v(z,w_nr(S_v)q_v)e_v(-\mathrm{Tr}\,\beta S_v)dS_v. \tag{11.3.1.a}$$

In particular, the integrals on the right-hand side converge absolutely for $Re(z) > n/2$.

This is well-known. The hypothesis on f_ℓ implies $E(f;z,q) = \sum_{\gamma\in w_nN_{Q_n}(\mathbf{Q})} f(z,\gamma q)$, and then the lemma follows from the absolute convergence of this series.

The following lemma will be used later to express certain Fourier-Jacobi coefficients of Siegel Eisenstein series on G_3 as products of Eisenstein series and theta functions.

Lemma 11.3.2. *Suppose $f \in I_3(\chi)$ and $\beta \in S_2(\mathbf{Q})$, $\beta > 0$. Let V be the two-dimensional $\mathcal{K}$-space of column vectors. If* $\mathrm{Re}(z) > 3/2$ *then*

$$E_\beta(f;z,g) = \sum_{\gamma\in Q_1(\mathbf{Q})\backslash G_1(\mathbf{Q}),\gamma\in U_1(\mathbf{Q})} \sum_{x\in V} \int_{S_2(\mathbf{A})} f(w_3 \begin{pmatrix} 1_3 & \begin{smallmatrix} S & x \\ {}^t\bar{x} & 0 \end{smallmatrix} \\ & 1_3 \end{pmatrix} \alpha_1(1,\gamma)g) e_{\mathbf{A}}(-\mathrm{Tr}\,\beta S) dS.$$

Proof. Let $H \subseteq G_3$ be the stabilizer of the two-dimensional subspace $\{(0,0,0,x_1,x_2,0) : x_i \in \mathcal{K}\}$ of W_3. By Bruhat decomposition

$$G_3(\mathbf{Q}) = Q_3(\mathbf{Q})\xi_0 H(\mathbf{Q}) \sqcup Q_3(\mathbf{Q})\xi_1 H(\mathbf{Q}) \sqcup Q_3(\mathbf{Q})\xi_2 H(\mathbf{Q}),$$

where

$$\xi_0 = 1 \qquad \xi_1 = \begin{pmatrix} & & & 1 & \\ & 1_2 & & & \\ -1 & & & & \\ & & & & 1_2 \end{pmatrix} \qquad \xi_2 = \alpha_1(w_2, 1).$$

From the series defining $E_\beta(f;z,g)$ we find

$$E_\beta(f;z,g) = \sum_{i=0}^{3} \sum_{\zeta\in Q_3(\mathbf{Q})\backslash Q_3(\mathbf{Q})\xi_i H(\mathbf{Q})} \int_{S_2(\mathbf{Q})\backslash S_2(\mathbf{A})} f(z, \zeta \begin{pmatrix} 1_n & \begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix} \\ & 1_n \end{pmatrix} g) e_{\mathbf{A}}(-\mathrm{Tr}\,\beta S) dS.$$

It is relatively easy to see that the summands vanish when $i = 0,1$; in these cases we have that the integral in the summand equals

$$\int_{S_2(\mathbf{Q})\backslash S_2(\mathbf{A})} f(z, \zeta \begin{pmatrix} 1_3 & \begin{smallmatrix} s_1 & s_2 & 0 \\ \bar{s}_2 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \\ & 1_3 \end{pmatrix} g) e(-\mathrm{Tr}\,\beta S) dS, \quad S = \begin{pmatrix} s_1 & s_2 \\ \bar{s}_2 & s_3 \end{pmatrix},$$

and the hypothesis $\beta > 0$ ensures that the integration over s_3 yields zero. The lemma then follows upon observing that

$$Q_3(\mathbf{Q})\xi_2 H(\mathbf{Q}) = \sqcup_{\zeta\in Q_1(\mathbf{Q})\backslash G_1(\mathbf{Q}),\gamma\in U_1(\mathbf{Q})} \sqcup_{x\in V} \sqcup_{S\in S_2(\mathbf{Q})} Q_3(\mathbf{Q})\xi_2 \begin{pmatrix} 1_2 & x & S & 0 \\ & 1 & 0 & 0 \\ & & 1_2 & \\ & & {}^t\bar{x} & 1 \end{pmatrix} \alpha_1(1, \eta^{-1}\zeta)$$

$$= \sqcup_{\gamma\in Q_1(\mathbf{Q})\backslash G_1(\mathbf{Q}),\zeta\in U_1(\mathbf{Q})} \sqcup_{x\in V} \sqcup_{S\in S_2(\mathbf{Q})} Q_3(\mathbf{Q}) w_3 \begin{pmatrix} 1_3 & \begin{smallmatrix} S & x \\ {}^t\bar{x} & 0 \end{smallmatrix} \\ & 1_3 \end{pmatrix} \alpha_1(1,\zeta).$$

∎

The following formula is key to identifying the integral in the preceding lemma as a product of a Schwartz function in $\mathcal{S}(V \otimes \mathbf{Q}_v)$ and a Siegel section in $I_1(\chi)$. Let $a \in \mathbf{A}_{\mathcal{K}}^\times, b \in \mathbf{A}$. Then

$$\begin{pmatrix} 1_3 & \begin{smallmatrix} S & x \\ {}^t\bar{x} & 0 \end{smallmatrix} \\ & 1_3 \end{pmatrix} \alpha_1(1, \begin{pmatrix} a & \bar{a}^{-1}b \\ & \bar{a}^{-1} \end{pmatrix}) = \begin{pmatrix} 1_2 & xb\bar{a}^{-1} & & \\ & \bar{a}^{-1} & & \\ 0 & 0 & 1_2 & \\ 0 & \bar{a}^{-1}b & -b{}^t\bar{x} & a \end{pmatrix} \begin{pmatrix} 1_3 & \begin{smallmatrix} S - x{}^t\bar{x}b & xa \\ \bar{a}{}^t\bar{x} & 0 \end{smallmatrix} \\ 0 & 1_3 \end{pmatrix}.$$

For any place v , $f \in I_3(\chi_v)$, $x \in V \otimes \mathbf{Q}_v$, $g \in U_1(\mathbf{Q}_v)$, $y \in \mathrm{GL}_2(\mathbf{Q}_v)$, and $\beta \in S_2(\mathbf{Q}_v)$, let

$$FJ_\beta(f;z,x,g,y) := \int_{S_2(\mathbf{Q}_v)} f(z, w_3 \begin{pmatrix} 1_3 & \begin{smallmatrix} S & x \\ {}^t\bar{x} & 0 \end{smallmatrix} \\ & 1_3 \end{pmatrix} \alpha_1(\mathrm{diag}(y, {}^t\bar{y}^{-1}), g)) e_v(-\mathrm{Tr}\,\beta S) dS.$$

Then it follows from the above matrix identity that

$$\text{(11.3.2.a)} \quad FJ_\beta(f;z,x, \begin{pmatrix} a & \bar{a}^{-1}b \\ & \bar{a}^{-1} \end{pmatrix} g, y) = \chi_v^c(a)^{-1} |a\bar{a}|_v^{z+3/2} e_v(-{}^t\bar{x}\beta x b) FJ_\beta(f;z,xa,g,y).$$

It is also clear that for $u \in U_\beta(\mathbf{A})$, U_β being the unitary group associated to β in 10.1,

$$FJ_\beta(f;z,x,g,uy) = \chi(\det u)|\det u\bar{u}|_{\mathbf{A}}^{-z+1/2} FJ_\beta(f;z,u^{-1}x,g,y). \tag{11.3.2.b}$$

If, as a function of x, $FJ_\beta(f;z,x,g,y) \in \mathcal{S}(V\otimes \mathbf{Q}_v)$ (this is nearly always the case, but we make it explicit when needed) then (11.3.2.a) can be written as

$$\begin{aligned} &FJ_\beta(f;z,x,\begin{pmatrix} a & \bar{a}^{-1}b \\ & \bar{a}^{-1}\end{pmatrix} g,y) \\ &\qquad = (\lambda_v/\chi_v^c)(a)|a\bar{a}|_v^{z+1/2}\omega_\beta(\begin{pmatrix} a & -\bar{a}^{-1}b \\ & \bar{a}^{-1}\end{pmatrix})FJ_\beta(f;z,x,g,y)\end{aligned} \tag{11.3.2.c}$$

and (11.3.2.b) becomes

$$FJ_\beta(f;z,x,g,uy) = \chi(\det u)|\det u\bar{u}|_{\mathbf{A}}^{-z+1/2}\omega_\beta(u,1)FJ_\beta(f;z,x,g,y). \tag{11.3.2.d}$$

The character λ in (11.3.2.c) is the character implicit in the definition of the Weil representation ω_β.

11.4. **Some good Siegel sections.** We define some explicit functions in the various $I_n(\chi)$'s and compute their (local) Fourier-Jacobi coefficients. We also compute various local analogs of the pull-back integral (11.2.2.a).

11.4.1. *Archimedean Siegel sections.* Let $\kappa \geq 0$ be an integer. Then $\chi(x) = (x/|x|)^{-\kappa}$ is a character of $\mathbf{C}^\times$.

<u>The sections.</u> We let $f_{\kappa,n} \in I_n(\chi)$ be defined by $f_{\kappa,n}(k) := J_n(k,\mathbf{i})^{-\kappa}$. Then

$$f_{\kappa,n}(z,qk) = J_n(k,\mathbf{i})^{-\kappa}\chi(\det D_q)|\det A_qD_q^{-1}|^{z+n/2}, \quad q\in Q_n(\mathbf{R}), k\in K_{n,\infty}. \tag{11.4.1.a}$$

If $g \in U_n(\mathbf{R})$ then $f_{\kappa,n}(z,g) = J_n(g,\mathbf{i})^{-\kappa}|J_n(g,\mathbf{i})|^{\kappa-2z-n}$. As usual, if n is understood we may drop the subscript from our notation.

<u>Fourier-Jacobi coefficients.</u> Given a matrix $\beta \in S_n(\mathbf{R})$ we consider the local Fourier coefficient

$$f_{\kappa,n,\beta}(z,g) := \int_{S_n(\mathbf{R})} f_\kappa(z,w_n\begin{pmatrix} 1_n & S \\ & 1_n\end{pmatrix} g)e_\infty(-\mathrm{Tr}\,\beta S)dS.$$

This converges absolutely and uniformly for z in compact sets of $\{\mathrm{Re}(z) > n/2\}$.

Lemma 11.4.2. *Suppose $\beta \in S_n(\mathbf{R})$. The function $z \mapsto f_{\kappa,n,\beta}(z,g)$ has a meromorphic continuation to all of $\mathbf{C}$. Furthermore, if $\kappa \geq n$ then $f_{\kappa,n,\beta}(z,g)$ is holomorphic at $z_\kappa := (\kappa-n)/2$ and for $y \in \mathrm{GL}_n(\mathbf{C})$, $f_{\kappa,n,\beta}(z_\kappa,\mathrm{diag}(y,{}^t\bar{y}^{-1})) = 0$ if $\det\beta \leq 0$, and if $\det\beta > 0$ then*

$$f_{\kappa,n,\beta}(z_\kappa,\mathrm{diag}(y,{}^t\bar{y}^{-1})) = \frac{(-2)^{-n}(2\pi i)^{n\kappa}(2/\pi)^{n(n-1)/2}}{\prod_{j=0}^{n-1}(\kappa-j-1)!}e(i\mathrm{Tr}\,(\beta y{}^t\bar{y}))\det(\beta)^{\kappa-n}\det\bar{y}^\kappa.$$

Proof. Since $G_n(\mathbf{R}) = Q_n(\mathbf{R})K_{n,\infty}$, it clearly suffices to prove the first two claims under the assumption that $g = \mathrm{diag}(y, {}^t\bar{y}^{-1})$. Let $x := w_n n(S)\mathrm{diag}(y, {}^t\bar{y}^{-1})$. Since $J_n(x, \mathbf{i}) = (-1)^n \det(iy{}^t\bar{y} + S)\det\bar{y}^{-1}$, $f_{\kappa,n,\beta}(z, \mathrm{diag}(y, {}^t\bar{y}^{-1}))$ equals

$$(-1)^{n\kappa}\det\bar{y}^\kappa|\det\bar{y}|^{2(z-z_\kappa)}\times\int_{S_n(\mathbf{R})} e(-\mathrm{Tr}\,(\beta S))\det(iy{}^t\bar{y}+S)^{-\kappa-z+z_\kappa}\det(-iy{}^t\bar{y}+S)^{-z+z_\kappa}dS.$$

This equals

$$2^{n(n-1)/2}(-1)^{n\kappa}\det\bar{y}^\kappa|\det\bar{y}|^{2(z-z_\kappa)}\xi(y{}^t\bar{y}, \beta; z - z_\kappa + \kappa, z - z_\kappa)$$

where $\xi(-,-;s,s')$ is the function in [Sh97, (18.11.4)] (bear in mind that the volume form dS is $2^{n(n-1)/2}$ times that in *loc. cit.*), which is meromorphic as a function of s and s' by, for example, [Sh97, Lemma 18.12]. From this same lemma it follows easily that $f_{\kappa,\beta}(z, g)$ is holomorphic at $z = z_\kappa$ if $\kappa \geq n$ and that the value at z_κ is zero if $\beta \leq 0$.

To complete the proof of the lemma it suffices to note that

$$2^{n(n-1)/2-2}\xi(y{}^t\bar{y}, \beta; s, 0) = (-1)^n(-2\pi i)^{ns}\int_{S_n(\mathbf{R})} e^{2\pi i\mathrm{Tr}\,(\beta S)}(2\pi y{}^t\bar{y} + 2\pi iS)^{-s}dS.$$

By [Sh82, (1.23)], if $\mathrm{Re}(s)$ sufficiently large this equals

$$(-1)^n 2^{n(n-1)/2}(-2\pi i)^{ns}\pi^{-n(n-1)/2}\prod_{j=0}^{n-1}\Gamma(s-j)^{-1}e(\mathrm{Tr}\,\beta y{}^t\bar{y})\det\beta^{s-n}.$$

The formula at $s = \kappa$ then follows from the meromorphicity of $\xi(-,-;s,s')$. ∎

Suppose now that $n = 3$. For $\beta \in S_2(\mathbf{R})$ let $FJ_{\beta,\kappa}(z, x, g, y) := FJ_\beta(f_\kappa; z, x, g, y)$.

Lemma 11.4.3. *Let $z_\kappa := (\kappa - 3)/2$. Let $\beta \in S_2(\mathbf{R})$, $\det\beta > 0$.*

(i) $FJ_{\beta,\kappa}(z_\kappa, x, \eta, 1) = f_{\kappa,2,\beta}(z_\kappa + 1/2, 1)e(i < x, x >_\beta)$.
(ii) *For $g \in U_1(\mathbf{R})$*

$$FJ_{\beta,\kappa}(z_\kappa, x, g, y) = e(i\mathrm{Tr}\,\beta y{}^t\bar{y})\det\bar{y}^\kappa c(\beta,\kappa)f_{\kappa-2,1}(z_\kappa, g')\omega_\beta(g')\Phi_{\beta,\infty}(x),$$

where

$$g' = \begin{pmatrix} 1 & \\ & -1\end{pmatrix} g \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \text{and} \quad c(\beta,\kappa) = \frac{(2\pi i)^{2\kappa}(2/\pi)}{4(\kappa-1)!(\kappa-2)!}\det\beta^{\kappa-2}$$

and the Weil representation ω_β is defined using the character $\lambda_\infty(z) = (z/|z|)^{-2}$.

Proof. If $x = 0$ then part (i) is clear. Suppose then that $x \neq 0$. Then $x = A^t(a, 0)$ for some $A \in U(2)$ and $a \in \mathbf{R}_{>0}$. It then easily follows that

$$FJ_{\beta,\kappa}(z, x, \eta, 1) = FJ_{{}^t\bar{A}\beta A,\kappa}(z, {}^t(a, 0), \eta, 1),$$

and so, upon noting that $f_{\kappa,2,\beta}(z, 1) = f_{\kappa,2,{}^t\bar{A}\beta A}(z, 1)$, to complete the proof of part (i) we may assume $x = {}^t(a, 0)$ with $a \in \mathbf{R}_{>0}$.

Let $u := \sqrt{a^2+1}$. Then

$$\begin{pmatrix} 1_3 & \begin{smallmatrix} 0&0&a\\ 0&0&0\\ a&0&0 \end{smallmatrix} \\ & 1_3 \end{pmatrix} = \begin{pmatrix} u &&&&& \\ & 1 &&&& \\ && u &&& \\ && a/u & 1/u && \\ &&&& 1 & \\ a/u &&&&& 1/u \end{pmatrix} \begin{pmatrix} 1/u &&&&& a/u \\ & 1 &&&& \\ && 1/u & a/u && \\ && -a/u & 1/u && \\ &&&& 1 & \\ -a/u &&&&& 1/u \end{pmatrix}.$$

Write $k(a)$ for the second matrix in the product on the right; this belongs to $K^+_{3,\infty}$. It then follows (since $x = {}^t(a,0)$) that

$$\begin{aligned} w_3 \begin{pmatrix} 1_3 & \begin{smallmatrix} S & x \\ {}^tx & 0 \end{smallmatrix} \\ & 1_3 \end{pmatrix} &= w_3 \begin{pmatrix} u && * & s_1/u & s_2 & \\ & 1 & * & \bar{s}_2/u & s_3 & \\ && u &&& \\ && a/u & 1/u && \\ &&&& 1 & \\ a/u &&& * & * & 1/u \end{pmatrix} k(a) \\ &= \begin{pmatrix} 1/u &&& * & * & * \\ & 1 && * & * & * \\ * & * & 1/u & * & * & * \\ &&& u & * & * \\ &&&& 1 & \\ &&&&& u \end{pmatrix} w_3 \begin{pmatrix} 1 &&& s_1/u^2 & s_2/u & \\ & 1 && \bar{s}_2/u & s_3 & \\ && 1 &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} k(a). \end{aligned}$$

Substituting into the integral defining $FJ_{\beta,\kappa}(z, {}^t(a,0), \eta, 1)$ we find

$$\begin{aligned} FJ_{\beta,\kappa}(z_\kappa, {}^t(a,0), \eta, 1) &= u^{4-2\kappa} FJ_{\beta',\kappa}(z_\kappa, 0, \eta, 1), \quad \beta' = \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) \beta \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) \\ &= u^{4-2\kappa} f_{\kappa,2,\beta'}(z_\kappa, 1) \\ &= f_{\kappa,2,\beta}(z_\kappa, 1) e(ia^2\beta_1), \quad \beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \bar{\beta}_2 & \beta_3 \end{pmatrix}, \end{aligned}$$

the last equality following from Lemma 11.4.2. This proves part (i).

To prove part (ii) we first note that

$$FJ_{\beta,\kappa}(z_\kappa, x, g, y) = \det \bar{y}^\kappa \det y\bar{y}^{2-\kappa} FJ_{{}^t\bar{y}\beta y,\kappa}(z_\kappa, y^{-1}x, g, 1).$$

Let $\beta' := {}^t\bar{y}\beta y$ and $x' := y^{-1}x$. Let $g \in U_1(\mathbf{R})$ and write $g = p\eta k$ with $p = \begin{pmatrix} a & \bar{a}^{-1}b \\ & \bar{a}^{-1} \end{pmatrix} \in B_1(\mathbf{R})$ and $k \in K'_{1,\infty}$. Then $g' = \begin{pmatrix} a & -\bar{a}^{-1}b \\ & \bar{a}^{-1} \end{pmatrix} k^{-1}$. It follows from (11.3.2.d) and part (i) of this lemma that

$$\begin{aligned} FJ_{\beta',\kappa}(z_\kappa, x', g, 1) &= \bar{\chi}^c(a)|a|^\kappa e(-b{}^t\bar{x}'\beta' x') J_1(k, \mathbf{i})^\kappa FJ_{\beta',\kappa}(z_\kappa, x'a, \eta, 1) \\ &= \bar{\chi}^c(a)|a|^\kappa e(-b{}^t\bar{x}'\beta' x') J_1(k, \mathbf{i})^\kappa f_{\kappa,2,\beta'}(\kappa/2-1, 1) e(ia\bar{a}{}^t\bar{x}'\beta' x') \\ &= \bar{\chi}^c(a)|a|^\kappa J_i(k,i)^\kappa e((-b+ia\bar{a}){}^t\bar{x}\beta x) f_{\kappa,2,\beta'}(\kappa/2-1, 1) \\ &= \omega_{\beta,\infty}(g')\Phi_{\beta,\infty}(x) f_{\kappa-2,1}(z_\kappa, g') f_{\kappa,2,\beta'}(\kappa/2-1, 1). \end{aligned}$$

Part (ii) then follows from Lemma 11.4.2. ∎

<u>Pull-back integrals.</u> We now consider analogs of (11.2.3.a), but only in the case $n = 1$. Let (π, V), (π_1, H), and $\psi = \tau$ be as in 9.1.1 and 9.2.1. We extend π_1 to a representation π_ψ of $G_1(\mathbf{R})$ on H by setting $\pi_\psi((a,g))h = \psi(a)\pi_1(g)h$. Let $\phi \in V$ be the unique (up

to scalar) non-zero vector such that $\pi(k)\phi = j(k,i)^{-\kappa}\phi$ for all $k \in K'_{\infty,+}$. Let $f_\kappa \in I_3(\tau)$ be as above and let

$$F_\kappa(z,g) := \int_{U_1(\mathbf{R})} f_\kappa(z, S^{-1}\alpha_1(g, g_1h))\bar{\tau}(\det g_1h)\pi_1(g_1h)\phi dg_1, \quad g \in G_2(\mathbf{R}), h \in G_1(\mathbf{R}), \quad \mu_1(h) = \mu_2(g). \tag{11.4.3.a}$$

If this integral converges for some z and g, then a simple calculation shows that $F_\kappa(z,pg)$ converges for all $p \in P(\mathbf{R})$ and $F_\kappa(z,pg) = \rho(p)\delta(p)^{z+3/2}F_\kappa(z,g)$, where ρ is the representation associated with the triple (π,ψ,τ) as in 9.1.1. Letting ξ_κ be the representation of K_∞ defined in 9.2.1, another easy calculation shows that $F_\kappa(z,gk)$ converges for all $k \in K_\infty$ and equals $\xi_\kappa(k)F_\kappa(z,g)$. Since $G(\mathbf{R}) = P(\mathbf{R})gK_\infty$, this proves that $F_\kappa(z,g')$ converges for all $g' \in G(\mathbf{R})$.

Similarly, for $f_\kappa \in I_2(\tau)$ and $g \in G_1(\mathbf{R})$ we let

$$F'_\kappa(z,g) := \int_{U_1(\mathbf{R})} f_\kappa(z, S^{-1}\alpha'_1(g, g_1h))\bar{\tau}(\det g_1h)\pi_1(g_1h)\phi dg_1, \quad g,h \in G_1(\mathbf{R}), \quad \mu_1(h) = \mu_1(g). \tag{11.4.3.b}$$

Note that $F'_\kappa(z+1/2,g) = F_\kappa(z, m(g,1))$.

Lemma 11.4.4. *The integrals* (11.4.3.a) *and* (11.4.3.b) *converge if* $\mathrm{Re}(z) \geq (\kappa-m-1)/2$ *and* $\mathrm{Re}(z) > (m-1-\kappa)/2$, $m = 2$ *and* 1, *respectively, and for such* z *we have:*

(i) $F_\kappa(z,g) = \pi 2^{-2z-1}\frac{\Gamma(z+(1+\kappa)/2)}{\Gamma(z+(3+\kappa/2))}F_{\kappa,z}(g)$;
(ii) $F'_\kappa(z,g) = \pi 2^{-2z}\frac{\Gamma(z+\kappa/2)}{\Gamma(z+1+\kappa/2)}\pi_\psi(g)\phi$.

On the right-hand side of the equality in part (i), $F_\kappa \in I(\rho)$ is the function so denoted in 9.3.1 and defined using ϕ.

Proof. Since $G(\mathbf{R}) = P(\mathbf{R})K_\infty$, it follows from the observations preceding the lemma that it suffices to show that $F_\kappa(z,1)$ converges for the indicated values of z and satisfies $F_\kappa(z,1) = \pi 2^{-2z-1}\Gamma(z+(1+\kappa)/2)\Gamma(z+(3+\kappa)/2)^{-1}\phi$.

We fix a $(\mathfrak{gl}_2, K'_\infty)$-module embedding $\iota : V \hookrightarrow \mathcal{A}^0(\mathrm{GL}_2)$ of π such that if $\varphi = \iota(\phi)$ then $f(h) := j(g,i)^\kappa \det(g)^{-\kappa/2}\varphi(g)$, $g \in \mathrm{GL}_2^+(\mathbf{R})$, $h = g(i) \in \mathfrak{h}$, is a holomorphic weight κ cuspform. Then ι extends to a Hilbert-space identification of H with the $\mathrm{GL}_2(\mathbf{R})$-submodule of $L^2(\mathrm{GL}_2(\mathbf{Q})\backslash\mathrm{GL}_2(\mathbf{A}))$ generated by φ: $\varphi(xg) = \iota(\pi_1(g)\phi)(x)$. We extend φ to a function on $G_1(\mathbf{R})$ by setting $\varphi((a,g)) := \psi(a)\varphi(g)$, so $\varphi(xg) = \iota(\pi_\psi(g)\phi)(x)$. If

$$\int_{U_1(\mathbf{R})} f_\kappa(z, S^{-1}\alpha_1(1,g))\bar{\tau}(\det g)\varphi(xg)dg \tag{11.4.4.a}$$

converges for all x, then $F_\kappa(z,1)$ also converges and (11.4.4.a) equals $\iota(F_\kappa(z,1))(x)$.

Without loss of generality we may assume $x \in \mathrm{SL}_2(\mathbf{R})$. Let $h := g(i) = a+bi$. Then it is straightforward to check that the integrand in (11.4.4.a) equals

$$J_3(S^{-1}\alpha_1(1,g),\mathbf{i})^{-\kappa}|J_3(S^{-1}\alpha_1(1,g),\mathbf{i})|^{\kappa-2z-3}\bar{\tau}(\det g)j(xg,i)^{-\kappa}f(x(h)). \tag{11.4.4.b}$$

Since

$$J_3(S^{-1}\alpha_1(1,g),\mathbf{i}) = J_3(S^{-1},\alpha_1(1,g)(\mathbf{i}))J_3(\alpha_1(1,g),\mathbf{i}) = (1-i/\bar{h})J_3(\alpha_1(1,g),\mathbf{i}),$$

and since $j(xg,i) = j(x,h)j(g,i)$, $j(g,i)J_3(\alpha_1(1,g),\mathbf{i})\bar{\tau}(\det g) = ib^{-1}\bar{h}$, and $|J_3(\alpha_1(1,g),\mathbf{i})| = b^{-1/2}|\bar{h}|$, we find that (11.4.4.b) equals

$$i^{-\kappa}(\bar{h}-i)^{-\kappa}|\bar{h}-i|^{\kappa-2z-3}b^{z+3/2+\kappa/2}j(x,h)^{-\kappa}f(x(h)).$$

Therefore (11.4.4.a) equals

$$\int_{\mathfrak{h}} i^{\kappa}(i-\bar{h})^{-\kappa}|i-\bar{h}|^{\kappa-2z-3}b^{z+3/2+k/2}j(x,h)^{-\kappa}f(x(h))\frac{dadb}{b^2}, \quad h = a+ib.$$

This last integral is well-known to converge for z as in the statement of the lemma and to equal $\pi 2^{-2z-1}\Gamma(z+(1-\kappa)/2)\Gamma(z+(3-\kappa)/2)^{-1}j(x,i)^{-\kappa}f(x(i))$ (cf. [Sh95, Lemma 4.7]). Since $j(x,i)^{-\kappa}f(x(i)) = \varphi(x)$, the lemma follows. ∎

11.4.5. *ℓ-adic Siegel sections: the unramified case.* Let χ be a unitary character of $\mathcal{K}_\ell^\times$. For $f \in I_n(\chi)$ and $\beta \in S_{n-r}(\mathbf{Q}_\ell)$, $0 \le r < n$, we consider the local Fourier coefficient

$$f_\beta(z,g) := \int_{S_{n-r}(\mathbf{Q}_\ell)} f(z, w_n \begin{pmatrix} 1_n & \begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix} \\ & 1_n \end{pmatrix} g)e_\ell(-\mathrm{Tr}\,\beta S)dS.$$

This converges absolutely and uniformly for z in compact subsets of $\{\mathrm{Re}(z) > n/2\}$.

<u>Fourier-Jacobi coefficients.</u> Suppose now that χ is unramified.

Lemma 11.4.6. *Let $\beta \in S_n(\mathbf{Q}_\ell)$ and let $r := \mathrm{rank}(\beta)$. Then for $y \in \mathrm{GL}_n(\mathcal{K}_\ell)$*

$$f^{sph}_{\ell,\beta}(z,\mathrm{diag}(y,{}^t\bar{y}^{-1})) = \chi(\det y)|\det y\bar{y}|_\ell^{-z+n/2}D_\ell^{-n(n-1)/4} \times \frac{\prod_{i=r}^{n-1}L(2z+i-n+1,\bar{\chi}'\chi^i_{\mathcal{K}})}{\prod_{i=0}^{n-1}L(2z+n-i,\bar{\chi}'\chi^i_{\mathcal{K}})}h_{\ell,{}^t\bar{y}\beta y}(\bar{\chi}'(\ell)\ell^{-2z-n}),$$

where $h_{\ell,{}^t\bar{y}\beta y} \in \mathbf{Z}[X]$ is a monic polynomial depending on ℓ and ${}^t\bar{y}\beta y$ but not on χ.

This is well-known; see Propositions 18.14 and 19.2 of [Sh97].

If ℓ is unramified in $\mathcal{K}$, then a matrix $\beta \in S_2(\mathbf{Z}_\ell)$ will be called *ℓ-primitive* if $\det\beta \in \mathbf{Z}_\ell^\times$. If β is ℓ-primitive then $h_{\ell,\beta} = 1$ (cf. [Sh97, 19.2]).

Lemma 11.4.7. *Suppose ℓ is unramified in $\mathcal{K}$. Let $\beta \in S_2(\mathbf{Q}_\ell)$ such that $\det\beta \neq 0$. Let $y \in \mathrm{GL}_2(\mathcal{K}_\ell)$ such that ${}^t\bar{y}\beta y \in S_2(\mathbf{Z}_\ell)$. Let λ be an unramified character of $\mathcal{K}_\ell^\times$ such that $\lambda|_{\mathbf{Q}_\ell^\times} = 1$.*

(i) *If $\beta, y \in \mathrm{GL}_2(\mathcal{O}_\ell)$ then for $u \in U_\beta(\mathbf{Q}_\ell)$*

$$FJ_\beta(f_3^{sph};z,x,g,uy) = \chi(\det u)|\det u\bar{u}|_\ell^{-z+1/2}\frac{f_1^{sph}(z,g)\omega_\beta(u,g)\Phi_{0,y}(x)}{\prod_{i=0}^{1}L(2z+3-i,\bar{\chi}'\chi^i_{\mathcal{K}})}.$$

(ii) *If* ${}^t\bar{y}\beta y \in \mathrm{GL}_2(\mathcal{O}_\ell)$ *and* $g = \left(\begin{smallmatrix} 1 & \\ n & 1 \end{smallmatrix}\right)$, $n \in \mathbf{Q}_\ell$, *then for* $u \in U_\beta(\mathbf{Q}_\ell)$

$$FJ_\beta(f_3^{sph}; z, x, g, uy) = \chi(\det uy)|\det uy|_{\mathcal{K}}^{-z+1/2} \frac{f_1^{sph}(z,g)\omega_\beta(u,g)\Phi_{0,y}(x)}{\prod_{i=0}^{1} L(2z+3-i, \bar{\chi}'\chi_{\mathcal{K}}^i)}.$$

Here $f_1^{sph} \in I_1(\chi/\lambda)$ *is the unramified spherical function, the Weil representation* ω_β *is defined using* λ, *and* $\Phi_{0,y} \in \mathcal{S}(V \otimes \mathbf{Q}_\ell)$ *is as in 10.2.3.*

Proof. We first note that the argument proving part (i) of Lemma 11.4.3 can be adapted to show

$$\begin{aligned} FJ_\beta(f_3^{sph}; z, x, 1, uy) &= \chi(\det uy)|\det uy|_{\mathcal{K}}^{-z+1/2} FJ_{{}^t\bar{y}\beta y}(f_3^{sph}; z, y^{-1}u^{-1}x, 1, 1) \\ &= \chi(\det uy)|\det uy|_{\mathcal{K}}^{-z+1/2} \Phi_0(y^{-1}u^{-1}x) f_{2,{}^t\bar{y}\beta y}^{sph}(z+1/2, 1) \\ &= \chi(\det uy)|\det uy|_{\mathcal{K}}^{-z+1/2} \omega_\beta(u,1)\Phi_{0,y}(x) f_{2,{}^t\bar{y}\beta y}^{sph}(z+1/2, 1), \end{aligned}$$

where $f_2^{sph} = f_\ell^{sph} \in I_2(\chi)$. Let $p \in B_1(\mathbf{Q}_\ell) \cap U_1(\mathbf{Q}_\ell)$ and $k \in U_1(\mathbf{Z}_\ell)$. Then by (11.3.2.c) and the preceding equalities

$$FJ_\beta(f_3^{sph}; z, x, pk, uy) = f_1^{sph}(pk)\omega_\beta(p)\Phi_{0,y}(x) f_{2,{}^t\bar{y}\beta y}^{sph}(z+1/2, 1),$$

with $f_1^{sph} = f_\ell^{sph} \in I_1(\chi/\lambda)$. Part (i) of the lemma then follows upon appeal to the preceding lemma and part (i) of Lemma 10.2.4. The proof of part (ii) is a simple calculation appealing to the equality

$$FJ_\beta(f_3^{sph}; z, x, g, uy) = \chi(\det uy)|\det uy|_{\mathcal{K}}^{-z+1/2} FJ_{{}^t\bar{y}\beta y}(f_3^{sph}; z, y^{-1}u^{-1}x, g, 1)$$

and to part (i). ■

Pull-back integrals. Suppose now that $n = 1$. We return to the setup and notation of 9.1.2. In particular, we let (π, V), ψ, τ, ρ, and $\xi := \psi/\tau$ be as in 9.1.2. Then the pair (π, ψ) determines a representation of $G_1(\mathbf{Q}_\ell)$ on V, which we denote π_ψ: if $g = (a, b)$ then $\pi_\psi(g)v := \psi(a)\pi(b)v$. Let $\phi \in V$. Let $m = 1$ or 2. Given $f \in I_{m+1}(\tau)$ we consider the local analog of (11.2.2.a):

$$F_\phi(f; z, g) := \int_{U_1(\mathbf{Q}_\ell)} f(z, \gamma(g, g_1 h))\bar{\tau}(\det g_1 h)\pi_\psi(g_1 h)\phi \, dg_1, \tag{11.4.7.a}$$

where $\gamma = \gamma_1$ or γ_1' depending on whether $m = 2$ or $m = 1$. If $m = 2$ and the integral converges for some z and g, then a simple calculation shows that $F_\phi(f; z, pg)$ converges for all $p \in P(\mathbf{Q}_\ell)$ and equals $\rho(p)\delta(p)^{z+3/2}F_\phi(f; z, g)$. If $F_\phi(f; z, g)$ converges for all g then, since f is $K_{m+1,\ell}$-finite, $F_\phi(f; z, pg)$ is $K_{m,\ell}$-finite; in particular, if $m = 2$ then $F_\phi(f; z, g) = F_z(g)$ for some $F \in I(\rho)$.

Lemma 11.4.8. *Suppose π, ψ, and τ are unramified and ϕ is a newvector. If* $\mathrm{Re}(z) > (m+1)/2$ *then* (11.4.7.a) *converges and*

$$F_\phi(f_\ell^{sph}; z, g) = \begin{cases} \frac{L(\tilde{\pi},\xi,z+1/2)}{\prod_{i=0}^{1} L(2z+2-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}\pi_\psi(g)\phi & m=1 \\ \frac{L(\tilde{\pi},\xi,z+1)}{\prod_{i=0}^{1} L(2z+3-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}F_{\rho,z}(g) & m=2. \end{cases}$$

Here, F_ρ is the function defined in 9.1.2 using the newvector ϕ. The lemma is well-known (cf. [LR05, Prop. 3.3]).

11.4.9. *ℓ-adic Siegel sections: ramified cases.*

The sections. Let χ be a unitary character of $\mathcal{K}_\ell^\times$. We let $f_n^\dagger \in I_n(\chi)$ be the function supported on $Q_n(\mathbf{Z}_\ell)w_nN_{Q_n}(\mathbf{Z}_\ell)$ $(= Q_n(\mathbf{Z}_\ell)w_nK_{Q_n}(\lambda^t)$ for any $t>0)$ such that $f_n^\dagger(w_nr) = 1$, $r \in N_{Q_n}(\mathbf{Z}_\ell)$. Given $(\lambda^u) \subseteq \mathcal{O}_\ell$ contained in the conductor of χ, we let $f_{u,n} \in I_n(\chi)$ be the function such that $f_{u,n}(k) = \chi(\det D_k)$ if $k \in K_{Q_n}(\lambda^u)$ and $f_{u,n}(k) = 0$ otherwise.

When n is understood we sometimes drop it from our notation for these sections.

Lemma 11.4.10. *Suppose ℓ is not ramified in $\mathcal{K}$ and suppose χ is such that $\mathcal{O}_\ell \neq \mathrm{cond}(\chi) \supseteq \mathrm{cond}(\chi\chi^c)$. Let $(\ell^u) := \mathrm{cond}(\bar{\chi}^c)$. Then $M(z, f_n^\dagger) = f_{u,n} \in I_n(\bar{\chi}^c)$ for all $z \in \mathbf{C}$.*

Proof. It suffices by analytic continuation to prove the lemma for $\mathrm{Re}(z) > n/2$, so we also assume this. Let $f := M(z, f^\dagger)$ (which exists since $\mathrm{Re}(z) > 3/2$). Let $t := u$ if ℓ is inert and let $t := \max\{u', u''\}$, $u = (u', u'')$, if ℓ splits. Note that $K_{Q_n}(\ell^t) = K_{Q_n}(\ell^u)$ in both cases. The hypotheses on χ imply that $t > 0$.

Let $F(g) := f(z, gx_t)$ with $x_t := \mathrm{diag}(1, \ell^t)w_n^{-1}$. We will show that F is supported on $Q_n(\mathbf{Q}_\ell)w_nK_{Q_n}(\ell^t)$. It will then follow that f is supported on $K_{Q_n}(\ell^t)$. This will prove the lemma since it is easily seen that $f(1) = 1$.

Let $v \in 1 + \ell^{t-1}\mathcal{O}_\ell$ be such that $\chi(v) \neq \bar{\chi}^c(v)$, which is possible by the hypotheses on χ and χ^c. We have

$$G_n(\mathbf{Q}_\ell) = \sqcup_{x\in W_{Q_n}\backslash W_{G_n}/W_{Q_n}} \sqcup_{r\in N_{x,t}} Q_n(\mathbf{Q}_\ell)xrK_{Q_n}(\ell^t),$$

with each $N_{x,t}$ a subset of $K_{Q_n}(\ell) \cap N_{Q_n}^{opp}(\mathbf{Z}_\ell)$ and $N_{w_n,t} = \{1\}$. Let $w_n \neq x \in W_{Q_n}\backslash W_{G_n}/W_{Q_n}$. Let $0 \leq m \leq n$, be such that if $y = \mathrm{diag}(1_m, v, 1_{n-m-1})$ and $d = \mathrm{diag}(\bar{y}^{-1}, y)$ then $x^{-1}dx = d$ (such an m always exists). Then for $r \in N_{t,x}$, $\bar{\chi}^c(v)F(xr) = F(dxr) = F(xrkd) = \chi(v)F(xr)$ since $x^{-1}d^xr = rkx^{-1}dx$ for some $k \in N_{Q_n}^{opp}(\mathbf{Z}_\ell) \cap K_{Q_n}(\ell^t)$. Since $\bar{\chi}(v) \neq \chi(v)$ it must be that $F(xr) = 0$. Thus F is supported on $Q_n(\mathbf{Q}_\ell)w_nK_{Q_n}(\ell^t)$. ∎

Fourier-Jacobi coefficients. We now prove a number of results about the Fourier-Jacobi coefficients of these sections and their images under interwining operators.

Let

(11.4.10.a) $$S_n(\mathbf{Z}_\ell)^* := \{\beta \in S_n(\mathbf{Q}_\ell) \ : \ \mathrm{Tr}\,(\beta S) \in \mathbf{Z}_\ell, S \in S_n(\mathbf{Z}_\ell)\}.$$

Lemma 11.4.11. *Let $A \in \mathrm{GL}_n(\mathcal{K}_\ell)$. If $\det \beta \neq 0$ then*

$$f^\dagger_{n,\beta}(z, \mathrm{diag}(A, {}^t\bar{A}^{-1})) = \begin{cases} \chi(\det A)|\det A\bar{A}|_\ell^{-z+n/2} D_\ell^{-n(n-1)/4} & {}^t\bar{A}\beta A \in S_n(\mathbf{Z}_\ell)^* \\ 0 & \textit{otherwise}. \end{cases}$$

Here D_ℓ is the absolute discriminant of $\mathcal{K}_\ell$ over $\mathbf{Q}_\ell$. This is a straightforward calculation.

Lemma 11.4.12. *Suppose ℓ splits in $\mathcal{K}$ and $\beta \in S_n(\mathbf{Q}_\ell)$, $\det \beta \neq 0$.*

(i) *If $\beta \notin S_n(\mathbf{Z}_\ell)^*$ then $M(z, f_n^\dagger)_\beta(-z, 1) = 0$.*
(ii) *Suppose $\beta \in S_n(\mathbf{Z}_\ell)^*$. Let $c := \mathrm{ord}_\ell(\mathrm{cond}(\chi'))$. Then*

$$M(z, f_n^\dagger)_\beta(-z, 1) = \chi'(\det \beta)|\det \beta|_\ell^{-2z}\mathfrak{g}(\bar{\chi}')^n c_n(\chi', z).$$

where

$$c_n(\chi', z) := \begin{cases} \chi'(\ell^{nc})\ell^{2ncz - cn(n+1)/2} & c > 0 \\ \ell^{2nz - n(n+1)/2} & c = 0. \end{cases}$$

Proof. By meromorphic continuation it suffices to prove the lemma for $\mathrm{Re}(z) > n/2$, so we assume this. Then

$$M(z, f^\dagger)_\beta(-z, 1) = \int_{S_n(\mathbf{Q}_\ell)} \int_{S_n(\mathbf{Q}_\ell)} f^\dagger(z, w_n r(S) w_n r(X)) e_\ell(-\mathrm{Tr}\,\beta X) dX dS.$$

Since

$$w_n r(S) w_n r(X) = \left(\begin{smallmatrix} -1 & -X \\ S & -1+SX \end{smallmatrix}\right),$$

we see that $w_n r(S) w_n r(X) \in Q_n(\mathbf{Q}_\ell) w_n N_{Q_n}(\mathbf{Z}_\ell)$ if and only if $S \in \mathrm{GL}_n(\mathcal{K}_\ell)$ and $S^{-1} - X \in S_n(\mathbf{Z}_\ell)$, in which case

$$w_n r(S) w_n r(X) = \begin{pmatrix} -S^{-1} & 1 \\ & -S \end{pmatrix} w_n \left(\begin{smallmatrix} 1 & -S^{-1}+X \\ & 1 \end{smallmatrix}\right)$$

and

$$f^\dagger(z, w_n r(S) w_n r(X)) e_\ell(-\mathrm{Tr}\,\beta X) = \chi(\det(-S))|\det S|_\ell^{-2z-n} e_\ell(-\mathrm{Tr}\,\beta S^{-1}) e_\ell(\mathrm{Tr}\,\beta(S^{-1} - X)).$$

It follows that, upon making the obvious change of variables,

$$M(z, f^\dagger)_\beta(-z, 1) = \int_{S_n(\mathbf{Q}_\ell)\cap \mathrm{GL}_n(\mathcal{K}_\ell)} \bar{\chi}(\det S)|\det S|_\ell^{2z-n} e_\ell(\mathrm{Tr}\,\beta S) dS \times \int_{S_n(\mathbf{Z}_\ell)} e_\ell(\mathrm{Tr}\,\beta Y) dY.$$

Since the integral over Y vanishes if $\beta \notin S_n(\mathbf{Z}_\ell)^*$ we have that $M(s, f^\dagger)_\beta(-z, 1) = 0$ if $\beta \notin S_n(\mathbf{Z}_\ell)^*$.

Assume then that $\beta \in S_n(\mathbf{Z}_\ell)$. Then the integral over Y equals 1. Using the hypothesis that ℓ splits (so $S_n(\mathbf{Z}_\ell)^* = S_n(\mathbf{Z}_\ell)$) and the identification $\mathcal{K}_\ell = \mathbf{Q}_\ell \times \mathbf{Q}_\ell$ to write $S = (g, {}^t g)$ yields

$$M(z, f^\dagger)_\beta(-z, 1) = \int_{\mathrm{GL}_n(\mathbf{Q}_\ell)} \bar{\chi}'(\det g)|\det g|_\ell^{2z} e_\ell(\mathrm{Tr}\,\beta_1 g) d^\times g,$$

where $\beta = (\beta_1, {}^t\beta_1)$ and $d^\times x$ is $|\det x|_\ell^{-n}$ times the Haar measure on $M_n(\mathbf{Q}_\ell)$. Let $\mathcal{B} \subset \mathrm{GL}_n(\mathbf{Q}_\ell)$ be a set of coset representatives for $\mathrm{GL}_n(\mathbf{Q}_\ell)/\mathrm{GL}_n(\mathbf{Z}_\ell)$. Then $M(z, f^\dagger)_\beta(-z, 1)$ equals

$$\sum_{b \in \mathcal{B}} |\det b|_\ell^{2z} \int_{\mathrm{GL}_n(\mathbf{Z}_\ell)} \bar{\chi}'(\det bg) e_\ell(\mathrm{Tr}\,\beta_1 bg) dg,$$

Let $I(b)$ denote the integral in this expression. Such integrals (generalized Gauss sums) have been evaluated in [HLS]. There it is shown that if $c > 0$ then $I(b) = 0$ unless $\beta_1 b \in \ell^{-\max\{c,1\}}\mathrm{GL}_n(\mathbf{Z}_\ell)$ (i.e., unless $b \in \beta_1^{-1}\ell^{-\max\{c,1\}}\mathrm{GL}_n(\mathbf{Z}_\ell)$), in which case $I(b)$ is shown to equal $\chi'(\det \ell^c \beta)\ell^{-\max\{c,1\}n(n+1)/2}\mathfrak{g}(\bar{\chi}')^n$. ∎

Pull-back integrals. We return to the setup and notation of 9.1.2 and of the pull-back integrals in 11.4.5. We conclude this section with a result that relates $F_\phi(f; z, -)$ and $F_\phi(M(z, f); -z, -)$, where $F_\phi(-, -, -)$ is as in (11.4.7.a).

Let $\rho^\vee$ be the representation associated with the triple $(\pi_1, \psi_1, \tau_1) := (\pi^\vee \otimes \tau', \psi\tau\tau^c, \bar{\tau}^c)$ as in 9.1.2. For $\phi \in V$ we let $\phi^\vee = \pi(\eta)\phi$.

Proposition 11.4.13. *Let $m = 1$ or 2. There exists a meromorphic function $\gamma^{(m)}(\rho, z)$ on $\mathbf{C}$ such that*

(i) *if $m = 1$ then $F_{\phi^\vee}(M(z, f); -z, g) = \gamma^{(1)}(\rho, z)\tau(\mu_1(g))F_\phi(f; z, \eta g)$; moreover, if $\pi \cong \pi(\chi_1, \chi_2)$ and ℓ splits in $\mathcal{K}$ then*

$$\gamma^{(1)}(\rho, z) = \psi(-1)c_2(\tau', z)\mathfrak{g}(\bar{\tau}', \ell^e)^2\epsilon(\tilde{\pi} \otimes \xi^c, z + 1/2)\frac{L(\pi \otimes \bar{\xi}^c, 1/2 - z)}{L(\tilde{\pi} \otimes \xi^c, z + 1/2)},$$

where $c_2(\tau', z)$ is as in Lemma 11.4.12 and $(\ell^e) = \mathrm{cond}(\tau')$;

(ii) *if $m = 2$ and π, ψ, and τ are the ℓ-constituents of a global triple as in 9.1.4, then $F_{\phi^\vee}(M(z, f); -z, g) = \gamma^{(2)}(\rho, z)A(\rho, z, F_\phi(f; z, -))_{-z}(g)$.*

Each of these equalities is an identity of meromorphic functions of z.

Proof. Part (i) essentially follows from [LR05]. Let $\mathcal{P} : V \otimes V \to \mathbf{C}$ be the unique (up to scalar) $G_1(\mathbf{Q}_\ell)$-pairing for $\tilde{\pi}_{\bar{\psi}} \times \pi_\psi$. The equality asserted is equivalent to

$$\mathcal{P}(\phi_1 \otimes \tau(\mu_1(g))F_{\phi^\vee}(M(z, f); -z, g)) = \gamma^{(1)}(\rho, z)\mathcal{P}(\phi_1 \otimes F_\phi(f; z, \eta g)) \tag{11.4.13.a}$$

for all $\phi, \phi_1 \in V$, $g \in G_1(\mathbf{Q}_\ell)$. The left-hand side can be rewritten as

$$\int_{U_1(\mathbf{Q}_\ell)} M(z, f)(-z, \gamma_1'(g, g_1 h))\bar{\tau}_1(\det g_1 h)\bar{\tau}(\mu_1(g))\mathcal{P}(\phi_1 \otimes \pi_{1,\psi_1}(g_1 h)\phi^\vee)dg_1,$$

which equals

$$\int_{U_1(\mathbf{Q}_\ell)} M(z,f)(-z,\gamma_1'(1,g_1))\bar{\tau}_1(\det g_1)\mathcal{P}(\phi_1'\otimes\pi_\psi(g_1)\phi)dg_1, \quad \phi_1' = \tilde{\pi}_{\bar{\psi}}(g^{-1}\eta^{-1}),$$

which in the notation of [LR05] is $Z^{\mathcal{V}}(M(z)f_z,\phi_1'\otimes\phi)$ (with $\mathcal{V}=(V_1,w_1)$). Similarly, the pairing on the right-hand side of (11.4.13.a) equals

$$\int_{U_1(\mathbf{Q}_\ell)} f(z,\gamma_1'(1,g_1))\bar{\tau}(\det g_1)\mathcal{P}(\phi_1'\otimes\pi_\psi(g_1)\phi)dg_1,$$

which is $Z^{\mathcal{V}}(f_z,\phi_1'\otimes\phi)$. Theorem 3 of [LR05] asserts that there exists a meromorphic function $\Gamma^{\mathcal{V}}(z,\tilde{\pi}_{\bar{\psi}},\bar{\tau}^c)$ (in the notation of *loc. cit.*) such that $Z^{\mathcal{V}}(M(z)f_z,\phi_1'\otimes\phi) = \Gamma^{\mathcal{V}}(z,\tilde{\pi}_{\bar{\psi}},\bar{\tau}^c)Z^{\mathcal{V}}(f_z,\phi_1'\otimes\phi)$, thus $\gamma^{(1)}(\rho,z) := \Gamma^{\mathcal{V}}(z,\tilde{\pi}_{\bar{\psi}},\bar{\tau}^c)$ has the asserted property.

Suppose now that $\pi\simeq\pi(\chi_1,\chi_2)$. Then by [LR05, (19), (25), and Thm. 4], in the notation of *loc. cit.*,

$$\begin{aligned}\Gamma^{\mathcal{V}}(z,\tilde{\pi}_{\bar{\psi}},\bar{\tau}^c) &= c(z,\bar{\tau}^c,1,e_\ell)\Gamma(z,\tilde{\pi}_{\bar{\psi}},\bar{\tau}^c,1,e_\ell)\\ &= c(z,\bar{\tau}^c,1,e_\ell)\psi(-1)\epsilon(\tilde{\pi}\otimes\xi^c,z+1/2)\frac{L(\pi\otimes\bar{\xi}^c,1/2-z)}{L(\tilde{\pi}\otimes\xi^c,z+1/2)}.\end{aligned}$$

The factor $c(z,\bar{\tau}^c,1,e_\ell)$ is a constant appearing in a functional equation. In our case, it follows from its definition that this constant equals $M(z,f^\dagger)_1(-z,1)/f_1^\dagger(z,1)$, where the subscript '1' denotes the local Fourier coefficient at $1\in S_2(\mathbf{Q}_\ell)$ (take f in (14) of [LR05] to be the section $f(z,g)=f^\dagger(z,gw_2)$ of $I(\tau)$). If ℓ splits in $\mathcal{K}$ this ratio is equal to $c_2(\tau',z)\mathfrak{g}(\bar{\tau}')^2$ by Lemma 11.4.12, and we obtain the desired formula for $\gamma^{(1)}(\rho,z)$.

We deduce part (ii) from the functional equations for Eisenstein series. The conclusion of part (ii) very likely holds without the hypothesis of being part of a global triple, but we have settled for the weaker result as we can make do with it. To prove part (ii) we change our notation: suppose $(\pi,V)=(\otimes\pi_v,\otimes V_v)$, $\psi=\otimes\psi_v$, and $\tau=\otimes\tau_v$ are global objects as in 9.1.4. Let ρ be the representation associated with this triple. Let $(\pi_1,\psi_1,\tau_1) := (\pi^\vee\otimes\tau',\psi\tau\tau^c,\bar{\tau}^c)$, so $\rho^\vee$ is the representation associated to this triple. Let $f\in I_3(\tau)$ and $\phi\in V$. Let $\phi^\vee := \pi(\eta)\phi$ and let $F_\phi(f;z,-)\in I(\rho)$ be as in (11.2.2.a). The integral (11.2.2.a) converges if $\mathrm{Re}(z)>3/2$ and (11.2.3.d) provides a meromorphic continuation in z to all of $\mathbf{C}$. Writing $F_\phi(f)$ for the (meromorphic) function $z\mapsto F_\phi(f;z,-)\in I(\rho)$, we have by (11.2.3.c)

$$\int_{U_1(\mathbf{Q})\backslash U_1(\mathbf{A})} E(f;z,\gamma_1(g,g_1h))\bar{\tau}(\det g_1h)\phi_\psi(g_1h)dg_1 = E(F_\phi(f),z,g), \quad \mu_1(h)=\mu_2(g).$$

Here ϕ has been extended to $G_1(\mathbf{A})$: $\phi_\psi((a,g))=\psi(a)\phi(g)$, $a\in\mathbf{A}_{\mathcal{K}}^\times$, $g\in\mathrm{GL}_2(\mathbf{A})$. From the functional equation for Siegel Eisenstein series (see (11.1.1.d)) we deduce that

$$\int_{U_1(\mathbf{Q})\backslash U_1(\mathbf{A})} E(M(z,f);-z,\gamma_1(g,g_1h))\bar{\tau}_1(\det g_1h)\phi_{\psi_1}^\vee(g_1h)dg_1 = E(F_\phi(f),z,g).$$

By (11.2.3.c) the left-hand side equals $E(F_{\phi^\vee}(M(z,f)),-z,g)$ while by the functional equation for Klingen-Eisenstein series the right-hand side equals $E(A(\rho,z,F_\phi(f)),-z,g)$.

This implies

$$F_{\phi^\vee}(M(z,f);-z,g) = A(\rho,z,F_\phi(f;z,-))_{-z}(g). \tag{11.4.13.b}$$

Let Σ be a set of places of $\mathbf{Q}$ including ∞, ℓ, and those where $\mathcal{K}$, π, ψ, or τ is ramified. Suppose $\phi = \otimes\phi_v$ with ϕ_v a newvector if $v \notin \Sigma$, and suppose $f = \otimes f_v$ with $f_v = f_v^{sph}$ if $v \notin \Sigma$. Let $g \in G(\mathbf{A})$ be such that $g_v = 1$ if $v \notin \Sigma$. From (11.1.1.b), (11.1.1.c), and Lemma 11.4.8 we deduce that the left-hand side of (11.4.13.b) equals $c_1(\rho,z) \otimes_{v\notin\Sigma} \phi_v \otimes F_{\phi_\ell^\vee}(M(z,f_\ell);-z,g_\ell)$ where

$$c_1(\rho,z) := \frac{L^\Sigma(\tilde\pi_1 \otimes \xi, -z+1)\prod_{i=0}^{n-1} L^\Sigma(2z+1-i,\bar\tau'\chi_{\mathcal{K}}^i)}{L^\Sigma(-2z+2,\tau'\chi_{\mathcal{K}})\prod_{i=0}^{n-1} L^\Sigma(2z+2-i,\bar\tau'\chi_{\mathcal{K}}^i)} \otimes_{v\in\Sigma, v\neq\ell} F_{\phi_v^\vee}(M(z,f_v);-z,g_v).$$

Clearly $c_1(\rho,z)$ defines a meromorphic function from $\mathbf{C}$ to a finite-dimensional subspace W of $\otimes_{v\in\Sigma,v\neq\ell}V_v$. The f_v, ϕ_v, and g_v for $v\in\Sigma, v\neq\ell$ can be chosen so that $c_1(\rho,z)$ is not identically zero. For example, take $f_\infty = M(-z,f_\kappa)$, and for $v\nmid\infty$ take $f_v = M(-z,f_v')$ with f_v' having sufficiently small support. We assume such a choice has been made.

From (9.1.5.c) and (9.1.2.c) we see that the right-hand side of (11.4.13.b) equals $c_2(\rho,z) \otimes_{v\notin\Sigma} \phi_v \otimes A(\rho_\ell, z, F_{\phi_\ell}(f_\ell;z,-))_{-z}(g_\ell)$ where

$$c_2(\rho,z) := \frac{L^\Sigma(\tilde\pi\otimes\xi,z)L^\Sigma(2z,\bar\tau')}{L^\Sigma(\tilde\pi\otimes\xi,z+1)L^\Sigma(2z+1,\bar\tau')} \otimes_{v\in\Sigma,v\neq\ell} A(\rho_v,z,F_{\phi_v^\vee}(f_v;z,-))_{-z}(g_v).$$

Clearly $c_2(\rho,z)$ also defines a meromorphic function from $\mathbf{C}$ into a finite-dimensional subspace W' of $\otimes_{v\in\Sigma,v\neq\ell}V_v$. Since

$$c_1(\rho,z)\otimes F_{\phi_\ell^\vee}(M(z,f_\ell),-z,g_\ell) = c_2(\rho,z)\otimes A(\rho_\ell,z,F_{\phi_\ell}(f_\ell;z,-))_{-z}(g_\ell),$$

we may take $W' = W$. Let $w_1,\dots,w_n$ be a basis for W and write $c_i(\rho,z) = \sum_{j=1}^n a_{j,i}(z)w_i$ with the $a_{j,i}(z)$'s meromorphic function. Then

$$a_{j,1}F_{\phi_\ell^\vee}(M(z,f_\ell);-z,g_\ell) = a_{j,2}A(\rho_\ell,z,F_{\phi_\ell}(f_\ell;z,-))_{-z}(g_\ell).$$

Since $c_1(\rho,z)$ is not identically zero, some $a_{j_0,1}(z)$ is not identically zero. We may then take $\gamma^{(2)}(\rho,z) = a_{j_0,2}(z)/a_{j_0,1}(z)$, proving part (ii). ■

11.4.14. *ℓ-adic Siegel sections: ramified cases again.*

The sections. We define modified version of the sections $f^\dagger$ that are well-adapted to the pull-back formula.

Let $m = 1$ or 2. For $x \in \mathcal{O}_\ell \cap \mathcal{K}_\ell^\times$ let

$$f_x^{\dagger,(m)}(z,g) := f_{m+1}^\dagger(z, g\begin{pmatrix} 1 & & & & & 1/x \\ & 1_{m-1} & & & 0_{m-1} & \\ & & 1 & 1/\bar x & & \\ & & & 1_{m-1} & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}).$$

Fourier-Jacobi coefficients.

Lemma 11.4.15. *Let $\beta = (b_{i,j}) \in S_{m+1}(\mathbf{Q}_\ell)$. Then for all $z \in \mathbf{C}$, $f^{\dagger,(m)}_{x,\beta}(z,1) = 0$ if $\beta \notin S_{m+1}(\mathbf{Z}_\ell)^*$. If $\beta \in S_{m+1}(\mathbf{Z}_\ell)^*$ then*

$$f^{\dagger,(m)}_{x,\beta}(z,1) = D_\ell^{-m(m+1)/4} e_\ell(\mathrm{Tr}\, b_{m+1,1}/x).$$

This is a straightforward calculation using Lemma 11.4.11.

Lemma 11.4.16. *Let $\beta \in S_2(\mathbf{Q}_\ell)$, $\det\beta \neq 0$. Let $y \in \mathrm{GL}_2(\mathcal{K}_\ell)$ and suppose ${}^t\bar{y}\beta y \in S_2(\mathbf{Z}_\ell)^*$. Let λ, θ be characters of $\mathcal{K}_\ell^\times$ and suppose $\lambda|_{\mathbf{Q}_\ell^\times} = 1$. Let $(c) := \mathrm{cond}(\lambda) \cap \mathrm{cond}(\theta) \cap (\ell)$. Let $x \in \mathcal{K}_\ell^\times$ be such that $D_\ell | x$, $\mathrm{cond}(\chi^c)|x$, and $cD_\ell \det {}^t\bar{y}\beta y | x$. Suppose $y^{-1}\beta^{-1}{}^t\bar{y}^{-1} = \left(\begin{smallmatrix} * & * \\ * & d\end{smallmatrix}\right)$ with $d \in \mathbf{Z}_\ell^\times$. Then for $h \in U_\beta(\mathbf{Q}_\ell)$*

$$\sum_{a\in(\mathcal{O}_\ell/x)^\times} \theta\bar{\chi}^c(a) FJ_\beta(f_x^{\dagger,(2)}; z, u, g\left(\begin{smallmatrix} a^{-1} & \\ & \bar{a}\end{smallmatrix}\right), hy)$$
$$= \chi(\det hy)|\det hy|_{\mathcal{K}}^{-z+1/2} D_\ell^{-1/2} \sum_{b\in\mathbf{Z}_\ell/D_\ell} f_b(z, g\eta)\omega_\beta(h, g\left(\begin{smallmatrix} 1 & \\ b & 1\end{smallmatrix}\right))\Phi_{\theta,x,y}(u),$$

where $\Phi_{\theta,x,y}$ is as in 10.2.3, ω_β is defined using λ, and $f_b \in I_1(\chi/\lambda)$ is the section defined by

$$f_b(g) := \begin{cases} \chi\lambda^{-1}(d_p) & g = b_1\eta\left(\begin{smallmatrix} 1 & m \\ & 1\end{smallmatrix}\right), b_1 \in B_1(\mathbf{Z}_\ell), m - b \in D_\ell\mathbf{Z}_\ell \\ 0 & \textit{otherwise.}\end{cases}$$

Recall that D_ℓ is the discriminant of $K_\ell/\mathbf{Q}_\ell$.

Proof. Let $K := \{k \in U_1(\mathbf{Z}_\ell) : a_k - 1, d_k - 1, c_k \in (x\bar{x})\}$ and $K' := \{k \in K : b_k \in D_\ell\mathcal{O}_\ell\}$. Since $D_\ell|x$, K' is a normal subgroup of K. We also have

(11.4.16.a) $$K = \sqcup_{b\in\mathbf{Z}_\ell/D_\ell} \left(\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right) K'$$

and a decomposition

(11.4.16.b) $$U_1(\mathbf{Q}_\ell) = B_1'(\mathbf{Q}_\ell)\eta K \sqcup_r B_1'(\mathbf{Q}_\ell)x_r K, \quad B_1' := B_1 \cap U_1,\ x_r = \left(\begin{smallmatrix} 1 & \\ s & 1\end{smallmatrix}\right),\ \ell | s.$$

Let

$$F(g) := \sum_{a\in(\mathcal{O}_\ell/x)^\times} \theta\bar{\chi}^c(a) FJ_\beta(f_x^{\dagger,(2)}; z, u, g\eta^{-1}\left(\begin{smallmatrix} a^{-1} & \\ & \bar{a}\end{smallmatrix}\right), hy).$$

Then it follows from the definition of $f_x^{\dagger,(2)}$ that

(11.4.16.c) $$F(gk) = F(g), \quad k \in K.$$

In light of this, (11.4.16.b), and (11.3.2.a), to determine $F(g)$ for all g we only need to determine $F(\eta)$ and each $F(x_r)$. We claim that

(a) $F(x_r) = 0$ for all x_r, and
(b) $F(\eta) = \chi(\det hy)|\det hy|_{\mathcal{K}}^{-z+1/2} D_\ell^{-1/2}\Phi_{\theta,x,y}(h^{-1}u)$.

To prove this claim, we first observe that

$$\left(\begin{smallmatrix} 1 & \\ s & 1 \end{smallmatrix}\right)\eta^{-1} = \begin{pmatrix} -s^{-1} & -1 \\ & -s \end{pmatrix}\left(\begin{smallmatrix} 1 & \\ -s^{-1} & 1 \end{smallmatrix}\right).$$

It then follows from (11.3.2.a) that $FJ_\beta(f_x^{\dagger,(2)}; z, u, x_r\eta^{-1}, hy)$ is a multiple of

$$FJ_\beta(f_x^{\dagger,(2)}; z, -s^{-1}u, \left(\begin{smallmatrix} 1 & \\ -s^{-1} & 1 \end{smallmatrix}\right), hy).$$

As $s^{-1} \notin \mathbf{Z}_\ell$, in the integrand of the formula defining this last expression $f^\dagger$ is evaluated at an element that does not belong to $Q_3(\mathbf{Q}_\ell)\eta N_{Q_3}(\mathbf{Z}_\ell)$, and so the value of the expression is 0. Part (a) follows. Part (b) is clear from the definitions and the observation that $FJ_\beta(f; z, u, g, y) = \chi(\det hy)|\det hy|_{\mathcal{K}}^{-z+1/2} FJ_{{}^t\bar{y}\beta y}(f; z, y^{-1}h^{-1}u, g, 1)$.

From (a) and (11.4.16.b) it follows that $F(g)$ is supported on $B_1'(\mathbf{Q}_\ell)\eta K$. If $g = p\eta\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)k$, $b \in \mathbf{Z}_\ell$ and $k \in K'$, then by (b) and (11.3.2.a)

$$\begin{aligned}
F(g) &= \chi(\det y)|\det y\bar{y}|_\ell^{-z+1/2} D_\ell^{-1/2} f_1^\dagger(z, g)\omega_\beta(p)\Phi_{\theta,x,y}(h^{-1}u) \\
&= \chi(\det y)|\det y\bar{y}|_\ell^{-z+1/2} D_\ell^{-1/2} f_1^\dagger(z, g)\omega_\beta(h, gk^{-1}\left(\begin{smallmatrix} 1 & -b \\ & 1 \end{smallmatrix}\right)\eta^{-1})\Phi_{\theta,x,y}(u) \\
&= \chi(\det y)|\det y\bar{y}|_\ell^{-z+1/2} D_\ell^{-1/2} f_1^\dagger(z, g)\omega_\beta(h, g\left(\begin{smallmatrix} 1 & -b \\ & 1 \end{smallmatrix}\right)\eta^{-1})\Phi_{\theta,x,y}(u) \\
&= \chi(\det y)|\det y\bar{y}|_\ell^{-z+1/2} D_\ell^{-1/2} f_1^\dagger(z, g)\omega_\beta(h, g\eta^{-1}\left(\begin{smallmatrix} 1 & \\ b & 1 \end{smallmatrix}\right))\Phi_{\theta,x,y}(u),
\end{aligned}$$

where $f_1^\dagger \in I_1(\chi/\lambda)$. The next to last equation follows from the one above it by part (i) of Lemma 10.2.5 and the fact that K' is normal in K. In view of (11.4.16.a), the lemma is a simple consequence of this final expression for $F(g)$. ∎

Pull-back integrals. We again return to the setup and notation of 9.1.2 and of the pull-back integrals in 11.4.5. To unify some of our formulas, in the following we write S for S' when $m = 1$.

Let $\mathcal{T}$ denote a triple (ϕ, ψ, τ) with $\phi \in V$ having a conductor with respect to $\tilde{\pi}$. Let $\phi_x := \pi_\psi(\eta\mathrm{diag}(\bar{x}^{-1}, x))\phi$ and let

$$F_{\mathcal{T},x}^{(m)}(z, g) := \int_{U_1(\mathbf{Q}_\ell)} f_x^{\dagger,(m)}(z, S^{-1}\alpha(g, g'h))\bar{\tau}(\det g'h)\pi_\psi(g'h)\phi_x dg',$$

where $\alpha = \alpha_1$ or α_1' depending on whether $m = 2$ or 1. If $f(z, g) = f_x^{\dagger,(m)}(z, gS^{-1})$ then $F_{\mathcal{T},x}^{(m)}(z, g) = F_{\phi_x}(f; z, g)$.

Proposition 11.4.17. *Suppose $x = \lambda^t$, $t > 0$, is contained in the conductors of τ and ψ and $x\bar{x} \in (\lambda^{r_\phi}) = \mathrm{cond}_{\tilde{\pi}}(\phi)$. Then $F_{\mathcal{T},x}^{(m)}(z, g)$ converges for all z and g and*

$$F_{\mathcal{T},x}^{(1)}(z, \eta) = [U_1(\mathbf{Z}_\ell) : K_x]^{-1}\tau(x)|x\bar{x}|_\ell^{-z-1}\phi$$

and

$$F_{\mathcal{T},x}^{(2)} = [U_1(\mathbf{Z}_\ell) : K_x]^{-1}\tau(x)|x\bar{x}|_\ell^{-z-3/2} F_{\phi,r,t}$$

for any $r \geq \max\{r_\phi, t\}$. Here K_x is the subgroup defined in (11.4.17.c) *below.*

Proof. We first note that

$$f_x^{\dagger,(2)}(z, S^{-1}\alpha_1(g, g_1)) = f^\dagger(z, S^{-1}\alpha_1(g, g_1)w_3^{-1}d_xSd_x^{-1}w_3) \tag{11.4.17.a}$$
$$= \tilde{f}_x(\gamma_1(gw^{-1}, g_1\eta\mathrm{diag}(\bar{x}^{-1}, x))),$$

where $d_x = \mathrm{diag}(1_2, x, 1_2, \bar{x}^{-1})$ and $\tilde{f}_x(g) := f^\dagger(z, gd_x^{-1}w_3)$. Note that $\tilde{f}_x$ is supported on

$$Q(x) := Q_3(\mathbf{Q}_\ell)\{r'(T) = \left(\begin{smallmatrix} 1 & \\ T & 1\end{smallmatrix}\right) \ : \ T \in S_x = \mathrm{diag}(1_2, \bar{x})S_3(\mathbf{Z}_\ell)\mathrm{diag}(1_2, x)\}$$

and satisfies $\tilde{f}_x(r'(T)) = f^\dagger(z, d_x^{-1}w_3) = \tau^c(x)|x\bar{x}|_\ell^{-z-3/2}$ for $T \in S_x$. We claim that

$$\gamma_1(g, 1) \in Q(x) \Rightarrow g \in P(\mathbf{Q}_\ell)N_Q^{opp}(\mathbf{Z}_\ell)K(x), \tag{11.4.17.b}$$

where $K(x) := \{\mathrm{diag}({}^t\bar{A}^{-1}, A) \ : \ A - 1 \in xM_2(\mathcal{O}_\ell)\}$, and

$$\gamma_1(1, g) \in Q(x) \Leftrightarrow g \in K_x := \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in U_1(\mathbf{Z}_\ell) \ : \ a - 1 \in (\bar{x}), b \in (x\bar{x}), \tag{11.4.17.c}$$
$$c \in \mathcal{O}_\ell, d - 1 \in (x)\}.$$

For $g \in U(\mathbf{Q}_\ell)$ write

$$A_g = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4\end{smallmatrix}\right)$$

and similarly denote the matrix entries of B_g, C_g, and D_g. Then

$$\gamma_1(g, 1) = \begin{pmatrix} a_1 & a_2 & -b_1 & b_1 & b_2 & 0 \\ a_3 & a_4 & -b_3 & b_3 & b_4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ c_1 & c_2 & 1-d_1 & d_1 & d_2 & 0 \\ c_3 & c_4 & -d_3 & d_3 & d_4 & 0 \\ a_1 - 1 & a_2 & -b_1 & b_1 & b_2 & 1 \end{pmatrix}.$$

So $h := \gamma_1(g, 1) \in Q(x)$ if and only if D_h is invertible and $D_h^{-1}C_h \in S_x$. This implies that we must have (1) D_g is invertible, (2) $D_g^{-1}C_g \in S(\mathbf{Z}_\ell)$, and (3) $d_4 - \det D_g, d_3 \in \det D_g(x)$. Conditions (1) and (2) imply that $g \in Q(\mathbf{Q}_\ell)N_Q^{opp}(\mathbf{Z}_\ell)$. Since $\gamma_1(N_Q^{opp}(\mathbf{Z}_\ell), 1)\} \subseteq r'(S_x)$, without loss of generality we may assume that $g \in Q(\mathbf{Q}_\ell)$ (i.e., that $C_g = 0$). Since $g' = m(1, \det(D_g)^{-1})g$ is such that $\det D_{g'} = 1$ and since $\gamma_1(m(1, *), 1) \in Q_3(\mathbf{Q}_\ell)$, we may also assume that $\det(D_g) = 1$. Then (3), together with the hypothesis $t > 0$, implies that $d_4 \in \mathcal{O}_\ell^\times$. So there exists $n \in \mathbf{Q}_\ell$ such that

$$XD_g = \left(\begin{smallmatrix} * & 0 \\ * & *\end{smallmatrix}\right), \quad X := \left(\begin{smallmatrix} 1 & n \\ & 1\end{smallmatrix}\right).$$

Since $Y := \mathrm{diag}({}^t\bar{X}^{-1}, X) \in N_P(\mathbf{Q}_\ell)$, $\gamma_1(Y, 1) \in Q_3(\mathbf{Q}_\ell)$. So upon replacing g with Yg we may assume that $g \in Q(\mathbf{Q}_\ell)$, $\det D_g = 1$, $d_3, d_4 - 1 \in (x)$, and $d_2 = 0$. It follows that $d_1 - 1 \in (x)$ as well. So $D_g - 1 \in xM_2(\mathcal{O}_\ell)$. Since $A_g = {}^t\bar{D}_g^{-1}$, we also have $A_g - 1 \in \bar{x}M_2(\mathcal{O}_\ell)$. Thus $U := \mathrm{diag}(A_g^{-1}, D_g^{-1}) \in K(x)$. Since $gU \in N_Q(\mathbf{Q}_\ell) \subset P(\mathbf{Q}_\ell)$ it follows that $g \in P(\mathbf{Q}_\ell)K(x)N_Q^{opp}(\mathbf{Z}_\ell) = P(\mathbf{Q}_\ell)N_Q^{opp}(\mathbf{Z}_\ell)K(x)$, proving (11.4.17.b).

Next, observe that if

$$g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in U_1(\mathbf{Q}_\ell)$$

then

$$\gamma_1(1,g) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & c \\ -c & 0 & d-1 & 1 & 0 & c \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1-a & 0 & b & 0 & 0 & a \end{pmatrix}.$$

As before, $h = \gamma_1(1,g) \in Q(x)$ only if D_h is invertible and $D_h^{-1}C_h \in S_x$. These two conditions imply that (1) $a \neq 0$, (2) $a^{-1} - 1 \in (\bar{x})$, (3) $a^{-1}b \in (x\bar{x})$, (4) $a^{-1}c \in \mathcal{O}_\ell$, and (5) $1 - d - a^{-1}bc \in (x)$. From (2) it follows that $a, a^{-1} \in 1 + \bar{x}\mathcal{O}_\ell$, which combined with (3), (4), and (5) implies that $b \in (x\bar{x})$, $c \in \mathcal{O}_\ell$, $1 - d \in (x)$. Thus $g \in K_x$ as claimed, proving (11.4.17.c). Moreover, for such g it follows that

$$\tilde{f}_x(\gamma_1(1,g)) = \tau(a)\tau^c(x)|x\bar{x}|_\ell^{-z-3/2}.$$

From (11.4.17.a) and (11.4.17.b) and the observation that

$$S^{-1}\alpha_1(g,g_1) = \gamma_1(m(g_1,1),g_1)S^{-1}\alpha_1(m(g_1^{-1},1)g,1) \in Q_3(\mathbf{Q}_\ell)S^{-1}\alpha_1(m(g_1^{-1},1)g,1)$$

it follows that

$$f_x^{\dagger,(2)}(z, S^{-1}\alpha_1(g,g_1)) \neq 0 \Rightarrow g \in P(\mathbf{Q}_\ell)wK_B(x) = P(\mathbf{Q}_\ell)wN_B(\mathbf{Z}_\ell). \tag{11.4.17.d}$$

and

$$\begin{aligned} F^{(2)}_{\mathcal{T},x}(z,w) &= \int_{U_1(\mathbf{Q}_\ell)} \tilde{f}_x(\gamma_1(1, g\eta\mathrm{diag}(\bar{x}^{-1}, x)))\bar{\tau}(\det g)\pi_\psi(g)\phi_x dg \\ &= \tau(x/\bar{x}) \int_{K_x} \tilde{f}_x(\gamma_1(1,g))\bar{\tau}(\det g)\pi_\psi(g)\phi dg \\ &= \tau(x)|x\bar{x}|_\ell^{-z-3/2}(\int_{K_x} \tau(a_g)\bar{\tau}(\det g)\psi(d_g)dg)\phi \\ &= [U_1(\mathbf{Z}_\ell) : K_x]^{-1}\tau(x)|x\bar{x}|_\ell^{-z-3/2}\phi, \end{aligned} \tag{11.4.17.e}$$

the last line following from the one before by the definition of K_x and the hypotheses on x. A simple variation on this argument proves the assertion of the lemma for $F^{(1)}_{\mathcal{T},x}(z,\eta)$.

It follows from (11.4.17.d), (11.4.17.e), and the hypothesis that $x\bar{x} \in \mathrm{cond}_{\tilde{\pi}}(\phi)$ that to complete the proof of part (ii) it suffices to show that $F_{\mathcal{T},x}(z,wn) = F_{\mathcal{T},x}(z,w)$ for all $n \in N_B(\mathbf{Z}_\ell) \cap K_{t,t}$. But such an n is of the form

$$n = \begin{pmatrix} 1 & & & \\ a & 1 & & \\ & & 1 & -\bar{a} \\ & & & 1 \end{pmatrix}, \quad a \in (\lambda^t).$$

So

$$\alpha(n,1)\begin{pmatrix} 1 & & & & & 1/x \\ & 1 & & & 0 & \\ & & 1 & 1/\bar{x} & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \in \begin{pmatrix} 1 & & & & & 1/x \\ & 1 & & & 0 & \\ & & 1 & 1/\bar{x} & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} N_B(\mathbf{Z}_\ell),$$

and so $f_x^{\dagger,(2)}(z, g\alpha(n,1)) = f_x^{\dagger,(2)}(z,g)$, from which the desired property of $F_{\mathcal{T},z}(z,g)$ follows. ∎

Using the preceding proposition we can relate the constants $\gamma^{(2)}$ and $\gamma^{(1)}$ in Proposition 11.4.13.

Proposition 11.4.18. *For $m = 1$ or 2, let $\gamma^{(m)}(\rho, z)$ be as in Proposition 11.4.13. If $\mathcal{O}_\ell \neq \mathrm{cond}(\tau) \supseteq \mathrm{cond}(\tau\tau^c)$ then $\gamma^{(2)}(\rho,z) = \gamma^{(1)}(\rho, z-1/2)$.*

Proof. Let $\phi \in V$ be an eigenform for $\tilde{\pi}$. Let $(x) := (\lambda^t) \subset \mathcal{O}_\ell$ satisfy the conditions of part (ii) of Proposition 11.4.17 relative to ϕ. For a fixed $z \in \mathbf{C}$, let $f_m \in I_{m+1}(\tau)$ be defined by $f_m(z,g) = f_x^{\dagger,(m)}(z, gS^{-1})$. Then $F_{\phi_x}(f_m; z, g) = F^{(m)}_{\mathcal{T},x}(z,g)$ with $\mathcal{T} = (\phi,\psi,\tau)$. It then follows from Lemma 9.2.4 and Proposition 11.4.17 that

$$\begin{aligned} A(\rho, z, F_{\phi_x}(f_2; z, -))(1) &= A(\rho, z, F^{(m)}_{\mathcal{T},x}(z,-))(1) \\ &= [K_{1,\ell} : K_x]^{-1}\tau(x)|x\bar{x}|_\ell^{-z-1/2}\phi \\ &= F^{(1)}_{\mathcal{T},x}(z-1/2, \eta) \\ &= F_{\phi_x}(f_1; z-1/2, \eta). \end{aligned} \tag{11.4.18.a}$$

On the other hand, it follows easily from the definition of $f_x^{\dagger,(m)}$ and from Lemma 11.4.10 (which applies because of the hypotheses on τ) that

$$M(z, f_2)(-z, \gamma_1(m(g,1), g')) = M(z-1/2, f_1)(-z+1/2, \gamma_1'(g,g')).$$

From this it follows that

$$F_{\phi^\vee}(M(z,f_2); -z, 1) = F_{\phi^\vee}(M(z-1/2, f_1); -z+1/2, 1).$$

By Proposition 11.4.13(ii) the left-hand side equals $\gamma^{(2)}(\rho,z)A(\rho, z, F_{\phi_x}(f_2; z, -))(1)$, and by part (i) of the same proposition and (11.4.18.a) the right-hand side equals $\gamma^{(1)}(\rho, z-1/2)A(\rho,z,F_{\phi_x}(f_2;z,-))(1)$. Since $A(\rho, z, F_{\phi_x}(f_2; z,-))(1) \neq 0$ it then follows that $\gamma^{(2)}(\rho,z) = \gamma^{(1)}(\rho, z-1/2)$. ∎

11.4.19. *p-adic sections.* Suppose now that $\ell = p$. Let (π, V), ψ, τ, ξ, and ρ be as in 9.1.2 and 9.2.3 with the additional assumption that they are the p-constituents of global objects as in 9.1.4. Let $(\pi_1, \psi_1, \tau_1) := (\tilde{\pi}, \bar{\psi}, \psi^c\bar{\psi}\bar{\tau}^c)$ and $\xi_1 := \psi_1/\tau_1 = \bar{\xi}^c$ be as in 9.2.5. Let ρ_1 be the representation associated with this triple. Then $\rho_1^\vee$ is the representation associated with the triple $(\pi \otimes \bar{\psi}'\bar{\tau}', \bar{\psi}\bar{\tau}\bar{\tau}^c, \tau\bar{\psi}\psi^c)$.

Let $x \in \mathcal{O}_p$ be such that $(x) = \mathrm{cond}(\xi_1)$ (so $(\bar{x}) = \mathrm{cond}(\xi)$).

<u>The sections.</u> For $m = 1$ or 2 let $f_x^{\dagger,(m)} \in I_{m+1}(\tau_1)$. Let $f_z^{0,(m)} \in I_{m+1}(\tau)$ be defined by

$$f_z^{0,(m)}(k) := \psi(\det k)\bar{\psi}^{m+1}(\mu_{m+1}(k))M(z, f_x^{\dagger,(m)})(k).$$

A priori this is only defined for z in an open set of $\mathbf{C}$. More generally, $z \mapsto f_z^{0,(m)}$ should be viewed as a meromorphic section from $\mathbf{C}$ to $I_{m+1}(\tau)$.

Let $\tilde{f}_{u,n} \in I_n(\tau)$ be the section

$$\tilde{f}_{u,n}(k) := \psi(\det k)\bar{\psi}^n(\mu_n(k))f_{u,n}(k), \quad f_{u,n} \in I_n(\bar{\tau}_1^c),$$

where $f_{u,n}$ is defined in 11.4.9. Note that $\tilde{f}_{u,n}(k)$ depends on ψ as well as τ.

Lemma 11.4.20. *If ψ and τ are as in the Generic Case of 9.2.5 then*

$$f^{0,(m)}_{-z}(z,k) = \tilde{f}_{t,m+1}(z, k \begin{pmatrix} 1 & & & & & 1/x \\ & 1_{m-1} & & & 0_{m-1} & \\ & & 1 & 1/\bar{x} & & \\ & & & 1 & & \\ & & & & 1_{m-1} & \\ & & & & & 1 \end{pmatrix}) \in I_{m+1}(\tau),$$

where $(p^t) = \mathrm{cond}(\bar{\tau}_1^c)$.

This is immediate from the definition of $f^{0,(m)}_z$ and Lemma 11.4.10.

<u>Fourier-Jacobi coefficients.</u>

Lemma 11.4.21. *Let $\beta = (b_{i,j}) \in S_{m+1}(\mathbf{Q}_p)$, $\det\beta \neq 0$. If $\beta \in S_{m+1}(\mathbf{Z}_p)$ then*

$$f^{0,(m)}_{z,\beta}(-z,1) = \bar{\tau}'(\det\beta)|\det\beta|_p^{-2(m+1)z}\mathfrak{g}(\tau')^{m+1}c_{m+1}(\bar{\tau}',z)e_p(\mathrm{Tr}\,(b_{m+1,1}/x)),$$

where $c_{m+1}(\bar{\tau}',z)$ is as in Lemma 11.4.12.

Proof. We have

$$f^{0,(m)}_{z,\beta}(-z,1) = M(z,f^{\dagger})_{\beta}(-z, \begin{pmatrix} 1 & & & & & 1/x \\ & 1_{m-1} & & & 0_{m-1} & \\ & & 1 & 1/\bar{x} & & \\ & & & 1 & & \\ & & & & 1_{m-1} & \\ & & & & & 1 \end{pmatrix}),$$

and so the formula in the lemma follows easily from Lemma 11.4.12. ∎

Lemma 11.4.22. *Suppose ψ and τ are as in the Generic Case of 9.2.5. Let $(p^t) := \mathrm{cond}(\tau')$. Let $\beta \in S_2(\mathbf{Q}_p)$, $\det\beta \neq 0$, and suppose $\beta \in \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$. Let $y \in \mathrm{GL}_2(\mathcal{O}_p)$. Let λ be an unramified character of $\mathcal{K}_p^\times$ such that $\lambda|_{\mathbf{Q}_p^\times} = 1$. Then for $g \in U_1(\mathbf{Q}_p)$ and $h \in U_\beta(\mathbf{Q}_p)$*

$$\sum_{a\in(\mathcal{O}_p/x)^\times} \xi^c\tau(a)FJ_\beta(f^{0,(2)}_z; -z, u, g\begin{pmatrix} a^{-1} & \\ & \bar{a}\end{pmatrix}, hy)$$
$$= \xi(-1)c(\beta,\tau,z)\tau(\det hy)|\det h\bar{h}|_p^{-z+1/2}\tilde{f}_{t,1}(z,g\eta)\omega_\beta(h,g)\Phi_{\xi^c,x,y}(u),$$

where $\tilde{f}_{t,1} \in I_1(\tau/\lambda) = I_1(\tau)$ is as defined above, ω_β is defined using λ, and

$$c(\beta,\tau,z) := \bar{\tau}'(-\det\beta)|\det\beta|_p^{2z+1}\mathfrak{g}(\tau')^2\bar{\tau}'(p^{2t})p^{-4tz-5t}.$$

Proof. Note that $\tau' = \bar{\tau}_1'$, so $(p^t) = \mathrm{cond}(\bar{\tau}_1')$. Let $(p^c) := \mathrm{cond}(\bar{\tau}_1^c)$, $c = (c_1,c_2)$. As τ and ψ are in the Generic Case, $t = c_1 > c_2 > 0$.

Recall that $(x) = \mathrm{cond}(\xi_1)$. Let $K := \{k \in U_1(\mathbf{Z}_p) \ : \ c_k, a_k - 1, d_k - 1 \in (x\bar{x})\}$. Then there is a decomposition of $U_1(\mathbf{Q}_p)$ of the form

(11.4.22.a)
$$U_1(\mathbf{Q}_p) = \sqcup B_1'(\mathbf{Q}_p) x_r K, \quad B_1' := B_1 \cap U_1, \\ \text{each } x_r = \left(\begin{smallmatrix} 1 & \\ d & 1 \end{smallmatrix}\right), \mathrm{ord}_p(d) \le \mathrm{ord}_p(x\bar{x}).$$

Let
$$F(g) := \sum_{a \in (\mathcal{O}_p/x)^\times} \xi^c \tau(a) FJ_\beta(f_z^{0,(2)}; -z, u, g\eta^{-1} \left(\begin{smallmatrix} a^{-1} & \\ & \bar{a} \end{smallmatrix}\right), hy).$$

Our hypotheses ensure that

(11.4.22.b)
$$F(gk) = F(g), \quad k \in K.$$

So, in light of (11.4.22.a) and (11.3.2.a), to determine $F(g)$ it suffices to determine the $F(x_r)$'s. We claim that

(a) $F(x_r) = 0$ if $p^t \nmid d$;
(b) if $p^t | d$ then

$$F(x_r) = c(\beta, \tau, z)\tau(\det hy) |\det h\bar{h}|_p^{-z+1/2} \\ \times \sum_{a \in (\mathcal{O}_p/x)^\times} \xi^c(a) \Phi_0(y^{-1}h^{-1}u - {}^t(\frac{ad}{x}, 0)) e_p(-\mathrm{Tr}_{\mathcal{K}_p/\mathbf{Q}_p}(\frac{\bar{a}}{\bar{x}}, 0){}^t\bar{y}{}^t\bar{h}\beta u + \frac{a\bar{a}d}{x\bar{x}});$$

(c)
$$\sum_{a \in (\mathcal{O}_p/x)^\times} \xi^c(a) \Phi_0(y^{-1}h^{-1}u - {}^t(\frac{ad}{x}, 0)) e_p(-\mathrm{Tr}_{\mathcal{K}_p/\mathbf{Q}_p}(\frac{\bar{a}}{\bar{x}}, 0){}^t\bar{y}{}^t\bar{h}\beta u - \beta_y \frac{a\bar{a}d}{x\bar{x}}) \\ = \xi(-1)\omega_\beta(\eta^{-1} \left(\begin{smallmatrix} 1 & -d \\ & 1 \end{smallmatrix}\right)) \Phi_{\xi^c, x, y}(h^{-1}u),$$

where $\beta_y := ({}^t\bar{y}\beta y)_{11}$. Since
$$FJ_\beta(f; z, u, g, hy) = \tau(\det hy) |\det h\bar{h}|_p^{-z+1/2} FJ_{{}^t\bar{y}\beta y}(f; z, y^{-1}h^{-1}u, g, 1),$$
we can reduce (a) and (b) to the case $h = y = 1$. Part (c) is an easy calculation using the explicit descriptions of the Weil representation.

We begin the proof of (a) and (b) by noting that

$$w_3 \begin{pmatrix} 1_3 & \begin{smallmatrix} T & u \\ {}^t\bar{u} & d \end{smallmatrix} \\ & 1_3 \end{pmatrix} \alpha(1, \eta^{-1}) \begin{pmatrix} 1_3 & & a/x \\ & \bar{a}/\bar{x} & \\ & & 1_3 \end{pmatrix} = w_3 \begin{pmatrix} 1 & a/x & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & -\bar{a}/\bar{x} & & 1 \end{pmatrix} \begin{pmatrix} 1_3 & \begin{smallmatrix} T' & v \\ {}^t\bar{v} & d \end{smallmatrix} \\ & 1_3 \end{pmatrix} \alpha(1, \eta^{-1})$$

where
$$T' = T - T'', \ T'' = \begin{pmatrix} \bar{a}u_1/\bar{x} + a\bar{u}_1/x - a\bar{a}d/x\bar{x} & \bar{a}u_2/\bar{x} \\ a\bar{u}_2/x & 0 \end{pmatrix}, \quad v = {}^t(u_1 - ad/x, u_2).$$

As ψ_1 and τ_1 are as in the Generic Case of 9.2.5, by Lemma 11.4.20
$$f_z^{0,(2)}(-z, g) = \tilde{f}_{c,3}(-z, g \begin{pmatrix} 1_3 & & 1/x \\ & 1/\bar{x} & 0_1 \\ & & 1_3 \end{pmatrix}),$$

where $(p^c) = \mathrm{cond}(\bar\tau_1^c)$. As $\tilde f_{c,3} = \tilde f_{t,3}$ by the previously noted relation between t and c, it then follows that

$$FJ_\beta(f_z^{0,(2)}; -z, u, x_r\eta^{-1}\begin{pmatrix} a^{-1} & \\ & \bar a\end{pmatrix}, 1) = \bar\tau(a)FJ_\beta(f_t; -z, v, x_r\eta^{-1}, 1)e_p(-\mathrm{Tr}\,\beta T''),$$

where $f_t := f_{t,3} \in I_3(\bar\tau_1^c)$. To prove (a) and (b) it then suffices to show

(a′) $FJ_\beta(f_t; -z, v, x_r\eta^{-1}, 1) = 0$ if $p^t \nmid d$, and
(b′) $FJ_\beta(f_t; -z, v, x_r\eta^{-1}, 1) = c(\beta, \tau, z)\Phi_0(v)$ if $p^t|d$.

We have

$$w_3\begin{pmatrix} 1_3 & \begin{matrix} S & v \\ {}^t\bar v & d\end{matrix} \\ & 1_3\end{pmatrix}\alpha(1, \eta^{-1}) = \begin{pmatrix} & & 1_2 & \\ & -1 & & \\ -1_2 & v & -S & \\ & d & -{}^t\bar v & -1\end{pmatrix}.$$

This belongs to $Q_3(\mathbf{Q}_p)K_{Q_3}(p^t)$ if and only if S is invertible, $S^{-1} \in p^tM_2(\mathcal{O}_p)$, $S^{-1}v \in p^tM_{2\times1}(\mathcal{O}_p)$, and ${}^t\bar vS^{-1}v - d \in p^t\mathbf{Z}_p$. When these conditions hold the integrand in our formula for $FJ_\beta(f_t; z, v, \begin{pmatrix} 1 & \\ d & 1\end{pmatrix}\eta^{-1}, 1)$ equals $\tau(-\det S)|\det S|_p^{3-2z}e_p(-\mathrm{Tr}\,\beta S)$.

Since $v = \gamma^t(b, 0)$ for some $\gamma \in \mathrm{SL}_2(\mathcal{O}_p)$ and $b \in \mathcal{K}_p$, we can reduce to the case $v = {}^t(b, 0)$. Writing $b = (b_1, b_2)$ with $b_i \in \mathbf{Q}_p$ and $S = (T, {}^tT)$ with $T \in M_2(\mathbf{Q}_p)$ and $T^{-1} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4\end{pmatrix}$, the conditions on S and v can be rewritten as

(11.4.22.c) $$\det T \neq 0 \quad \text{and} \quad a_i, a_1b_1, a_3b_1, a_1b_2, a_2b_2, a_1b_1b_2 - d \in p^t\mathbf{Z}_p.$$

When these conditions hold, the integrand in the formula for $FJ_\beta(f_t; z, v, \begin{pmatrix} 1 & \\ d & 1\end{pmatrix}\eta^{-1}, 1)$ equals $\tau'(-\det T)|\det T|_p^{3-2z}e_p(-\mathrm{Tr}\,\beta T)$.

Let

$$\Gamma := \{\begin{pmatrix} h & j \\ k & l\end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p) \ : \ h, l \in \mathbf{Z}_p^\times, d(h-1) \in p^t\mathbf{Z}_p, j \in \mathbf{Z}_p, k \in p^t\mathbf{Z}_p, jb_2 \in \mathbf{Z}_p\}.$$

This is a group. If T satisfies (11.4.22.c), then so does $T\gamma$ for $\gamma \in \Gamma$. Let $\mathcal{T}$ denote the set of $T \in M_2(\mathbf{Q}_p)$ satisfying (11.4.22.c). Then $FJ_\beta(f_t; z, v, \begin{pmatrix} 1 & \\ d & 1\end{pmatrix}\eta^{-1}, 1)$ equals

(11.4.22.d) $$\sum_{T \in \mathcal{T}/\Gamma} |\det T|_p^{5-2z}\int_\Gamma \tau'(-\det T\gamma)e_p(-\mathrm{Tr}\,\beta T\gamma)d\gamma.$$

Let $T' := \beta T$ and write $T' = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4\end{pmatrix}$. Then the integral in (11.4.22.d) is zero unless $c_1, c_2, c_4 \in p^{-t}\mathbf{Z}_p$ and $c_3 \in b_2\mathbf{Z}_p + \mathbf{Z}_p$. But then

$$\beta\begin{pmatrix} b_1 \\ 0\end{pmatrix} = T'T^{-1}\begin{pmatrix} b_1 \\ 0\end{pmatrix} = \begin{pmatrix} c_1a_1b_1 + c_2a_3b_1 \\ c_3a_1b_1 + c_4a_3b_1\end{pmatrix} \in M_{2\times1}(\mathbf{Z}_p).$$

As $\beta \in \mathrm{GL}_2(\mathcal{O}_p)$ by hypothesis, this implies that (11.4.22.d) is zero unless $b_1 \in \mathbf{Z}_p$. Letting

$$\Gamma' := \{\gamma = \begin{pmatrix} h & j \\ k & l\end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p) \ : \ h, l \in \mathbf{Z}_p^\times, d(h-1) \in p^t\mathbf{Z}_p, k \in \mathbf{Z}_p, j \in p^t\mathbf{Z}_p, kb_1 \in \mathbf{Z}_p\},$$

and noting that if T satisfies (11.4.22.c) then so does γT for any $\gamma \in \Gamma'$, and considering $T'' := T\beta$, we similarly find that $FJ_\beta(f_t; z, v, \begin{pmatrix} 1 & \\ d & 1\end{pmatrix}\eta^{-1}, 1)$ is zero unless

$$(b_2, 0)\beta = (b_2, 0)T^{-1}T'' \in M_{1\times2}(\mathbf{Z}_p),$$

which implies that $b_2 \in \mathbf{Z}_p$. Finally, for $b \in \mathcal{O}_p$ the conditions $S^{-1} \in p^t M_2(\mathcal{O}_p)$ and ${}^t\bar{v}S^{-1}\bar{v} - d \in p^t\mathbf{Z}_p$ imply that $d \in p^t\mathbf{Z}_p$; the claim (a′) follows.

Note that when $v = {}^t(v_1, v_2)$ with $v_1, v_2 \in \mathcal{O}_p$ and $p^t|d$ we have

$$\begin{aligned} FJ_\beta(f_t; z, v, \left(\begin{smallmatrix} 1 & \\ d & 1 \end{smallmatrix}\right) \eta^{-1}, 1) &= \tau(-1) f_{t,2,\beta}(z + 1/2, 1) \\ &= \tau(-1) M(-z - 1/2, f_2^\dagger)_\beta(z + 1/2, 1), \end{aligned}$$

with $f_2^\dagger \in I_2(\tau_1)$. By Lemma 11.4.12, the last displayed line equals

$$\bar{\tau}'(-\det\beta)|\det\beta|_p^{2z+1}\mathfrak{g}(\tau')^2\bar{\tau}'(p^{2t})p^{-4tz-5t},$$

which equals $c(\beta, \tau, z)$ by definition. This proves (b′).

Returning to the proof of the lemma we note that (a) implies that $F(g)$ is supported on $B_1'(\mathbf{Q}_p)K'$ with $K' = \{k \in U_1(\mathbf{Z}_p) : c_k, a_k - 1, d_k - 1 \in p^t\}$. Then (b) and (c) together with (11.3.2.c) imply that for $g = b\left(\begin{smallmatrix} 1 & \\ d & 1 \end{smallmatrix}\right) k \in B_1'(\mathbf{Q}_p)\left(\begin{smallmatrix} 1 & \\ d & 1 \end{smallmatrix}\right) K \subset B_1'(\mathbf{Q}_p)K'$,

$$\begin{aligned} F(g) &= \xi(-1)c(\beta,\tau,z)\tau(\det h)|\det h\bar{h}|_p^{-z+1/2}\tilde{f}_{t,1}(g)\omega_\beta(b\eta^{-1}\left(\begin{smallmatrix} 1 & -d \\ & 1 \end{smallmatrix}\right))\Phi_{\xi^c,x,y}(h^{-1}u) \\ &= \xi(-1)c(\beta,\tau,z)\tau(\det h)|\det h\bar{h}|_p^{-z+1/2}\tilde{f}_{t,1}(g)\omega_\beta(h, g\eta^{-1})\Phi_{\xi^c,x,y}(u), \end{aligned}$$

the second equality following from the first since $\omega_\beta(\eta K\eta^{-1})\Phi_{\xi^c,x,y} = \Phi_{\xi^c,x,y}$ by part (i) of Lemma 10.2.5. Then

$$\begin{aligned} \sum_{a\in(\mathcal{O}_p/x)^\times} & \xi^c\tau(a)FJ_\beta(f_z^{0,(2)}; -z, u, g\left(\begin{smallmatrix} a^{-1} & \\ & \bar{a} \end{smallmatrix}\right), hy) \\ &= F(g\eta) \\ &= \xi(-1)c(\beta,\tau,z)\tau(\det h)|\det h\bar{h}|_p^{-z+1/2}\tilde{f}_{t,1}(g\eta)\omega_\beta(h,g)\Phi_{\xi^c,x,y}(u). \end{aligned}$$

∎

Pull-back integrals. As previously, we will write S for S' when $m = 1$ to unify some of the formulas.

Proposition 11.4.23. *Let $\phi \in V$ be an eigenvector for π such that $p|\mathrm{cond}_\pi(\phi)$. Let $(x) := \mathrm{cond}(\xi_1) = (p^t)$. Suppose $t > 0$ and that x is contained in* $\mathrm{cond}(\tau_1)$ *and* $\mathrm{cond}(\psi_1)$ *and that $x\bar{x} \in \mathrm{cond}_\pi(\phi)$. Let $\phi_x^\vee := \psi_p(-1)\pi(\mathrm{diag}(x, \bar{x}^{-1}))\phi$. Let $\tilde{f}_z^{0,(m)}(g) := f_z^{0,(m)}(gS^{-1})$. Then*

$$F_{\phi_x^\vee}(\tilde{f}_{-z}^{0,(m)}; z, g) = \gamma^{(m)}(\rho_1, -z)[U_1(\mathbf{Z}_p) : K_x]^{-1}\bar{\tau}^c(x)|x\bar{x}|_p^{z-(m+1)/2}\begin{cases} F_{\phi,z}^0(g) & m = 2 \\ \pi_\psi(g)\phi & m = 1, \end{cases}$$

where K_x is as in (11.4.17.c) *and $\gamma^{(m)}(\rho_1, -z)$ is as in Proposition 11.4.13.*

Proof. Let $\tilde{f}_x^{\dagger,(m)}(g) := f_x^{\dagger,(m)}(gS^{-1})$. We have

(11.4.23.a) $$F_{\phi_x^\vee}(\tilde{f}_{-z}^{0,(m)}; z, g) = \psi(\det g)\tau(\mu_m(g))\bar{\psi}(\mu_m(g))^{m-1}F_{\phi_x^\vee}(M(-z, \tilde{f}_x^{\dagger,(m)}); z, g),$$

where the left-hand side is defined with respect to (π,ψ,τ) and the right-hand with respect to $(\pi\otimes\bar\psi'\bar\tau',\bar\psi\bar\tau\bar\tau^c,\tau\bar\psi\psi^c)$. Suppose $m=1$. By part (i) of Proposition 11.4.13 the right-hand side of (11.4.23.a) equals

$$\gamma^{(1)}(\rho_1,-z)\psi(x/\bar x)\psi(\det g)\tau(\mu_1(g))\tau_1(\mu_1(g))F_{\phi_x}(\tilde f_x^{\dagger,(1)};-z,\eta g),$$
$$\phi_x=\tilde\pi(\eta\mathrm{diag}(\bar x^{-1},x))\phi,$$

which in turn equals

$$\gamma^{(1)}(\rho_1,-z)\psi(x/\bar x)\pi_\psi(g)F_{\phi_x}(\tilde f_x^{\dagger,(1)};-z,\eta).$$

By Proposition 11.4.17 this equals

$$\gamma^{(1)}(\rho_1,-z)[U_1(\mathbf{Z}_p):K_x]^{-1}\psi(x/\bar x)\tau_1(x)|x\bar x|_\ell^{z-1}\pi_\psi(g)\phi.$$

Since $\tau_1=\bar\tau^c\psi^c\bar\psi$, the proposition is true in the case $m=1$.

Suppose then that $m=2$. By part (ii) of Proposition 11.4.13 the right-hand side of (11.4.23.a) equals

$$\gamma^{(2)}(\rho_1,-z))\psi(x/\bar x)\psi(\det g)\bar\xi(\mu(g))A(\rho_1,-z,F_{\phi_x}(\tilde f_x^{\dagger,(2)};-z,-))_z(g). \tag{11.4.23.b}$$

Since

$$F_{\phi_x}(\tilde f_x^{\dagger,(2)})=\tau_1(x)|x\bar x|^{z-3/2}[U_1(\mathbf{Z}_p):K_x]^{-1}F_{\phi,t,t}\in I(\rho_1)$$

by Proposition 11.4.17, it then follows from the definition of $F^0_{\phi,z}$ that (11.4.23.b) equals

$$\gamma^{(2)}(\rho_1,-z)\bar\tau^c(x)|x\bar x|_p^{z-3/2}[U_1(\mathbf{Z}_p):K_x]^{-1}F^0_{\phi,z}.$$

■

11.5. **Good Siegel Eisenstein series.** Let $(\pi,V)=(\otimes\pi_v,\otimes V_v)$ be as in 9.3.1 and let $\mathcal{D}=(\Sigma,\varphi,\psi,\tau)$ be an Eisenstein datum for π (also as in 9.3.1). We augment this datum with a choice of an integer $M_{\mathcal{D}}$ satisfying

(11.5.0.c)
- $M_{\mathcal{D}}$ is divisible only by primes in $\Sigma\backslash\{p\}$;
- for $\ell\in\Sigma\backslash\{p\}$, $M_{\mathcal{D}}$ is contained in $\delta_{\mathcal{K}}\mathrm{cond}(\xi_\ell)$, $\mathrm{cond}(\psi_\ell)$, $\mathrm{cond}(\tau_\ell)$,and $\mathrm{cond}_{\tilde\pi_\ell}(\phi_\ell)$.

All constructions to follow and subsequent formulas depend on this choice. Of course, in our applications we are free to choose a suitable $M_{\mathcal{D}}$.

Let $x_p:=p^{t_p}\in\mathcal{O}_p$ be such that $(x_p)=\mathrm{cond}(\xi_p^c)$. Let

$$U_{\mathcal{D}}:=K_{x_p,p}\prod_{\ell\in\Sigma\backslash\{p\}}K_{M_{\mathcal{D}},\ell}\prod_{\ell\notin\Sigma}U_1(\mathbf{Z}_\ell),$$

with $K_{x,\ell}$ as in (11.4.17.c).

For $m=1$ or 2 we define a meromorphic section $f_{\mathcal{D}}^{(m)}:\mathbf{C}\to I_{m+1}(\tau)$ as follows: $f_{\mathcal{D}}^{(m)}(z)=\otimes f_{\mathcal{D},w}^{(m)}(z)$ where

- $f_\infty^{(m)}(z):=f_\kappa\in I_{m+1}(\tau_\infty)$;

- if $\ell \notin \Sigma$ then $f^{(m)}_{\mathcal{D},\ell}(z) := f^{sph}_\ell \in I_{m+1}(\tau_\ell)$;
- if $\ell \in \Sigma$, $\ell \neq p$, then $f^{(m)}_{\mathcal{D},\ell}(z) := f^{\dagger,(m)}_{M_\mathcal{D}} \in I_{m+1}(\tau_\ell)$;
- $f^{(m)}_{\mathcal{D},p}(z) := f^{0,(m)}_{-z} \in I_{m+1}(\tau_p)$, where x_p is used to define $f^{0,(m)}_{-z}$.

We let $H^{(m)}_\mathcal{D}(z,g) := E(f^{(m)}_\mathcal{D}; z, g)$.

Let

$$K^{(m)}_\mathcal{D} := \{k \in G_{m+1}(\widehat{\mathbf{Z}}) \ : \ 1 - k \in M^2_\mathcal{D}|x_p\bar{x}_p|^{-1}M_{2(m+1)}(\mathcal{O}_\ell)\}$$

Then it easily follows from the definition of the $f^{(m)}_{\mathcal{D},\ell}(z)$'s that

$$H^{(m)}_\mathcal{D}(z, gk) = H^{(m)}_\mathcal{D}(z,g), \quad k \in K^{(m)}_\mathcal{D}, \tag{11.5.0.d}$$

and that if $t_p > 0$, $x_p \in \mathrm{cond}(\psi)$, and $x_p\bar{x}_p \in \mathrm{cond}_\pi(\phi_p)$ then

$$H^{(m)}_\mathcal{D}(z, g\alpha(1,k)) = \tau(a_{k_p})H^{(m)}_\mathcal{D}(z,g), \quad k \in U_\mathcal{D}. \tag{11.5.0.e}$$

For $u \in \mathrm{GL}_{m+1}(\mathbf{A}_{\mathcal{K},f})$ let

$$L^{(m)}_u := \{\beta \in S_{m+1}(\mathbf{Q}) \ : \ \beta \geq 0, \ \mathrm{Tr}\,\beta\gamma \in \widehat{\mathbf{Z}}, \ \gamma \in uS_{m+1}(\widehat{\mathbf{Z}})^t\bar{u}\}.$$

Lemma 11.5.1.

(i) *If $\kappa \geq m+1$ then $H^{(m)}_\mathcal{D}$ is holomorphic at $z_\kappa := (\kappa - m - 1)/2$.*
(ii) *If $\kappa \geq m+1$ and if $g \in Q_{m+1}(\mathbf{A})$ then*

$$H^{(m)}_\mathcal{D}(z_\kappa, g) = \sum_{\beta \in S_{m+1}(\mathbf{Q}), \beta > 0} H^{(m)}_{\mathcal{D},\beta}(z_\kappa, g).$$

Furthermore, if $\beta > 0$, $g_\infty = r(X)\mathrm{diag}(Y, {}^t\bar{Y}^{-1})$ and $g_f = r(a)\mathrm{diag}(u, {}^t\bar{u}^{-1}) \in G_{m+1}(\mathbf{A}_f)$, then $H^{(m)}_{\mathcal{D},\beta}(z_\kappa, g) = 0$ if $\beta \notin L^{(m)}_u$ and otherwise

$$\begin{aligned} H^{(m)}_{\mathcal{D},\beta}(z_\kappa, g) = & e_\mathbf{A}(\mathrm{Tr}\,\beta a)\frac{(-2)^{-m-1}(2\pi i)^{(m+1)\kappa}(2/\pi)^{m(m+1)/2}\det\beta^{\kappa-m-1}\det\bar{Y}^\kappa}{\prod_{j=0}^m(\kappa-j-1)!\prod_{j=0}^m L^S(\kappa-j, \bar{\tau}'\chi^j_\mathcal{K})} \\ & \times e(\mathrm{Tr}\,(\beta(X+iY^t\bar{Y}))\prod_{\ell\in S} f_{\mathcal{D},\beta_u,\ell}(z_\kappa, 1) \\ & \times \tau(\det u)|\det u\bar{u}|^{m+1-\kappa/2}_\mathbf{Q}\prod_{\ell\notin S} h_{\ell,\beta}(\bar{\tau}'_\ell(\ell)\ell^{-\kappa}), \end{aligned}$$

$$\beta_u = {}^t\bar{u}\beta u,$$

for any finite set of places $S \supseteq \Sigma$ such that $g_\ell \in K_{m+1,\ell}$ if $\ell \notin S$.

Proof. It follows from Lemma 11.3.1 that if $\mathrm{Re}(z) > (m+1)/2$ and $q \in Q_{m+1}(\mathbf{A})$ then $H^{(m)}_{\mathcal{D},\beta}(z,q) = \prod f^{(m)}_{\mathcal{D},w}(z, q_w)$. Let $S \supseteq \Sigma$ be any finite set of places containing ∞ and all primes ℓ that ramify in $\mathcal{K}$ or are such that $q_\ell \notin K_{m+1,\ell}$. From the definition of the $f_{\mathcal{D},w}^{(m)}$'s and Lemma 11.4.6 it follows that if $r = \mathrm{rank}\beta$ then

$$(11.5.1.a)\qquad \begin{aligned} H_{\mathcal{D},\beta}^{(m)}(z,q) = &\frac{\prod_{i=r}^m L^S(2z+i-m,\bar\tau'\chi_{\mathcal{K}}^i)}{\prod_{i=0}^m L^S(2z+m+1-i,\bar\tau'\chi_{\mathcal{K}}^i)} \prod_{w\in S} f_{\mathcal{D},w,\beta}^{(m)}(z,q_w) \\ &\times \prod_{\ell\notin S} h_{\ell,\beta}(\bar\tau'_\ell(\ell)\ell^{-2z-m-1}). \end{aligned}$$

Assuming $\kappa \geq m+1$, to establish the holomorphy of $H_{\mathcal{D}}^{(m)}(z,q)$ at $z = z_\kappa$ it suffices to prove that (11.5.1.a) is holomorphic at z_κ (by the usual argument; cf. [Sh97, Prop. 19.1]). Since any $g \in G_{m+1}(\mathbf{A})$ can be written as $g = \gamma q k$ with $\gamma \in G_{m+1}(\mathbf{Q})$, $q \in Q_{m+1}(\mathbf{A})$, and $k \in K_{m+1,\infty}^+ K_{\mathcal{D}}^{(m)}$, it would then follow that $H_{\mathcal{D}}^{(m)}(z,g)$ is holomorphic at z_κ.

The holomorphy at z_κ of the ratio of L-functions is clear, and that of $f_{\mathcal{D},\infty,\beta}(z,q_\infty)$ follows from Lemma 11.4.2. If $\ell \notin \Sigma$ then the desired holomorphy of $f_{\mathcal{D},\ell,\beta}(z,q_\ell)$ follows directly from (11.1.1.b), while if $\ell \in \Sigma$, $\ell \neq p$, then the holomorphy follows from Lemma 11.4.11. Finally, the holomorphy of $f_{\mathcal{D},p,\beta}(z,q_p)$ at z_κ follows from Lemma 11.4.21.

Suppose g is as in part (ii) of the lemma. By Lemma 11.4.2, $f_{\mathcal{D},\infty,\beta}^{(m)}(z_\kappa,g_\infty) = f_{\kappa,\beta}(z_\kappa,g_\infty) = 0$ unless $\beta > 0$, and so $H_{\mathcal{D},\beta}^{(m)}(z_\kappa,g) = 0$ by (11.5.1.a) unless $\beta > 0$. If $\beta > 0$ then the value asserted for $H_{\mathcal{D},\beta}^{(m)}(z_\kappa,g)$ is a consequence of (11.5.1.a) and Lemmas 11.4.2 and 11.4.6. ∎

As a consequence of part (i) of the preceding lemma, if $\kappa \geq m+1$ we can define a function $H_{\mathcal{D}}^{(m)}(Z,x)$ on $\mathbf{H}_{m+1}\times G_{m+1}(\mathbf{A}_f)$ by

$$H_{\mathcal{D}}^{(m)}(Z,x) := \mu_{m+1}(g_\infty)^{(m+1)\kappa/2} J_{m+1}(g_\infty,\mathbf{i})^{-\kappa} H_{\mathcal{D}}^{(m)}((\kappa-m-1)/2, g_\infty x),$$
$$g_\infty \in G_{m+1}^+(\mathbf{R}), g_\infty(\mathbf{i}) = Z.$$

Lemma 11.5.2. *Suppose $\kappa \geq m+1$. Then $H_{\mathcal{D}}^{(m)}(Z,x) \in M_\kappa(K_{\mathcal{D}}^{(m)})$.*

Proof. Since $G_{m+1}(\mathbf{A}) = \sqcup G_{m+1}(\mathbf{Q}) G_{m+1}^+(\mathbf{R}) q_i K_{m+1,\infty}^+ K_{\mathcal{D}}^{(m)}$ for some $q_i \in Q_{m+1}(\mathbf{A}_f)$, by the definition of $f_{\mathcal{D},\infty}^{(m)}(z,g)$ and (11.5.0.d) it suffices to show that each $H_{\mathcal{D}}^{(m)}(Z,q_i)$ is holomorphic as a function of Z. But if $Z = X + iY{}^t\bar Y$ and $q_i = r(a_i)\mathrm{diag}(u_i, {}^t\bar u_i^{-1})$, then it follows from part (ii) of Lemma 11.5.1 upon taking $g = r(X)\mathrm{diag}(Y, {}^t\bar Y^{-1})q_i$ that $H_{\mathcal{D}}^{(m)}(Z,q_i) = \sum_{\beta\in L_{u_i}^{(m)},\det\beta>0} c_i(\beta) e(\mathrm{Tr}\,(\beta Z))$ for some $c_i(\beta) \in \mathbf{C}$, and this series is visibly holomorphic in Z. ∎

Supposing $\kappa \geq m+1$, for $\beta \in S_{m+1}(\mathbf{Q})$, $\beta \geq 0$, we will denote by $A_{\mathcal{D},\beta}^{(m)}(x)$ the β-Fourier coefficient of $H_{\mathcal{D}}^{(m)}(Z,x)$.

Lemma 11.5.3. *Suppose $\kappa \geq m+1$ and that $x = \mathrm{diag}(u, \bar{u}^{-1})$, $u \in \mathrm{GL}_{m+1}(\mathbf{A}_{\mathcal{K},f})$ with $u_\ell = \mathrm{diag}(1_m, \bar{a}_\ell)$, $a_\ell \in \mathcal{O}_\ell^\times$, if $\ell \in \Sigma$. If $\beta \notin L_u^{(m)}$ or if $\det\beta = 0$ then $A_{\mathcal{D},\beta}(x) = 0$, and for $\beta = (\beta_{ij}) \in L_u^{(m)}$ with $\det\beta > 0$*

$$
\begin{aligned}
A^{(m)}_{\mathcal{D},\beta}(x) = D_{\mathcal{K}}^{-m(m+1)/4} & \frac{(-2)^{-m-1}(2\pi i)^{(m+1)\kappa}(2/\pi)^{m(m+1)/2}(\det\beta|\det\beta|_p)^{\kappa-m-1}}{\prod_{j=0}^m(\kappa-j-1)!\prod_{j=0}^m L^\Sigma(\kappa-j,\bar{\tau}'\chi_{\mathcal{K}}^j)} \\
& \times \bar{\tau}_p(a_p\det(\beta))\mathfrak{g}(\tau_p')^{m+1}c(\bar{\tau}_p',(m+1-\kappa)/2)e_p(\mathrm{Tr}_{\mathcal{K}_p/\mathbf{Q}_p}(a_p\beta_{m+1,1}/x_p)) \\
& \times \prod_{\ell\in\Sigma,\ell\neq p}\tau_\ell^c(a_\ell)e_\ell(\mathrm{Tr}_{\mathcal{K}_\ell/\mathbf{Q}_\ell}(a_\ell\beta_{m+1,1}/M_{\mathcal{D}})) \\
& \times \prod_{\ell\notin\Sigma}\tau_\ell(\det u_\ell)|u_\ell\bar{u}_\ell|_\ell^{m+1-\kappa/2}h_{\ell,{}^t\bar{u}_\ell\beta u_\ell}(\bar{\tau}_\ell(\ell)\ell^{-\kappa}).
\end{aligned}
$$

This is a straightforward consequence of part (ii) of Lemma 11.5.1, part (i) of Proposition 11.4.17, and Lemma 11.4.21.

11.6. $E_{\mathcal{D}}$ **via pull-back.** We continue with the notation of the preceding section and of 9.3.1. We extend φ to a cuspform φ_ψ on $G_1(\mathbf{A})$ by setting $\varphi_\psi((a,g)) := \psi(a)\varphi(g)$. Let φ_0 be the cuspform defined by

$$
\varphi_0(g) := \psi_p(-1)\varphi_\psi(gy), \quad y_v := \begin{cases} 1 & v=\infty, v\notin\Sigma \\ \eta\mathrm{diag}(M_{\mathcal{D}}^{-1}, M_{\mathcal{D}}) & v=\ell\in\Sigma, v\neq p \\ \mathrm{diag}(x_p,\bar{x}_p^{-1}) & v=p. \end{cases}
$$

Proposition 11.6.1. *Let $m=1$ or 2. Suppose that $(x_p) = (p^{t_p})$ with $t_p > 0$ and that $x_p \in \mathrm{cond}(\psi)$ and $x_p\bar{x}_p \in \mathrm{cond}_{\pi_p}(\phi_p)$ (where ϕ_p is defined by $\varphi = \otimes\phi_v$). Let $g \in G_m(\mathbf{A})$ and $h \in G_1(\mathbf{A})$ be such that $\mu_1(h) = \mu_m(g)$. If $\kappa \geq m+1$ then*

$$
\begin{aligned}
&\int_{U_1(\mathbf{Q})\backslash U_1(\mathbf{A})} H^{(m)}_{\mathcal{D}}(z,\alpha(g,g'h))\bar{\tau}(\det g'h)\varphi_0(g'h)dg' \\
&\qquad = [U_1(\widehat{\mathbf{Z}}):U_{\mathcal{D}}]^{-1}\begin{cases} c^{(1)}_{\mathcal{D}}(z)\varphi(g) & m=1 \\ c^{(2)}_{\mathcal{D}}(z)E_{\mathcal{D}}(z,g) & m=2, \end{cases}
\end{aligned}
\tag{11.6.1.a}
$$

where

$$
\begin{aligned}
c^{(m)}_{\mathcal{D}}(z) :=& \pi 2^{-2z-m+1}M_{\mathcal{D}}^{2z+(m+1)}|x_p\bar{x}_p|_p^{z-(m+1)/2}\bar{\tau}_p^c(M_{\mathcal{D}}x_p) \\
&\times \frac{\Gamma(z+(m-1+\kappa)/2)L^\Sigma(\tilde{\pi},\xi,z+m/2)}{\Gamma(z+(m+1+\kappa)/2)\prod_{i=0}^1 L^\Sigma(\bar{\tau}'\epsilon_{\mathcal{K}}^i,2z+m+1-i)}\gamma^{(m)}(\rho_{1,p},-z)
\end{aligned}
$$

with $\gamma^{(m)}(\rho_{1,p},-z)$ as in Proposition 11.4.23 and $\rho_{1,p}$ as in 11.4.9.

Of course, (11.6.1.a) is to be viewed as an equality of meromorphic functions in z; note that this shows that $c^{(2)}_{\mathcal{D}}(z)E_{\mathcal{D}}(z,g)$ is defined at $z = z_\kappa := (\kappa-3)/2$ even though $E_{\mathcal{D}}(z,g)$ may not be.

Proof. By uniqueness of meromorphic continuation it suffices to prove the proposition for $\mathrm{Re}(z)$ sufficiently large, so we may assume $\mathrm{Re}(z) > 2(m+1)$. Then

$$H_{\mathcal{D}}^{(m)}(z,\alpha(g,g'h)) = E(f_{\mathcal{D}}^{(m)};z,\alpha(g,g'h)) = E(\tilde{f}_{\mathcal{D}}^{(m)};z,\gamma(g,g'h)),$$
$$\tilde{f}_{\mathcal{D}}^{(m)}(z,g) = f_{\mathcal{D}}^{(m)}(z,gS^{-1}),$$

where we follow an earlier convention and write S for S' when $m=1$. It then follows from Proposition 11.2.3 that the left side of (11.6.1.a) equals

$$\begin{cases} F_{\varphi_0}(\tilde{f}_{\mathcal{D}}^{(1)};z,g) & m=1 \\ E(F_{\varphi_0}(\tilde{f}_{\mathcal{D}}^{(2)};z,-),z,g) & m=2. \end{cases}$$

To prove the proposition it therefore suffices to prove that $[U_1(\widehat{\mathbf{Z}}):U_{\mathcal{D}}]F_{\varphi_0}(\tilde{f}_{\mathcal{D}}^{(m)};z,-)$ equals $c_{\mathcal{D}}^{(1)}(z)\varphi$ if $m=1$ and $c_{\mathcal{D}}^{(2)}(z)\varphi_{\mathcal{D}}(z)$ if $m=2$, where $\varphi_{\mathcal{D}}:\mathbf{C}\to I(\rho)$ is as in 9.3.1.

Recall that φ corresponds to some $\phi = \otimes\phi_v \in \otimes V_v$ and so φ_0 corresponds to $\phi_0 = \psi_p(-1)\otimes\phi_{0,v}$, $\phi_{0,v} = \pi_v(y_v)\phi_v$. Then via the identification $V = \otimes V_v$, the automorphic form

$$g_1 \mapsto \begin{cases} F_{\varphi_0}(\tilde{f}_{\mathcal{D}}^{(1)};z,g_1g) & m=1 \\ F_{\varphi_0}(\tilde{f}_{\mathcal{D}}^{(2)};z,m(g_1,1)g) & m=2 \end{cases}, \quad g_1\in\mathrm{GL}_2(\mathbf{A}),$$

is identified with

$$\psi_p(-1)\int_{U_1(\mathbf{A})}\tilde{f}_{\mathcal{D}}^{(m)}(z,\gamma(g,g'h))\bar{\tau}(\det g'h)\pi_{\psi}(g'h)\phi_0\,dg',$$

which factors as

$$\psi_p(-1)\prod_v\int_{U_1(\mathbf{Q}_v)}f_{\mathcal{D},v}^{(m)}(z,S^{-1}\alpha(g_v,g_v'h_v))\bar{\tau}_v(\det g_v'h_v)\pi_{v,\psi_v}(g_v'h_v)\phi_{0,v}\,dg_v'.$$

The proposition then follows from the definitions of the $f_{\mathcal{D},v}^{(m)}$'s, Lemmas 11.4.4 and 11.4.8, and Propositions 11.4.17 and 11.4.23; the hypotheses on x_p ensure that we can appeal to Proposition 11.4.23 when $v=p$. ■

Let $\beta\in S_m(\mathbf{Q})$. If $m=1$ let

$$\phi_\beta(g) := \int_{S_1(\mathbf{Q})\backslash S_1(\mathbf{A})}\varphi(\left(\begin{smallmatrix}1 & S\\ & 1\end{smallmatrix}\right)g)\,dS$$

be the β-Fourier coefficient of φ. If $m=2$, recall that $\mu_{\mathcal{D}}(\beta,g)$ is the β-Fourier coefficient of $E_{\mathcal{D}}(z,g)$ (see 9.3.4). Let $H_{\mathcal{D},\beta}^{(m)}(z,g)$ be the β-Fourier-Jacobi coefficient of $H_{\mathcal{D}}^{(m)}(z,g)$ (see 11.3). The following is an immediate consequence of the preceding proposition.

Proposition 11.6.2. *Let $m=1$ or 2. Suppose that $(x_p)=(p^{t_p})$ with $t_p>0$ and that $x_p\in\mathrm{cond}(\psi)$ and $x_p\bar{x}_p\in\mathrm{cond}_{\pi_p}(\phi_p)$ (where ϕ_p is defined by $\varphi=\otimes\phi_v$). Let $g\in G_m(\mathbf{A})$*

and $h \in G_1(\mathbf{A})$ be such that $\mu_1(h) = \mu_m(g)$. Let $\beta \in S_m(\mathbf{Q})$. If $\kappa \geq m+1$ then

(11.6.2.a)

$$\int_{U_1(\mathbf{Q})\backslash U_1(\mathbf{A})} H^{(m)}_{\mathcal{D},\beta}(z, \alpha(g, g'h))\bar{\tau}(\det g'h)\varphi_0(g'h)dg'$$

$$= [U_1(\widehat{\mathbf{Z}}) : U_{\mathcal{D}}]^{-1} \begin{cases} c^{(1)}_{\mathcal{D}}(z)\varphi_\beta(g) & m = 1 \\ c^{(2)}_{\mathcal{D}}(z)\mu_{\mathcal{D}}(\beta, z, g) & m = 2, \end{cases}$$

where $c^{(m)}_{\mathcal{D}}(z)$ is as in Proposition 11.6.1.

Still assuming that x_p satisfies the hypotheses in Proposition 11.6.2, the integral in (11.6.2.a) can be transformed into an expression that is more easily given a classical interpretation.

Let $h_{\mathcal{K}}$ be the class number of $\mathcal{K}$. Let $a_1, ..., a_{h_{\mathcal{K}}} \in \widehat{\mathcal{O}}$ be representatives for the class group of $\mathcal{K}$. We can and do assume that each $a_i = (q_i, 1) \in \mathcal{O}_{q_i}$ for some prime $q_i \notin \Sigma$ that splits in $\mathcal{K}$. Let

$$\Gamma_{\mathcal{D}} := U_1(\mathbf{Q}) \cap U_{\mathcal{D}}, \quad \Gamma_{\mathcal{D},i} := U_1(\mathbf{Q}) \cap \begin{pmatrix} a_i^{-1} & \\ & \bar{a}_i \end{pmatrix} U_{\mathcal{D}} \begin{pmatrix} a_i & \\ & \bar{a}_i^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ & q_i \end{pmatrix} \Gamma_{\mathcal{D}} \begin{pmatrix} 1 & \\ & q_i^{-1} \end{pmatrix}.$$

If $\gamma \in \Gamma_{\mathcal{D}}$ then $\det\gamma \equiv 1$ modulo some odd rational prime (since p is odd). Since $\det\gamma \in \mathcal{O}^\times$ it follows that $\det\gamma = 1$, and since $\gamma = (a, m)$ with $a\bar{a}\det m = \mu_1(g) = 1 = \det(\gamma) = a^2 \det m$, it must be that $a = \bar{a}$. So without loss of generality, $a = 1$ and $\det m = \det\gamma = 1$. In particular

$$\Gamma_{\mathcal{D}} = \eta(\Gamma_1(p^{u_p} M_{\mathcal{D}}) \cap \Gamma_0(p^{r_p} M^2_{\mathcal{D}}))\eta^{-1},$$

$$(p^{u_p}) := (x_p) \cap \mathbf{Z}_p, \quad (p^{r_p}) := (x_p\bar{x}_p).$$

Let $\mathcal{Y} \subset \widehat{\mathcal{O}}$ be any set of representatives for $(\widehat{\mathcal{O}}/\bar{x}_p M_{\mathcal{D}})^\times/(\widehat{\mathbf{Z}}/p^{u_p} M_{\mathcal{D}})^\times$. Then

$$U_1(\mathbf{A}) = \sqcup_{i=1}^{h_{\mathcal{K}}} \sqcup_{a\in\mathcal{Y}} U_1(\mathbf{Q})U_1(\mathbf{R}) \begin{pmatrix} a_i^{-1}a^{-1} & \\ & \bar{a}_i\bar{a} \end{pmatrix} U_{\mathcal{D}}.$$

Letting

$$\tilde{H}^{(m)}_{\mathcal{D},\beta}(z, g) := \sum_{a\in(\widehat{\mathcal{O}}/(x_p M_{\mathcal{D}}))^\times} \chi_{\pi,p}\bar{\chi}_\pi\xi^c\tau(a) H^{(m)}_{\mathcal{D},\beta}(z, g\alpha(1, \mathrm{diag}(a^{-1}, \bar{a}))),$$

(this is independent of the choice of the a's) we then have as a consequence of (11.5.0.e) and the fact that $\varphi_0(gk) = \varphi_0(g)$ for $k \in U_{\mathcal{D}}$ that if $g \in U_m(\mathbf{A})$, then the left-hand side of (11.6.2.a) equals

$$(\#(\mathbf{Z}/p^{u_p} M_{\mathcal{D}})^\times)^{-1}[U_1(\widehat{\mathbf{Z}}) : U_{\mathcal{D}}]^{-1}$$

$$\times \sum_{i=1}^{h_{\mathcal{K}}} \bar{\tau}^c\tau(a_i) \int_{\Gamma_{\mathcal{D},i}\backslash U_1(\mathbf{R})} \tilde{H}^{(m)}_{\mathcal{D},\beta}(z, \alpha(g, g'\mathrm{diag}(a_i^{-1}, \bar{a}_i)))$$

$$\times \bar{\tau}(\det g')\varphi_0(g'\mathrm{diag}(a_i^{-1}, \bar{a}_i))dg'.$$

Since

(11.6.2.b) $$\tilde{H}^{(m)}_{\mathcal{D},\beta}(z, g\alpha(1, \mathrm{diag}(a^{-1}, a))) = \bar{\chi}^2_{\pi,p}\chi_\pi(a)\tilde{H}^{(m)}_{\mathcal{D},\beta}(z, g), \quad a \in \widehat{\mathbf{Z}}^\times,$$

and

$$\varphi_0(g\mathrm{diag}(a^{-1}, a)) = \chi^2_{\pi,p}\bar{\chi}_\pi(a)\varphi_0(g), \quad a \in \widehat{\mathbf{Z}}^\times,$$

we can rewrite the integral as

(11.6.2.c)

$$[U_1(\widehat{\mathbf{Z}}) : U_{\mathcal{D}}]^{-1} \times \sum_{i=1}^{h_{\mathcal{K}}} \bar{\tau}^c\tau(a_i) \int_{\Gamma'_{\mathcal{D},i}\backslash U_1(\mathbf{R})/K^+_{1,\infty}} \tilde{H}^{(m)}_{\mathcal{D},\beta}(z, \alpha(g, g'\mathrm{diag}(a_i^{-1}, \bar{a}_i))) \\ \times \bar{\tau}(\det g')\varphi_0(g'\mathrm{diag}(a_i^{-1}, \bar{a}_i))dg',$$

where $\Gamma'_{\mathcal{D},i}$ is defined by replacing $\Gamma_{\mathcal{D}}$ with $\Gamma'_{\mathcal{D}} := \eta\Gamma_0(p^{r_p}M^2_{\mathcal{D}})\eta^{-1}$ in the definition of $\Gamma_{\mathcal{D},i}$.

Since $g' \in U_1(\mathbf{R})$ in (11.6.2.c), we have

$$\varphi_0(g'\mathrm{diag}(a_i^{-1}, \bar{a}_i)) = \bar{\psi}(a_i)\bar{\chi}_p(q_i)\varphi_0(\mathrm{diag}(1, q_i^{-1})_\infty g').$$

If we further assume that $g_f = \mathrm{diag}(u, {}^t\bar{u}^{-1})$, $u \in \mathrm{GL}_m(\mathbf{A}_{\mathcal{K},f})$, then we also have

$$\tilde{H}^{(m)}_{\mathcal{D},\beta}(z, \alpha(g, g'\begin{pmatrix} a_i^{-1} & \\ & \bar{a}_i\end{pmatrix})) \\ = \bar{\tau}_p^{m-1}(q_i)\bar{\tau}^m\tau^c(a_i)\tilde{H}^{(m)}_{\mathcal{D},\beta}(z, \alpha(\begin{pmatrix} 1_m & \\ & q_i^{-1}1_m\end{pmatrix}_\infty g\begin{pmatrix} a_i1_m & \\ & \bar{a}_i^{-1}1_m\end{pmatrix}, \begin{pmatrix} 1 & \\ & q_i^{-1}\end{pmatrix}_\infty g'))).$$

Substituting the right-hand sides of these last two equalities into (11.6.2.c) yields the following corollary of Proposition 11.6.1.

Corollary 11.6.3. *Let $m = 1$ or 2 and $\beta \in S_m(\mathbf{Q})$. Suppose that x_p is as in Proposition 11.6.1. If $\kappa \geq m+1$ and $g \in U_m(\mathbf{A})$ is such that $g_f = \mathrm{diag}(u, {}^t\bar{u}^{-1})$, then*

(11.6.3.a)

$$\sum_{j=1}^{h_{\mathcal{K}}} \bar{\tau}_p^m\bar{\xi}_p(q_j)\bar{\tau}^m\bar{\xi}(a_j)\int_{\Gamma'_{\mathcal{D},j}\backslash U_1(\mathbf{R})} \tilde{H}^{(m)}_{\mathcal{D},\beta}(z, \alpha(\begin{pmatrix} 1_m & \\ & q_j^{-1}1_m\end{pmatrix}_\infty g\begin{pmatrix} a_j1_m & \\ & \bar{a}_j^{-1}1_m\end{pmatrix}, \begin{pmatrix} 1 & \\ & q_j^{-1}\end{pmatrix}_\infty g'))) \\ \times \bar{\tau}(\det g')\varphi_0(\begin{pmatrix} 1 & \\ & q_j^{-1}\end{pmatrix}_\infty g')dg' \\ = \begin{cases} c^{(1)}_{\mathcal{D}}(z)\varphi_\beta(g) & m = 1 \\ c^{(2)}_{\mathcal{D}}(z)\mu_{\mathcal{D}}(\beta, z, g) & m = 2,\end{cases}$$

where $c^{(m)}_{\mathcal{D}}$ is as in Proposition 11.6.1.

11.7. **The classical picture III.** Let $\mathfrak{D} = (f, \psi, \xi, \Sigma)$ be a classical datum as in 9.4 and let $\mathcal{D} = (\Sigma, \varphi, \psi_0, \tau_0)$ be the associated Eisenstein datum (also as in 9.4). So in particular f is an eigenform in $S_\kappa(N, \chi)$ with $N = Mp^r$, $p \nmid M$ and $r > 0$, and the central character of π (the representation generated by φ) is $\chi_\pi = \chi_1\bar{\chi}_0^2 = \bar{\chi}\chi_p^2$, χ being the adele class character associated with χ and χ_0 being as in 9.4. We continue with the notation used in the preceding section, assuming in addition that

(11.7.0.b) $$x_p = p^{t_p} \text{ with } t_p > 0 \quad \text{and} \quad x_p \in \mathrm{cond}(\psi_0),\ x_p\bar{x}_p \in \mathrm{cond}_{\pi_p}(\phi_p).$$

Let

$$p^{r_p} := x_p\bar{x}_p,$$

so $r_p > 0$ is an integer. Choose $M_{\mathcal{D}}$ as in the preceding section and let

$$\Gamma_{\mathfrak{D}} := \Gamma_0(p^{r_p}M) \quad \text{and} \quad \Gamma'_{\mathfrak{D}} := \Gamma_0(p^{r_p}M_{\mathcal{D}}^2).$$

Let $z_\kappa := (\kappa - m - 1)/2$. Let $\beta \in S_m(\mathbf{Q})$. For $w \in \mathbf{H}_1$ and $x \in G_{m+1}(\mathbf{A}_f)$ let

$$h_{\mathcal{D},\beta}^{(m)}(w;x) := J_{m+1}(g,\mathbf{i})^\kappa \mu_{m+1}(g)^{-m\kappa/2} H_{\mathcal{D},\beta}^{(m)}(z_\kappa, gx),$$
$$g \in G_{m+1}(\mathbf{R}), \quad g(\mathbf{i}) = \left(\begin{smallmatrix} \mathbf{i} & \\ & w \end{smallmatrix}\right).$$

Then

(11.7.0.c) $$h_{\mathcal{D},\beta}^{(m)}(w;x) = e(\mathrm{Tr}\,\beta\mathbf{i}) \sum_{n\in\mathbf{Q}} \left(\sum_{\beta' \in S_{m+1}(\beta,n)} A_{\mathcal{D},\beta'}^{(m)}(x) \right) e(nw),$$
$$S_{m+1}(\beta,n) := \{\beta' = \left(\begin{smallmatrix} \beta & a \\ {}^t\bar{a} & n \end{smallmatrix}\right) \in S_{m+1}(\mathbf{Q}) \ : \ a \in \mathcal{K}\},$$

where $A_{\mathcal{D},\beta'}^{(m)}(x)$ is the β'-Fourier coefficient of $H_{\mathcal{D}}^{(m)}(Z;x)$ as in 11.5. From (11.5.0.e) it follows that for $x \in \alpha(U_m(\mathbf{A}_f),1)$,

$$h_{\mathcal{D},\beta}^{(m)}(w;x) \in M_\kappa(\eta\Gamma_{\mathcal{D}}\eta^{-1}).$$

Suppose $x \in \alpha(U_m(\mathbf{A}_f),1)$. Let

$$\begin{aligned}\tilde{h}_{\mathcal{D},\beta}^{(m)}(w;x) : &= J_{m+1}(g,\mathbf{i})^\kappa \mu_{m+1}(g)^{-m\kappa/2} \tilde{H}_{\mathcal{D},\beta}^{(m)}(z_\kappa, gx), \quad g \in G_{m+1}(\mathbf{R}), g(\mathbf{i}) = \left(\begin{smallmatrix} \mathbf{i} & \\ & w \end{smallmatrix}\right) \\ &= \sum_{a \in (\widehat{\mathcal{O}}/x_pM_{\mathcal{D}})^\times} \chi_0\xi_0^c\tau_0(a) h_{\mathcal{D},\beta}^{(m)}(w; x\alpha(1,\mathrm{diag}(a^{-1},\bar{a}))).\end{aligned}$$

This belongs to $M_\kappa(p^{r_p}M_{\mathcal{D}}^2,\chi)$, and so for $x \in U_m(\mathbf{A}_f)$

$$\tilde{g}_{\mathfrak{D},\beta,j}^{(m)}(w;x) := e(-\mathrm{Tr}\,\mathbf{i}\beta/q_j)\tilde{h}_{\mathcal{D},\beta/q_j}^{(m)}(w; \alpha(x\left(\begin{smallmatrix} a_j 1_m & \\ & \bar{a}_j^{-1}1_m \end{smallmatrix}\right),1)) \in M_\kappa(p^{r_p}M_{\mathcal{D}}^2,\chi).$$

Let

$$\tilde{g}_{\mathfrak{D},\beta}^{(m)}(w;x) := \sum_{j=1}^{h_{\mathcal{K}}} q_j^{m\kappa}\bar{\psi}_p^m\xi_p^{m-1}(q_j)\chi_0\bar{\psi}^m\xi^{m-1}(a_j)\tilde{g}_{\mathfrak{D},\beta,j}^{(m)}(w;x) \in M_\kappa(p^{r_p}M_{\mathcal{D}}^2,\chi)$$

and

$$g_{\mathfrak{D},\beta}^{(m)}(w;x) := \sum_{a \bmod M_1M_{\mathcal{D}}} \tilde{g}_{\mathfrak{D},\beta}^{(m)}\left(\frac{w+a}{M_1M_{\mathcal{D}}};x\right) \in M_\kappa(p^{r_p}M,\chi), \quad M_1 := M_{\mathcal{D}}/M.$$

Let $f_{\mathfrak{D}}(w,x)$ be as in 9.4. Note that $f_{\mathfrak{D}}(w,1) = f_{\mathfrak{D}}(w)$. Let $a_{\mathfrak{D}}(\beta,x)$ be the β-Fourier coefficient of $f_{\mathfrak{D}}(w,x)$. Recall that $c_{\mathfrak{D}}(\beta,x)$ denotes the β-Fourier coefficient of $E_{\mathfrak{D}}(Z,x)$.

Proposition 11.7.1. *Assume* (11.7.0.b) *holds and let* $x = \mathrm{diag}(u, {}^t\bar{u}^{-1})$, $u \in \mathrm{GL}_m(\mathbf{A}_{\mathcal{K},f})$. *Suppose* $u \in \mathbf{A}_f^\times$ *if* $m = 1$. *Suppose also* $\kappa \geq 2$ *if* $m = 1$ *and* $\kappa > 6$ *if* $m = 2$. *Let* $\beta \in S_m(\mathbf{Q})$.

(i) *There exists a constant* $C_{\mathfrak{D}}^{(m)}$ *depending only on* $\mathfrak{D}$ *and* m *such that*

$$< g_{\mathfrak{D},\beta}^{(m)}(-;x), f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}} = C_{\mathfrak{D}}^{(m)} \begin{cases} a_{\mathfrak{D}}(\beta,x) & m=1 \\ c_{\mathfrak{D}}(\beta,x) & m=2. \end{cases} \tag{11.7.1.a}$$

(ii) *If* $a(p,f) \neq 0$ *and if* $p|\mathfrak{f}_\chi$ *and* $p|\mathfrak{f}_{\chi^{-1}\xi}$ *then*

$$\begin{aligned} C_{\mathfrak{D}}^{(1)} &= -\pi 2^{2-\kappa} i^{-\kappa} \bar{\chi}_p \xi_p^c(M_{\mathcal{D}}) M_{\mathcal{D}}^{2\kappa} (M p^{r_p})^{-\kappa/2} p^{r_p + n_p(\kappa-2)} \\ &\quad \times \frac{\Gamma(\kappa-1) L_{\mathcal{K}}^{\Sigma}(f, \chi^{-1}\xi, \kappa-1)}{\Gamma(\kappa) \prod_{j=0}^{1} L^{\Sigma}(\chi^{-1}\xi' \chi_{\mathcal{K}}^j, \kappa-j)} \\ &\quad \times \bar{\psi}_p(-1) c_2(\bar{\tau}'_{0,p}, 1-\kappa/2) \mathfrak{g}(\tau'_{0,p})^2 \\ &\quad \times a(p,f)^{r_p - n_p} \chi_p \bar{\xi}_p^c(y_p) \mathfrak{g}(\xi_p^c, x_p) \mathfrak{g}(\bar{\chi}_p \xi_p^c, y_p), \end{aligned} \tag{11.7.1.b}$$

where $(y_p) := \mathrm{cond}(\bar{\chi}_p \xi_p^c)$ *and* $(p^{n_p}) := (y_p \bar{y}_p)$.

(iii) *If* $\mathcal{O}_p \neq \mathrm{cond}(\xi_p \psi_p^{-2} \psi_p^c) \supseteq \mathrm{cond}(\bar{\chi}_p \xi_p \xi_p^c)$ *then* $C_{\mathfrak{D}}^{(2)} = C_{\mathfrak{D}}^{(1)}$.

Proof. From the definition of φ and φ_0 it follows that for $g \in \mathrm{SL}_2(\mathbf{R})$,

$$\begin{aligned} \varphi_0(\begin{pmatrix} 1 & \\ & q_j^{-1} \end{pmatrix}_\infty g) &= a_{\mathfrak{D},1} q_j^{\kappa/2} M_{\mathcal{D}}^{-\kappa} p^{-r_p\kappa/2} J_1(g,i)^{-\kappa} f(q_j M_{\mathcal{D}}^{-2} p^{-r_p} z) \\ &= a_{\mathfrak{D},1} J_1(g,i)^{-\kappa} (f|_\kappa \begin{pmatrix} q_j & \\ & M_{\mathcal{D}}^2 p^{r_p} \end{pmatrix})(z), \end{aligned}$$

$$z = g(i), \quad a_{\mathfrak{D},1} := (-1)^\kappa \psi_p(\bar{x}_p^{-1} M_{\mathcal{D}}) p^{r_p\kappa/2}.$$

Let $z_\kappa := (\kappa - m - 1)/2$. If $g \in U_m(\mathbf{A}_f)$ and $g' \in \mathrm{SL}_2(\mathbf{R})$ then

$$\begin{aligned} &\tilde{H}_{\mathcal{D},\beta}^{(m)}(z_\kappa, \alpha(\begin{pmatrix} 1_m & \\ & q_j^{-1}1_m \end{pmatrix}_\infty g \begin{pmatrix} a_j 1_m & \\ & \bar{a}_j^{-1} 1_m \end{pmatrix}, \begin{pmatrix} 1 & \\ & q_j^{-1} \end{pmatrix}_\infty g') \\ &= q_j^{(m-1)\kappa/2} J_1(g'', i)^{-\kappa} e(\mathrm{Tr}\, \beta \mathbf{i}) \tilde{g}_{\mathfrak{D},\beta,j}^{(m)}(\frac{1}{q_j \bar{w}}; g) \\ &= q_j^{m\kappa/2} i^{-\kappa} \overline{j(g',i)}^{-\kappa} e(\mathrm{Tr}\, \beta \mathbf{i}) (\tilde{g}_{\mathfrak{D},\beta,j}^{(m)}(-;g)|_\kappa \begin{pmatrix} & -1 \\ q_j & \end{pmatrix})(-\bar{w}), \end{aligned}$$

$$g'' = \begin{pmatrix} d_{g'} & c_{g'} \\ b_{g'} & a_{g'} \end{pmatrix}, \quad w = g'(i).$$

Since g will remain fixed in what follows we suppress it in our notation, writing $\tilde{g}_{\mathfrak{D},\beta,j}^{(m)}(w)$ for $\tilde{g}_{\mathfrak{D},\beta,j}^{(m)}(w;g)$, etc.

As $j(g',i)\overline{j(g',i)} = \mathrm{Im}(w)^{-1}$, it follows that for $g = \mathrm{diag}(u, {}^t\bar{u}^{-1})$, $u \in \mathrm{GL}_2(\mathbf{A}_{\mathcal{K},f})$, the integral in (11.6.3.a) at $z = z_\kappa$ equals

$$e(\mathrm{Tr}\,\beta\mathbf{i})a_{\mathfrak{D},2}q_j^{m\kappa/2}\int_{\Gamma'_{\mathcal{D},j}\backslash\mathfrak{h}}(\tilde{g}^{(m)}_{\mathfrak{D},\beta,j}|_\kappa\begin{pmatrix} & -1\\ q_j & \end{pmatrix})(-\bar{w})(f|_\kappa\begin{pmatrix} q_j & \\ & M_{\mathcal{D}}^2p^{r_p}\end{pmatrix})(w)\mathrm{Im}(w)^\kappa d\mathrm{vol}(w),$$

$$a_{\mathfrak{D},2} := i^{-\kappa}a_{\mathfrak{D},1},$$

$d\mathrm{vol}(w)$ being the standard invariant volume form on $\mathfrak{h}$. Upon making the substitution $u = -\bar{w}$ this becomes

$$-e(\mathrm{Tr}\,\beta\mathbf{i})a_{\mathfrak{D},2}q_j^{m\kappa/2} < \tilde{g}^{(m)}_{\mathfrak{D},\beta,j}|_\kappa\begin{pmatrix} & -1\\ q_j & \end{pmatrix}, f^c|_\kappa\begin{pmatrix} q_j & \\ & M_{\mathcal{D}}^2p^{r_p}\end{pmatrix} >_{\Gamma'_{\mathcal{D},j}},$$

which in turn equals

$$-e(\mathrm{Tr}\,\beta\mathbf{i})a_{\mathfrak{D},2}q_j^{m\kappa/2} < \tilde{g}^{(m)}_{\mathfrak{D},\beta,j}, f^c|_\kappa\begin{pmatrix} & -1\\ p^{r_p}M_{\mathcal{D}}^2 & \end{pmatrix} >_{\Gamma'_{\mathfrak{D}}} .$$

It then follows that the left-hand side of (11.6.3.a) equals

$$-e(\mathrm{Tr}\,\beta\mathbf{i})a_{\mathfrak{D},2} < \tilde{g}^{(m)}_{\mathfrak{D},\beta}, f^c|_\kappa\begin{pmatrix} & -1\\ p^{r_p}M_{\mathcal{D}}^2 & \end{pmatrix} >_{\Gamma'_{\mathfrak{D}}},$$

which equals

$$-e(\mathrm{Tr}\,\beta\mathbf{i})a_{\mathfrak{D},2}M_{\mathcal{D}}^{-\kappa}M^{\kappa/2} < g^{(m)}_{\mathfrak{D},\beta}, f^c|_\kappa\begin{pmatrix} & -1\\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}} .$$

Since $c_{\mathcal{D}}(\beta,g) = e(\mathrm{Tr}\,\beta\mathbf{i})\mu_{\mathcal{D}}(\beta,z_\kappa,g)$, it then follows from (11.6.3.a) that (11.7.1.a) holds with

$$C^{(m)}_{\mathfrak{D}} := -c^{(m)}_{\mathcal{D}}(z_\kappa)M_{\mathcal{D}}^\kappa M^{-\kappa/2}/a_{\mathfrak{D},2},$$

with $c^{(m)}_{\mathcal{D}}(z)$ as in Proposition 11.6.1, proving part (i). In particular,

$$C^{(m)}_{\mathfrak{D}} = -\pi 2^{2-\kappa}i^{-\kappa}\xi_p^c(x_p/M_{\mathcal{D}})\bar{\chi}_p(M_{\mathcal{D}}^2)M_{\mathcal{D}}^{2\kappa}(Mp^{r_p})^{-\kappa/2}p^{(m+1)r_p} \times \frac{\Gamma(\kappa-1)L^\Sigma_{\mathcal{K}}(f,\chi^{-1}\xi,\kappa-1)}{\Gamma(\kappa)\prod_{j=0}^1 L^\Sigma(\chi^{-1}\xi'\chi^j_{\mathcal{K}},\kappa-j)}\gamma^{(m)}(\rho_p,-z_\kappa), \tag{11.7.1.c}$$

where ρ_p is the representation associated with $(\tilde{\pi}_p,\bar{\psi}_{0,p},\bar{\tau}^c_{0,p}\psi^c_{0,p}\bar{\psi}_{0,p})$ and $\gamma^{(m)}(\rho_p,z)$ is as in Proposition 11.4.13.

If $a(p,f)\neq 0$ then $\pi_p\simeq\pi(\mu_1,\mu_2)$ with $\mu_1|_{\mathbf{Z}_p^\times} = 1$, $\mu_2|_{\mathbf{Z}_p^\times} = \chi_p|_{\mathbf{Z}_p^\times}$, and $p^{(\kappa-1)/2}\mu_1\chi_{0,p}(p) = a(p,f)$. It then follows from Proposition 11.4.13(i) that under the hypotheses of part (ii)

$$\gamma^{(1)}(\rho_p,1-\kappa/2) = \psi_{0,p}(-1)c_2(\bar{\tau}'_{0,p},1-\kappa/2)\mathfrak{g}(\tau'_{0,p})^2\prod_{j=1}^2\epsilon(\mu_j\bar{\xi}_{0,p},-z_\kappa). \tag{11.7.1.d}$$

Since for any character λ of $\mathcal{K}_p^\times$

$$\epsilon(\lambda,s) = \int_{c^{-1}\mathcal{O}_p^\times}\lambda^{-1}(a)|a|_{\mathcal{K}}^{-s}e_p(\mathrm{Tr}\,a)da = |c|^s_{\mathcal{K}}\lambda(c)\mathfrak{g}(\lambda^{-1},c),\quad (c) := \mathrm{cond}(\lambda),$$

we have

$$\begin{aligned}\epsilon(\mu_1\bar{\xi}_{0,p},-z_\kappa) &= p^{r_p(\kappa-3)/2}\mu_1\bar{\xi}^c_{0,p}(x_p)\mathfrak{g}(\mu_1^{-1}\xi^c_{0,p},x_p)\\ &= p^{r_p(\kappa-3)/2}(\mu_1\chi_{0,p}(p))^{r_p}\bar{\xi}_p(x_p)\mathfrak{g}(\mu_1^{-1}\xi^c_{0,p},x_p)\\ &= p^{-r_p}a(p,f)^{r_p}\bar{\xi}^c_p(x_p)\mathfrak{g}(\xi^c_p,x_p).\end{aligned} \tag{11.7.1.e}$$

Similarly, if $y = p^{v_p} \in \mathcal{O}_p$ is such that $(y) = \mathrm{cond}(\chi_p\bar{\xi}^c_p) = \mathrm{cond}(\mu_2\bar{\xi}^c_{0,p})$ and $p^{n_p} = y\bar{y}$ then

$$\epsilon(\mu_2\bar{\xi}_{0,p},-z_\kappa) = p^{n_p(\kappa-2)}a(p,f)^{-n_p}\chi_p\bar{\xi}^c_p(y_p)\mathfrak{g}(\bar{\chi}_p\xi^c_p,y_p). \tag{11.7.1.f}$$

Combining (11.7.1.c), (11.7.1.d), (11.7.1.e), and (11.7.1.f) yields the equality in part (ii) of the proposition. Part (iii) follows from (11.7.1.c) and part (iii) of Proposition 11.4.13. ■

For $n \in \mathbf{Q}_{\geq 0}$ let $\tilde{b}^{(m)}_{\mathfrak{D},\beta}(n,x)$ be the n-Fourier coefficient of $\tilde{g}^{(m)}_{\mathfrak{D},\beta}(w;x)$ and $b^{(m)}_{\mathfrak{D},\beta}(n,x)$ the n-Fourier coefficient of $g^{(m)}_{\mathfrak{D},\beta}(w;x)$. Then

$$b^{(m)}_{\mathfrak{D},\beta}(n,x) = \begin{cases} M_{\mathcal{D}}M_1\tilde{b}^{(m)}_{\mathfrak{D},\beta}(M_{\mathcal{D}}M_1 n,x) & n\in\mathbf{Z}\\ 0 & \text{otherwise.}\end{cases} \tag{11.7.1.g}$$

Proposition 11.7.2. *Suppose* $y = \mathrm{diag}(u,{}^t\bar{u}^{-1})$, $u \in \mathrm{GL}_m(\mathbf{A}^{\Sigma}_{\mathcal{K},f})$. *For* $i = 1,...,h_{\mathcal{K}}$ *let* $v_i := \mathrm{diag}(u\bar{a}_i,1)$. *For* $\beta \in S_m(\mathbf{Q})$, $\beta' \geq 0$, *and* $n \in \mathbf{Q}_{\geq 0}$ *let*

$$L^{(m)}_{v_i}(\beta,n) := \{T = \left(\begin{smallmatrix}\beta/q_i & c\\ \bar{c} & n\end{smallmatrix}\right) \in L^{(m)}_{v_i},\ T>0\}.$$

This is a finite set.

(i)

$$\tilde{b}^{(m)}_{\mathfrak{D},\beta}(n,y) = \sum_{j=1}^{h_{\mathcal{K}}} q_j^{m\kappa}\bar{\psi}^m_p\xi^{m-1}_p(q_j)\chi_0\bar{\psi}^m\xi^{m-1}(a_j) \times \sum_{T\in L^{(m)}_{v_j}(\beta,n)}\ \sum_{a\in(\widehat{\mathcal{O}}/(x_pM_{\mathcal{D}}))^\times}\chi_0\xi^c_0\tau_0(a)A^{(m)}_{\mathcal{D},T}(y_{j,a}),$$

where $y_{j,a} := \mathrm{diag}(ua_j,\bar{a},\bar{u}^{-1}\bar{a}_j^{-1},a^{-1})$.

(ii) *Let* $(x_\ell) := \mathrm{cond}(\xi^c_\ell)$. *Suppose* $x_p = p^{t_p}$ *with* $t_p > 0$. *If* $T \in L^{(m)}_{v_i}(\beta,n)$ *then* $A_{\mathcal{D},T}(y_{i,a}) = 0$ *unless* $T^*_p := T_{m+1,1} \in \mathcal{O}^\times_p$ *and* $T^*_\ell := T_{m+1,1}x_\ell\delta_{\mathcal{K}}/M_{\mathcal{D}} \in \mathcal{O}_\ell$ *for* *all* $\ell \in \Sigma\backslash\{p\}$ *, in which case*

$$\sum_{a\in(\widehat{\mathcal{O}}/x_pM_{\mathcal{D}})^\times} \chi_0\xi_0^c\tau_0(a)A_{\mathcal{D},T}(y_{j,a})$$
$$\begin{aligned}=&D_{\mathcal{K}}^{-m(m+1)/4}\frac{(-2)^{-m-1}(2\pi i)^{(m+1)\kappa}(2/\pi)^{m(m+1)/2}}{\prod_{i=0}^m(\kappa-i-1)!\prod_{i=0}^m L^\Sigma(\kappa-i,\bar\chi\xi\chi_{\mathcal{K}}^j)}\\ &\times(\det T|\det T|_p)^{\kappa-m-1}\bar\tau_{0,p}(\det T)\\ &\times\mathfrak{g}(\tau'_{0,p})^{m+1}c_{m+1}(\bar\tau'_{0,p},-z_\kappa)\\ &\times\bar\xi_p^c(T_p^*)\mathfrak{g}(\xi_p^c,x_p)\\ &\times M_{\mathcal{D}}^2\prod_{\ell\in\Sigma,\ell\neq p}|x_\ell|_{\mathcal{K}}\begin{cases}\bar\xi_\ell^c(T_\ell^*)\mathfrak{g}(\xi_\ell^c,x_\ell\delta_{\mathcal{K}}) & (x_\ell)\neq\mathcal{O}_\ell, T_\ell^*\in\mathcal{O}_\ell^\times\\ 0 & (x_\ell)\neq\mathcal{O}_\ell, T_\ell^*\notin\mathcal{O}_\ell^\times\\ 1 & (x_\ell)=\mathcal{O}_\ell\end{cases}\\ &\times\bar\chi_0\psi\bar\xi(a_j^m\det u)|\det u|_{\mathcal{K}}^{-\kappa/2}\prod_{\ell\notin\Sigma}h_{\ell,q_j{}^t\bar u_\ell Tu_\ell}(\bar\chi_\ell\xi'_\ell(\ell)\ell^{-\kappa}).\end{aligned}$$

Proof. By (11.7.0.c) we have

$$\begin{aligned}\tilde g^{(m)}_{\mathfrak{D},\beta,j}(w;y)&=e(-\mathrm{Tr}\,\beta\mathbf{i})\sum_{a\in(\widehat{\mathcal{O}}/x_pM_{\mathcal{D}})^\times}\chi_0\xi_0^c\tau_0(a)h^{(m)}_{\mathcal{D},\beta/q_j}(w;y_{j,a})\\ &=\sum_{n\in\mathbf{Q}_{\geq0}}\sum_{T\in S_{m+1}(\beta,n)}\sum_{a\in(\widehat{\mathcal{O}}/x_pM_{\mathcal{D}})^\times}\chi_0\xi_0^c\tau_0(a)A^{(m)}_{\mathcal{D},T}(y_{j,a})e(nw)\\ &=\sum_{n\in\mathbf{Q}_{\geq0}}\sum_{T\in L^{(m)}_{v_j}(\beta,n)}\sum_{a\in(\widehat{\mathcal{O}}/x_pM_{\mathcal{D}})^\times}\chi_0\xi_0^c\tau_0(a)A^{(m)}_{\mathcal{D},T}(y_{j,a})e(nw).\end{aligned}$$

Part (i) then follows from the definition of $\tilde g^{(m)}_{\mathfrak{D},\beta}(w;y)$. Part (ii) follows from part (i) and Lemma 11.5.3. ∎

Some normalizations

We normalize the preceding results so that the resulting formulas are better suited for p-adic interpolation.

Let $(y_\ell):=\mathrm{cond}(\bar\chi_\ell\xi_\ell^c)$ and let $(\ell^{e_\ell}):=\mathrm{cond}(\bar\tau'_{0,\ell})=\mathrm{cond}(\bar\chi_\ell\xi'_\ell)$. Put

$$\begin{aligned}B_{\mathfrak{D}}^{(m)}:=&\frac{\prod_{j=0}^m(\kappa-j-1)!\prod_{j=0}^m L^\Sigma(\kappa-j,\bar\chi\xi'\epsilon_{\mathcal{K}}^j)\prod_{\ell\in\Sigma\backslash\{p\}}\chi_\ell\bar\xi_\ell^c(y_\ell\delta_{\mathcal{K}})\mathfrak{g}(\bar\chi_\ell\xi_\ell^c,y_\ell\delta_{\mathcal{K}})|y_\ell\delta_{\mathcal{K}}|_{\mathcal{K}}^{2-\kappa}}{\bar\chi_p\xi_p^c(M_{\mathcal{D}})M_{\mathcal{D}}^{2\kappa}\psi_p(-1)c_{m+1}(\bar\tau'_{0,p},(m+1-\kappa)/2)\mathfrak{g}(\tau'_{0,p})^{m+1}\mathfrak{g}(\xi_p^c,x_p)}\\ &\times i(-1)^m2^{m(m+2)}(2\pi i)^{-(m+1)\kappa}(\pi/2)^{m(m+1)/2}\\ &\times\begin{cases}\prod_{\ell\in\Sigma\backslash\{p\}}\chi_\ell\bar\xi'_\ell(\ell^{e_\ell})\ell^{e_\ell(\kappa-2)}\mathfrak{g}(\chi_\ell\bar\xi'_\ell)^{-1} & m=2\\ 1 & m=1\end{cases}\end{aligned}$$

and let

$$f^{(m)}_{\mathfrak{D},\beta,x}(w) := B^{(m)}_{\mathfrak{D}} g^{(m)}_{\mathfrak{D},\beta}(w;x).$$

Let $\rho^{(m)}_{\mathfrak{D},\beta}(n,x)$ be the n-Fourier coefficient of $f^{(m)}_{\mathfrak{D},\beta,x}(w)$. The following is an immediate consequence of (11.7.1.g), Proposition 11.7.2, and the fact that $\tau_0 = \psi\bar{\xi}|\cdot|^{\kappa/2}_{\mathcal{K}}$.

Lemma 11.7.3. *Suppose $y = \mathrm{diag}(u, \bar{u}^{-1})$, $u \in \mathrm{GL}_m(\mathbf{A}^{\Sigma}_{\mathcal{K},f})$. Let $v_i := \mathrm{diag}(ua_i, 1)$. Suppose $x_p = p^{t_p}$ with $t_p > 0$. Then for $n \in \mathbf{Z}$*

$$\rho^{(m)}_{\mathfrak{D},\beta}(n,y) = -i2^{m(m+1)-1} D_{\mathcal{K}}^{-m(m+1)/4} M^{-1} \sum_{i=1}^{h_{\mathcal{K}}} q_i^{m\kappa} \bar{\psi}_p^m \xi_p^{m-1}(q_i) \chi_0 \bar{\psi}^m \xi^{m-1}(a_i) \times \sum_{T \in L^{(m)}_{v_i}(\beta, M_{\mathcal{D}} M_1 n)} R^{(m)}_{\mathfrak{D},T}$$

where $R_{\mathfrak{D},T} = 0$ unless $T_p^* \in \mathcal{O}_p^\times$ and $T_\ell^* \in \mathcal{O}_\ell$ for all $\ell \in \Sigma\backslash\{p\}$ (T_ℓ^* being as in Proposition 11.7.2) in which case

$$\begin{aligned}
R^{(m)}_{\mathfrak{D},T} = {} & (\det T|\det T|_p)^{-m-1} \bar{\xi}^c_p(T_p^*) \prod_{\ell \neq p} \psi_\ell \bar{\xi}_\ell(\det T) \\
& \times \psi_p(-1)\chi_p \bar{\xi}^c_p(M_{\mathcal{D}}) M_{\mathcal{D}}^{4-2\kappa} \\
& \times \prod_{\ell \in \Sigma, \ell \neq p} |x_\ell|_{\mathcal{K}} \begin{cases} \bar{\xi}^c_\ell(T^*_\ell)\mathfrak{g}(\xi^c_\ell, x_\ell\delta_{\mathcal{K}}) & (x_\ell) \neq \mathcal{O}_\ell, T^*_\ell \in \mathcal{O}_\ell^\times \\ 0 & (x_\ell) \neq \mathcal{O}_\ell, T^*_\ell \notin \mathcal{O}_\ell^\times \\ 1 & (x_\ell) = \mathcal{O}_\ell \end{cases} \\
& \times \prod_{\ell \in \Sigma, \ell \neq p} \chi_\ell \bar{\xi}^c_\ell(y_\ell \delta_{\mathcal{K}}) |y_\ell \delta_{\mathcal{K}}|^{2-\kappa}_{\mathcal{K}} \mathfrak{g}(\bar{\chi}_\ell \xi^c_\ell, y_\ell \delta_{\mathcal{K}}) \\
& \times \bar{\chi}_0 \psi \bar{\xi}(a_j^m \det u) |\det u|^{-\kappa/2}_{\mathcal{K}} \prod_{\ell \notin \Sigma} h_{\ell, q_j {}^t\bar{u}_\ell T u_\ell}(\bar{\chi}_\ell \xi'_\ell(\ell)\ell^{-\kappa}) \\
& \times \begin{cases} \prod_{\ell \in \Sigma\backslash\{p\}} \chi_\ell \bar{\xi}'_\ell(\ell^{e_\ell}) \ell^{e_\ell(\kappa-2)} \mathfrak{g}(\chi_\ell \bar{\xi}'_\ell)^{-1} & m = 2 \\ 1 & m = 1. \end{cases}
\end{aligned}$$

Let $\pi(f) = \otimes \pi_v(f)$ be the (unitary) automorphic representation generated by $f_{\mathbf{A}}$. Let $W'(f) := \prod_{\ell \neq p} \epsilon(\pi_\ell(f), 1/2)$ and $W(f)_p := \epsilon(\pi_p(f), 1/2)$. These are algebraic numbers and $W'(f)$ satisfies $|W'(f)|_p = 1$.

We now assume that

(11.7.3.a) $$a(p,f) \neq 0$$

and set

(11.7.3.b) $$S(f) := a(p,f)^{-r} p^{r(\kappa/2-1)} W'(f)^{-1}.$$

Assuming (11.7.0.b) let

$$L_{\mathfrak{D}}^{(m)} := \frac{2^{-3}(2i)^{\kappa+1}}{a(p,f)^{r_p} p^{r_p(1-\kappa/2)}} B_{\mathfrak{D}}^{(m)} C_{\mathfrak{D}}^{(m)} M^{\kappa/2}$$

with $C_{\mathfrak{D}}^{(m)}$ as defined in Proposition 11.7.1.

Proposition 11.7.4. *Suppose* $\kappa \geq 2$ *if* $m = 1$ *and* $\kappa > 6$ *if* $m = 2$. *Assume* (11.7.3.a) *holds. Suppose* $x = \mathrm{diag}(u, {}^t\bar{u}^{-1})$ *with* $u \in \mathrm{GL}_m(\mathbf{A}_{\mathcal{K},f})$. *Suppose* $p | \mathfrak{f}_{\bar{\chi}\xi}$ *and* $p^r | \mathrm{Nm}(\mathfrak{f}_\xi)$. *Suppose also* $\mathrm{cond}(\psi_p) | \mathfrak{f}_\xi^c \mathcal{O}_p$.

(i)

$$\frac{< f_{\mathfrak{D},\beta,x}^{(m)}, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}}}{< f, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}}} = \frac{L_{\mathfrak{D}}^{(m)}}{2^{-3}(2i)^{\kappa+1} S(f) < f, f^c|_\kappa \begin{pmatrix} & -1 \\ N & \end{pmatrix} >_{\Gamma_0(N)}}$$
$$\times M^{-\kappa/2} W'(f)^{-1} \begin{cases} a_{\mathfrak{D}}(T,x) & m=1 \\ c_{\mathfrak{D}}(T,x) & m=2. \end{cases}$$

(ii)

$$L_{\mathfrak{D}}^{(1)} = a(p,f)^{-\mathrm{ord}_p(\mathrm{Nm}(\mathfrak{f}_{\bar{\chi}\xi}))} \left(\frac{(k-2)!}{(-2\pi i)^{\kappa-1}}\right)^2 \mathfrak{g}(\bar{\chi}\xi) \mathrm{Nm}(\mathfrak{f}_{\bar{\chi}\xi}\mathfrak{d})^{\kappa-2} L_{\mathcal{K}}^{\Sigma}(f, \bar{\chi}\xi, \kappa-1).$$

In particular, if $\xi = \theta \circ \mathrm{Nm}$ *for some Dirichlet character* θ *then*

$$L_{\mathfrak{D}}^{(1)} = \prod_{j=0}^{1} a(p,f)^{-\mathrm{ord}_p(\mathrm{cond}(\bar{\chi}\theta\chi_{\mathcal{K}}^j))} \frac{(k-1)! \mathfrak{f}_{\bar{\chi}\theta\chi_{\mathcal{K}}^j}^{\kappa-1} L^{\Sigma}(f, \bar{\chi}\theta\chi_{\mathcal{K}}^j, \kappa-1)}{(-2\pi i)^{\kappa-1} G(\bar{\chi}\theta\chi_{\mathcal{K}}^j)}.$$

(iii) *Under the hypotheses of Proposition 11.4.18*

$$L_{\mathfrak{D}}^{(2)} = p^{r_p} L_{\mathfrak{D}}^{(1)} \times L(3-\kappa, \chi\bar{\xi}') \prod_{\ell \in \Sigma} (1 - \bar{\chi}\xi'(\ell)\ell^{2-\kappa}).$$

Proof. Our hypotheses on the conductors ensures that (11.7.0.b) holds. To deduce part (i) we first observe that since $a(p,f) \neq 0$,

$$< f, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}} = a(p,f)^{r_p - r} p^{(r-r_p)(\kappa/2-1)} < f, f^c|_\kappa \begin{pmatrix} & -1 \\ N & \end{pmatrix} >_{\Gamma_0(N)}$$
$$= W'(f) a(p,f)^{r_p} p^{-r_p(\kappa/2-1)} S(f) < f, f^c|_\kappa \begin{pmatrix} & -1 \\ N & \end{pmatrix} >_{\Gamma_0(N)}.$$

We next note that it follows from Proposition 11.7.1 that

$$< f_{\mathfrak{D},\beta,x}^{(m)}, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}} = B_{\mathfrak{D}}^{(m)} C_{\mathfrak{D}}^{(m)} \times \begin{cases} a_{\mathfrak{D}}(T,x) & m=1 \\ c_{\mathfrak{D}}(T,x) & m=2. \end{cases}$$

Combining these two equalities with the definition of $L_{\mathfrak{D}}^{(m)}$ yields part (i) of the proposition.

Part (ii) is a straightforward calculation. For part (iii) we note that under the hypotheses of Proposition 11.4.18, $C_{\mathfrak{D}}^{(2)} = p^{r_p} C_{\mathfrak{D}}^{(1)}$, so

$$\begin{aligned} L_{\mathfrak{D}}^{(2)}/L_{\mathfrak{D}}^{(1)} &= p^{r_p} B_{\mathfrak{D}}^{(2)}/B_{\mathfrak{D}}^{(1)} \\ &= p^{r_p} \frac{(\kappa-3)! L^{\Sigma}(\kappa-2, \bar{\chi}\xi') c_2(\bar{\tau}'_{0,p}, (\kappa-2)/2) \prod_{\ell \neq p} \ell^{e_\ell(\kappa-2)}}{2^{\kappa-3} i^{\kappa-2} \pi^{\kappa-2} c_3(\bar{\tau}'_{0,p}, (\kappa-3)/2) \mathfrak{g}(\chi_p \bar{\xi}'_p) \prod_{\ell \neq p} \bar{\chi}_\ell \xi'_\ell(\ell^{e_\ell}) \mathfrak{g}(\chi_\ell \bar{\xi}'_\ell)} \\ &= p^{r_p} \frac{(\kappa-3)! L^{\Sigma}(\kappa-2, \bar{\chi}\xi') \mathfrak{f}_{\bar{\chi}\xi'}^{\kappa-2}}{2^{\kappa-3} i^{\kappa} \pi^{\kappa-2} i^{\kappa-2} \chi(-1) G(\bar{\chi}\xi')} \\ &= p^{r_p} L(3-\kappa, \chi\bar{\xi}') \prod_{\ell \in \Sigma} (1 - \bar{\chi}\xi'(\ell) \ell^{2-\kappa}), \end{aligned}$$

the second displayed line following from the first by the definition of the $B_{\mathfrak{D}}^{(m)}$'s, the third from the second by the definition of the c_{m+1}'s and properties of Gauss sums, and the fourth following from the third by the functional equation for $L(s, \bar{\chi}\xi')$. ■

Recall that in 9.4 we put $f'_{\mathfrak{D}}(z) := M^{\kappa/2} W'(f)^{-1} f_{\mathfrak{D}}(Mz)$, so

$$a(n, f'_{\mathfrak{D}}) = M^{\kappa/2} W'(f)^{-1} a_{\mathfrak{D}}(n/M, 1).$$

Since $a(1, f'_{\mathfrak{D}}) = 1$, the following is an immediate consequence of the preceding proposition.

Corollary 11.7.5. *Under the hypotheses of Proposition 11.7.2*

$$\frac{< f_{\mathfrak{D},1/M,1}^{(1)}, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p} M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}}}{< f, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p} M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}}} = \frac{L_{\mathfrak{D}}^{(1)}}{2^{-3}(2i)^{\kappa+1} S(f) < f, f^c|_\kappa \begin{pmatrix} & -1 \\ N & \end{pmatrix} >_{\Gamma_0(N)}}.$$

For any $x \in G(\mathbf{A}_f)$ let

$$G_{\mathfrak{D}}(Z, x) := M^{\kappa/2} W'(f)^{-1} L_{\mathfrak{D}}^{(2)} |\mu(x)|_{\mathbf{Q}}^{-\kappa} E_{\mathfrak{D}}(Z, x) \tag{11.7.5.a}$$

and let $C_{\mathfrak{D}}(\beta, x)$ denote the β-Fourier coefficient of $G_{\mathfrak{D}}(Z, x)$.

Corollary 11.7.6. *Under the hypotheses of Proposition 11.7.2*

$$\frac{< f_{\mathfrak{D},\beta,x}^{(2)}, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p} M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}}}{< f, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p} M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}}} = \frac{C_{\mathfrak{D}}(\beta, x)}{2^{-3}(2i)^{\kappa+1} S(f) < f, f^c|_\kappa \begin{pmatrix} & -1 \\ N & \end{pmatrix} >_{\Gamma_0(N)}}.$$

11.8. **A formula for $C_{\mathfrak{D}}(\beta, x)$.** The aim of this section is to express certain Fourier coefficients of $G_{\mathfrak{D}}(Z, x)$ as essentially Rankin-Selberg convolutions of f and sums of theta functions. This is used in §13 below to prove various p-adic properties of these coefficients.

11.8.1. *The formula.* Let $\mathfrak{D} = (f, \psi, \xi, \Sigma)$ be a classical datum and let $\mathcal{D} = (\varphi, \psi_0, \tau_0, \Sigma)$ be its associated Eisenstein datum. We assume

(11.8.1.a) π_p, ϕ_p, $\psi_{0,p}$, and $\tau_{0,p}$ are as in the Generic Case of 9.2.5

($\pi = \pi(f_{\mathbf{A}}) \otimes \chi_0^{-1} = \otimes \pi_v$, $\varphi = \otimes \phi_v$). Let λ be an idele class character of $\mathbf{A}_{\mathcal{K}}^\times$ such that

(11.8.1.b)
- $\lambda|_{\mathbf{A}^\times} = 1$;
- $\lambda_\infty(x) = (x/|x|)^{-2}$;
- λ_ℓ is unramified if $\ell \notin \Sigma\backslash\{p\}$.

Let $a_1, ..., a_{h_{\mathcal{K}}} \in \mathbf{A}_{\mathcal{K}}^\times$ be representatives of the class group of $\mathcal{K}$ as in the previous sections; so $a_i = (q_i, 1)$ for some rational prime $q_i \notin \Sigma$. Let $\mathcal{Q} = \{q_1, ..., q_{h_{\mathcal{K}}}\}$.

Let $\beta \in S_2(\mathbf{Q})$, $\beta > 0$, and $u \in \mathrm{GL}_2(\mathbf{A}_{K,f})$ be such that

(11.8.1.c)
- $u_\ell \in \mathrm{GL}_2(\mathcal{O}_\ell)$ for $\ell \notin \mathcal{Q}$;
- ${}^t\bar{u}\beta u \in S_2(\mathbf{Z}_\ell)^*$ for all primes ℓ;
- ${}^t\bar{u}\beta u$ is ℓ-primitive for all $\ell \notin \Sigma\backslash\{p\}$;
- if $u^{-1}\beta^{-1}{}^t\bar{u}^{-1} = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right)$ then $d_\ell \in \mathbf{Z}_\ell^\times$ for all $\ell \in \Sigma\backslash\{p\}$.

Let $M_{\mathcal{D}}$ be as in 11.5 and also satisfying

(11.8.1.d) $\mathrm{cond}(\lambda)|M_{\mathcal{D}}$ and $D_{\mathcal{K}} \det {}^t\bar{u}\beta u|M_{\mathcal{D}}$.

In what follows we drop the superscript '(2)' from our previous notation. Additionally, all Weil representations that show up are defined using the splitting determined by the character λ (see 10.1). For the definitions of the Schwartz functions and Siegel sections that appear below the reader should consult 10.2.3 and 11.4.

Let $h \in S_2(\mathbf{Q})$, $h > 0$, and let $y \in \mathrm{GL}_2(\mathbf{A}_{\mathcal{K},f})$ be such that $h, y_p \in \mathrm{GL}_2(\mathcal{O}_p)$, ${}^t\bar{y}hy \in S_2(\mathbf{Z}_\ell)^*$ for all ℓ, ${}^t\bar{y}hy$ is ℓ-primitive for all $\ell \notin \Sigma$, and $y^{-1}h^{-1}{}^t\bar{y}^{-1} = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right)$ with $d_\ell \in \mathbf{Z}_\ell^\times$ for all $\ell \in \Sigma\backslash\{p\}$. Assume that $D_{\mathcal{K}} \det {}^t\bar{y}hy|M_{\mathcal{D}}$. Let $r \in U_h(\mathbf{A}_f)$. Then by Lemma 11.3.2, for $g \in U_1(\mathbf{A})$ and $g' = \mathrm{diag}(ry, {}^t\bar{r}^{-1}{}^t\bar{y}^{-1})$ we have for $\mathrm{Re}(z) > 3/2$

$$H_{\mathcal{D},h}(z, \alpha(g', g)) = \sum_{\gamma \in B_1(\mathbf{Q})\backslash G_1(\mathbf{Q}), \gamma \in U_1(\mathbf{Q})} \sum_{x \in V} F_h(z; x, \gamma g, ry)$$

where

$$F_h(z; x, g, ry) := \prod_v F_{h,v}(z; x, g_v, r_v y_v), \quad F_{h,v}(z; x, g_v, r_v y_v) := FJ_h(f_{\mathcal{D},v}, z; x, g_v, r_v y_v),$$

with FJ_h as in (11.3.2.c).

Let $z_\kappa := (\kappa - 3)/2$. By Lemma 11.4.3

$$F_{h,\infty}(z_\kappa; x, g_\infty, 1) = \frac{(2\pi i)^{2\kappa}(2/\pi)}{4(\kappa-1)!(\kappa-2)!} \det h^{\kappa-2} e(i\mathrm{Tr}\, h) f_{\kappa-2,1}(z_\kappa, g'_\infty)\omega_h(r_\infty, g'_\infty)\Phi_{h,\infty}(x),$$

with $f_{\kappa,1} \in I_1(\tau_{0,\infty}/\lambda_\infty)$ and $g'_\infty = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) g_\infty \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$. By Lemma 11.4.7 if $\ell \notin \Sigma$ and $y_\ell \in \mathrm{GL}_2(\mathcal{O}_\ell)$ then

$$F_{h,\ell}(z;x,g_\ell,r_\ell y_\ell) = \frac{\tau_{0,\ell}(r_\ell)|\det r_\ell \bar{r}_\ell|_\ell^{-z+1/2}}{\prod_{j=0}^{1} L(2z+3-j,\bar{\tau}'_{0,\ell}\chi^j_{\mathcal{K},\ell})} f_1^{sph}(z,g_\ell)\omega_h(r_\ell,g_\ell)\Phi_{0,y_\ell}(x),$$

with $f_1^{sph} \in I_1(\tau_{0,\ell}/\lambda_\ell)$. By Lemma 11.4.15 if $\ell \in \Sigma\backslash\{p\}$ then

$$\begin{aligned}
&\sum_{a\in(\mathcal{O}_\ell/M_{\mathcal{D}})^\times} \chi_0\xi_0^c\tau_0(a)F_{h,\ell}(z;x,g_\ell\left(\begin{smallmatrix} a^{-1} & \\ & \bar{a} \end{smallmatrix}\right),r_\ell y_\ell) \\
&\qquad = \sum_{a\in(\mathcal{O}_\ell/M_{\mathcal{D}})^\times} \chi\bar{\xi}\bar{\tau}_0^c(a)F_{h,\ell}(z;x,g_\ell\left(\begin{smallmatrix} a^{-1} & \\ & \bar{a} \end{smallmatrix}\right),r_\ell y_\ell) \\
&\qquad = \tau_0(\det r_\ell y_\ell)|\det r_\ell y_\ell|_{\mathcal{K}}^{-z+1/2}D_\ell^{-1/2}\sum_{b\in\mathbf{Z}_\ell/D_\ell} f_b(z,g_\ell\eta)\omega_h(r_\ell,g_\ell\left(\begin{smallmatrix} 1 & \\ b & 1 \end{smallmatrix}\right))\Phi_{\chi_\ell\bar{\xi}_\ell,M_{\mathcal{D}},y_\ell}(x)
\end{aligned}$$

with $f_b \in I_1(\tau_{0,\ell}/\lambda_\ell)$ as in the lemma. Similarly, by Lemma 11.4.22

$$\begin{aligned}
&\sum_{a\in(\mathcal{O}_p/\bar{x}_p)^\times} \chi_0\xi_0^c\tau_0(a)F_{h,p}(-z;x,g_p\left(\begin{smallmatrix} a^{-1} & \\ & \bar{a} \end{smallmatrix}\right),r_p y) \\
&\qquad = \sum_{a\in(\mathcal{O}_p/\bar{x}_p)^\times} \xi^c\tau(a)F_{h,p}(-z;x,g_p\left(\begin{smallmatrix} a^{-1} & \\ & \bar{a} \end{smallmatrix}\right),r_p y) \\
&\qquad = \psi_p(-1)\bar{\tau}'_{0,p}(\det h)|\det h|_p^{2z+1}\mathfrak{g}(\tau'_{0,p})\bar{\tau}'_{0,p}(p^{2u_p})p^{-4u_pz-5u_p}\tau_0(\det r_p y_p)|\det r_p|_{\mathcal{K}}^{-z+1/2} \\
&\qquad\qquad \times \tilde{f}_{u_p,1}(z,g_p\eta)\omega_h(r_p,g_p)\Phi_{\xi^c,x_p,y_p}(x),
\end{aligned}$$

with $(p^{u_p}) := \mathrm{cond}(\tau'_{0,p})$ and $\tilde{f}_{u_p,1} \in I_1(\tau_{0,p}/\lambda_p)$ as in (11.4.19). Finally, if $\ell \notin \Sigma$ but $y_\ell \notin \mathrm{GL}_2(\mathcal{O}_\ell)$, then by part (ii) of Lemma 11.4.7 if $g_\ell = \left(\begin{smallmatrix} 1 & \\ n & 1 \end{smallmatrix}\right)$, $n \in \mathbf{Q}_\ell$,

$$F_{h,\ell}(z;x,g_\ell,r_\ell y_\ell) = \frac{\tau_0(\det r_\ell y_\ell)|\det r_\ell y_\ell|_{\mathcal{K}}^{-z+1/2}}{\prod_{j=0}^{1} L(2z+3-j,\bar{\tau}'_{0,\ell}\chi^j_{\mathcal{K},\ell})} f_1^{sph}(g_\ell)\omega_\beta(r_\ell,g_\ell)\Phi_{0,y_\ell}(x).$$

Let $w \in \mathfrak{h}$ and let $\gamma_\infty \in \mathrm{SL}_2(\mathbf{R})$ such that $\gamma_\infty(i) = w$. Taking $h = \beta/q_j$, $y = ua_j$, and $g = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right) \gamma_\infty \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$ in the above we find that for $x = \mathrm{diag}(ru, {}^t\bar{r}^{-1}{}^t\bar{u}^{-1})$

$$\begin{aligned}
&\tilde{H}_{\mathcal{D},\beta/q_j}(z_\kappa, \alpha(\begin{pmatrix} a_j 1_2 & \\ & \bar{a}_j^{-1} 1_2 \end{pmatrix} x, g)) \\
&\quad = \sum_{a \in (\widehat{\mathcal{O}}/x_p M_{\mathcal{D}})^\times} \chi_0 \xi_0^c \tau_0(a) H_{\mathcal{D},\beta/q_j} \alpha(\begin{pmatrix} a_j 1_2 & \\ & \bar{a}_j^{-1} 1_2 \end{pmatrix} x, g \begin{pmatrix} a^{-1} & \\ & \bar{a} \end{pmatrix})) \\
&\quad = \sum_{\gamma \in B_1(\mathbf{Q}) \backslash G_1(\mathbf{Q})} \sum_{v \in V} \sum_{a \in (\widehat{\mathcal{O}}/x_p M_{\mathcal{D}})^\times} \chi_0 \xi_0^c \tau_0(a) F_{\beta/q_j}(z_\kappa; v, \gamma g \begin{pmatrix} a^{-1} & \\ & \bar{a} \end{pmatrix}, ry) \\
&\quad = \sum_{\gamma = \eta^{-1} \text{ or } \gamma = \eta \left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right) \eta^{-1}, n \in \mathbf{Q}} \sum_{v \in V} \sum_{a \in (\widehat{\mathcal{O}}/x_p M_{\mathcal{D}})^\times} \chi_0 \xi_0^c \tau_0(a) F_{\beta/q_j}(z_\kappa; v, \gamma g \begin{pmatrix} a^{-1} & \\ & \bar{a} \end{pmatrix}, ry) \\
&\quad = e(i \mathrm{Tr}\, \beta/q_j) \tau_{0,p}^2(q_j) \tau_0^2(a_j) q_j^{-\kappa} C_{\mathcal{D}}(\beta, r, u) \sum_{b \in \mathbf{Z}/D_{\mathcal{K}}} \sum_{n \in \mathbf{Q}} \tilde{f}_{\mathcal{D},b}(z_\kappa, \eta \left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right) \gamma_\infty) \\
&\qquad\qquad \times \sum_{v \in V} \omega_{r,\beta/q_j}(\eta \left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right) \gamma_\infty \left(\begin{smallmatrix} 1 & -b \\ & 1 \end{smallmatrix}\right)_f) \Phi_{\mathcal{D},\beta/q_j,y}(v),
\end{aligned}$$

where

$$\begin{aligned}
C_{\mathcal{D}}(\beta, r, u) := &\frac{(2\pi i)^{2\kappa} (2/\pi) D_{\mathcal{K}}^{-1/2} \chi\bar{\xi}(\det ru) |\det ru|_{\mathcal{K}}^2 \det \beta^{\kappa-2}}{4 \prod_{j=0}^1 (\kappa - 1 - j)! L^\Sigma(\kappa - j, \bar{\chi}\xi' \chi_{\mathcal{K}}^j)} \\
&\times \psi_p(-1) \bar{\chi}_p \xi_p'(\det \beta) |\det \beta|_p^{\kappa-2} \mathfrak{g}(\chi_p \bar{\xi}_p')^2 \bar{\chi}_p \xi_p'(p^{2u_p}) p^{u_p},
\end{aligned}$$

$\tilde{f}_{\mathcal{D},b}$ is the Siegel section

$$\tilde{f}_{\mathcal{D},b} := f_{\kappa-2,1} \tilde{f}_{u_p,1} \prod_{\ell \in \Sigma \backslash \{p\}} f_{b,\ell} \prod_{\ell \notin \Sigma} f_\ell^{sph} \in I_1(\tau_0/\lambda)$$

with $f_{b,\ell}$ the local Siegel section at ℓ previously denoted by f_b, and $\Phi_{\mathcal{D},h,y}$ is the Schwartz function

$$\Phi_{\mathcal{D},h,y} := \Phi_{h,\infty} \Phi_{h,\xi_p^c,x_p,y_p} \prod_{\ell \notin \Sigma \backslash \{p\}} \Phi_{h,\chi_\ell \bar{\xi}_\ell, M_{\mathcal{D}}, y_\ell} \prod_{\ell \notin \Sigma} \Phi_{0,y_\ell} \in \mathcal{S}(V \otimes \mathbf{A}).$$

Letting $\mathcal{E}_{\mathcal{D},b}(g) := E(\tilde{f}_{\mathcal{D},b}, z_\kappa; g)$ and $\Theta_{\mathcal{D},h,y}(r,g) := \Theta_h(r, g; \Phi_{\mathcal{D},h,y})$ we have

$$\begin{aligned}
&\tilde{H}_{\mathcal{D},\beta/q_j}(z_\kappa, \alpha(\begin{pmatrix} a_j 1_2 & \\ & \bar{a}_j^{-1} 1_2 \end{pmatrix} x, g)) \\
&\quad = e(i \mathrm{Tr}\, \beta/q_j) \tau_{0,p}^2(q_j) \tau_0^2(a_j) q_j^{-\kappa} C_{\mathcal{D}}(\beta, r, u) \sum_{b \in \mathbf{Z}/D_{\mathcal{K}}} \mathcal{E}_{\mathcal{D},b}(\gamma_\infty) \Theta_{\mathcal{D},\beta/q_j,ua_j}(r, \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right) \gamma_\infty).
\end{aligned}$$

Setting $\mathcal{E}_{\mathcal{D},b}(w) := j(\gamma_\infty, i)^{\kappa-2} \mathcal{E}_{\mathcal{D},b}(\gamma_\infty)$ and $\Theta_{\mathcal{D},h,y}(r,w) := j(\gamma_\infty, i)^2 \Theta_{\mathcal{D},h,y}(r, \gamma_\infty)$ we then have

$$\begin{aligned}
&\tilde{h}_{\mathfrak{D},\beta/q_j}(w; \alpha(\begin{pmatrix} a_j 1_2 & \\ & \bar{a}_j^{-1} 1_2 \end{pmatrix} x, 1)) \\
&\quad = e(i \mathrm{Tr}\, \beta/q_j) \tau_{0,p}^2(q_j) \tau_0^2(a_j) q_j^{-\kappa} C_{\mathcal{D}}(\beta, r, u) \sum_{b \in \mathbf{Z}/D_{\mathcal{K}}} \mathcal{E}_{\mathcal{D},b}(w) \Theta_{\mathcal{D},\beta/q_j,ua_j}(r, w + b),
\end{aligned}$$

and so

$$\tilde{g}_{\mathfrak{D},\beta}(w;x)=C_{\mathcal{D}}(\beta,r,u)\sum_{j=1}^{h_{\mathcal{K}}}\bar{\xi}_p(q_j)\bar{\xi}(a_j)\sum_{b\in\mathbf{Z}/D_{\mathcal{K}}}\mathcal{E}_{\mathcal{D},b}(w)\Theta_{\mathcal{D},\beta/q_j,ua_j}(r,w+b). \tag{11.8.1.e}$$

Let

$$\mathcal{F}_{\mathcal{D}}(k):=f_{\kappa-2,1}(k_\infty)\tilde{f}_{u_p,1}(k_p)\prod_{\ell\in\Sigma\setminus\{p\}}f_\ell^\dagger(k_\ell)\prod_{\ell\notin\Sigma}f_\ell^{sph}(k_\ell)\in I_1(\tau_0/\lambda)$$

and $\mathcal{E}_{\mathcal{D}}(g):=E(\mathcal{F}_{\mathcal{D}},z_\kappa;g)$. Put $\mathcal{E}_{\mathcal{D}}(w):=j(\gamma,i)^{\kappa-2}\mathcal{E}_{\mathcal{D}}(\gamma_\infty)$. It is easy to see that

$$\mathcal{E}_{\mathcal{D},b}(g)=\delta_{\mathcal{K}}^{2-\kappa}\mathcal{E}_{\mathcal{D}}(g\left(\begin{smallmatrix}1&-b\\&1\end{smallmatrix}\right)_f\begin{pmatrix}\delta_{\mathcal{K}}&\\&\bar{\delta}_{\mathcal{K}}^{-1}\end{pmatrix}_f),$$

and hence that

$$\mathcal{E}_{\mathcal{D},b}(w)=D_{\mathcal{K}}^{2-\kappa}\mathcal{E}_{\mathcal{D}}(\frac{w+b}{D_{\mathcal{K}}}).$$

Putting

$$\Theta_{\mathcal{D},\beta}(r,w;u):=\sum_{j=1}^{h_{\mathcal{K}}}\bar{\xi}_p(q_j)\bar{\xi}(a_j)\Theta_{\mathcal{D},\beta/q_j,ua_j}(r,D_{\mathcal{K}}w)$$

it then follows from (11.8.1.e) that

$$\tilde{g}_{\mathfrak{D},\beta}(w;x)=C_{\mathcal{D}}(\beta,r,u)D_{\mathcal{K}}^{2-\kappa}\sum_{b\in\mathbf{Z}/D_{\mathcal{K}}}\mathcal{E}_{\mathcal{D}}(\frac{w+b}{D_{\mathcal{K}}})\Theta_{\mathcal{D},\beta}(r,\frac{w+b}{D_{\mathcal{K}}};u), \tag{11.8.1.f}$$

and hence

$$\begin{aligned}&<g_{\mathfrak{D},\beta}(-,x),f^c|_\kappa\begin{pmatrix}&-1\\p^{r_p}M&\end{pmatrix}>_{\Gamma_{\mathfrak{D}}}\\&=(M_{\mathcal{D}}M_1)^{\kappa/2}<\tilde{g}_{\mathfrak{D},\beta}(-,x),f^c|_\kappa\begin{pmatrix}&-1\\p^{r_p}M_{\mathcal{D}}^2&\end{pmatrix}>_{\Gamma_0(p^{r_p}M_{\mathcal{D}}^2)}\\&=C_{\mathcal{D}}(\beta,r,u)D_{\mathcal{K}}^{2-\kappa/2}(M_{\mathcal{D}}M_1)^{\kappa/2}\\&\qquad\times<\mathcal{E}_{\mathcal{D}}(-)\Theta_{\mathcal{D},\beta}(r,-;u),f^c|_\kappa\begin{pmatrix}&-1\\p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}}&\end{pmatrix}>_{\Gamma_0(p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}})}.\end{aligned} \tag{11.8.1.g}$$

<u>*Some more normalizations*</u>

Let

$$B_{\mathfrak{D},1}:=\frac{(\kappa-3)!L^\Sigma(\kappa-2,\bar{\chi}\xi')}{-2(2\pi i)^{\kappa-2}\mathfrak{g}(\chi_p\bar{\xi}_p')\bar{\chi}_p\xi_p'(p^{u_p})p^{2u_p}},$$

$$B_{\mathfrak{D},2}:=\frac{2^3i^{-2}D_{\mathcal{K}}^{3/2-\kappa/2}(M_{\mathcal{D}}M_1)^{\kappa/2}\prod_{\ell\in\Sigma\setminus\{p\}}\chi_\ell\bar{\xi}_\ell^c(y_\ell\delta_{\mathcal{K}})\mathfrak{g}(\bar{\chi}_\ell\xi_\ell^c,y_\ell\delta_{\mathcal{K}})|y_\ell\delta_{\mathcal{K}}|_{\mathcal{K}}^{2-\kappa}}{\bar{\chi}_p\xi_p^c(M_{\mathcal{D}})M_{\mathcal{D}}^{2\kappa}\mathfrak{g}(\xi_p^c,x_p)},$$

and

$$B_{\mathfrak{D}}(\beta,r,u):=\frac{\psi\bar{\xi}(\det ru)|\det ru|_{\mathcal{K}}^2\bar{\chi}_p\xi_p'(\det\beta)|\det\beta|_p^\kappa\det\beta^{\kappa-2}}{\prod_{\ell\in\Sigma\setminus\{p\}}\bar{\chi}_\ell\xi_\ell'(\ell^{e_\ell})\ell^{e_\ell(2-\kappa)}\mathfrak{g}(\chi_\ell\bar{\xi}_\ell')}.$$

Then

$$B_{\mathfrak{D}}^{(2)}C_{\mathcal{D}}(\beta,r,u)D_{\mathcal{K}}^{2-\kappa/2}(M_{\mathcal{D}}M_1)^{\kappa/2}=B_{\mathfrak{D}}(\beta,r,u)B_{\mathfrak{D},1}B_{\mathfrak{D},2}.$$

Putting

$$\mathcal{E}_{\mathfrak{D}}(w) := B_{\mathfrak{D},1}\mathcal{E}_{\mathcal{D}}(w) \quad \text{and} \quad \Theta_{\mathfrak{D},\beta}(r,w;u) := B_{\mathfrak{D},2}\Theta_{\mathcal{D},\beta}(r,w;u),$$

we then have

$$\begin{aligned} < f_{\mathfrak{D},\beta,x}(-), & f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}} \\ &= B^{(2)}_{\mathfrak{D}} < g_{\mathfrak{D},\beta}(-;x), f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}} \\ &= B_{\mathfrak{D}}(\beta,r,u) < \mathcal{E}_{\mathfrak{D}}(-)\Theta_{\mathfrak{D},\beta}(r,-;u), f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}})} . \end{aligned}$$

Combining this with Corollary 11.7.6 yields the following.

Proposition 11.8.2. *Let $\mathfrak{D} = (f,\psi,\xi,\Sigma)$ be a classical datum such that* (11.8.1.a) *holds. Let $\beta \in S_2(\mathbf{Q})$, $\beta > 0$, and $u \in \mathrm{GL}_2(\mathbf{A}_{\mathcal{K},f})$ be such that* (11.8.1.c) *and* (11.8.1.d) *hold. Let $h \in U_\beta(\mathbf{A}_f)$. Then for $x := \mathrm{diag}(hu, {}^t\bar{h}^{-1t}\bar{u}^{-1})$*

$$\begin{aligned} &\frac{C_{\mathfrak{D}}(\beta,x)}{2^{-3}(2i)^{\kappa+1}S(f) < f, f^c|_\kappa \begin{pmatrix} & -1 \\ N & \end{pmatrix} >_{\Gamma_0(N)}} \\ &\qquad = B_{\mathfrak{D}}(\beta,h,u) \frac{< \mathcal{E}_{\mathfrak{D}}(-)\Theta_{\mathfrak{D},\beta}(h,-;u), f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}})}}{< f, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}}}. \end{aligned}$$

11.9. **Identifying $\mathcal{E}_{\mathfrak{D}}$ and $\Theta_{\mathfrak{D},\beta}(h,-;u)$.** Keeping to the conventions, assumptions, and notation of the preceding section, we identify $\mathcal{E}_{\mathfrak{D}}(w)$ and $\Theta_{\mathfrak{D},\beta}(h,w;u)$ as essentially familiar and much-studied modular forms and highlight some of their properties (for the theta functions we only do this for special β's).

11.9.1. *Identifying $\mathcal{E}_{\mathfrak{D}}$.* Let $\mathfrak{D} = (f,\psi,\xi,\Sigma)$ be as in 11.8.1 (so in particular, ψ_0 and τ_0 are as in the Generic Case of 9.2.5). We first note that

(11.9.1.a)
$$\begin{aligned} \mathcal{E}_{\mathfrak{D}}(w) &= \sum_{n=1}^{\infty} a(n)e(nw), \\ a(n) = \bar{\chi}_p\xi'_p(n) &\sum_{\substack{d|n_1 \\ (\ell,d)=1\ \forall \ell\in\Sigma}} \bar{\chi}\xi'(d)(n_1/d)^{\kappa-3}, \quad n = p^m n_1, p\nmid n_1. \end{aligned}$$

Here $\bar{\chi}\xi'$ is being viewed as a Dirichlet character modulo $p^{u_p}M_{\mathcal{D}}$. This formula for the Fourier coefficients of $\mathcal{E}_{\mathfrak{D}}(w)$ follows from the computations of the local Fourier coefficients of $\mathcal{E}_{\mathcal{D}}(g) = E(\mathcal{F}_{\mathcal{D}}, z_\kappa; g)$ in 11.4.

Let $a \in \mathrm{GL}_2(\mathbf{A}_f)$ be defined by $a_\ell = \begin{pmatrix} & -1 \\ M_{\mathcal{D}}^2D_{\mathcal{K}} & \end{pmatrix}$ if $\ell \in \Sigma\backslash\{p\}$ and $a_\ell = 1$ otherwise. For $m \geq 0$ let $b_m \in \mathrm{GL}_2(\mathbf{A}_f)$ be defined by $b_{m,p} = \begin{pmatrix} & -1 \\ p^m & \end{pmatrix}$ and $b_{m,\ell} = 1$ if $\ell \neq p$. Let

$\alpha \in \mathrm{GL}_2^+(\mathbf{Q}) \cap U'(p^{r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}})a$ and $\rho_m \in \mathrm{GL}_2(\mathbf{Q})^+ \cap U'(p^{r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}})b_m$. We have

$$\begin{aligned}\mathcal{E}_{\mathfrak{D}}|_{\kappa-2}\alpha(w) &= B_{\mathfrak{D},1} j(\alpha, w)^{2-\kappa} \det(\alpha)^{\kappa/2-1} \mathcal{E}_{\mathcal{D}}(\alpha(w)) \\ &= B_{\mathfrak{D},1} j(\gamma_\infty, i)^{\kappa-2} \mathcal{E}_{\mathcal{D}}(\alpha\gamma_\infty) \\ &= B_{\mathfrak{D},1} j(\gamma_\infty, i)^{\kappa-2} E(\mathcal{F}'_{\mathcal{D}}, z_\kappa; \gamma_\infty),\end{aligned}$$

where $\mathcal{F}'_{\mathcal{D}}(z,g) := \mathcal{F}_{\mathcal{D}}(z, g\alpha_f^{-1}) \in I_1(\tau_0/\lambda)$. It follows that $\mathcal{F}'_{\mathcal{D}}(z,g)$ is supported on $B_1(\mathbf{A})\eta K_{1,\infty}^+ N_{B_1}(\widehat{\mathbf{Z}})\alpha_f = B_1(\mathbf{A}) K_{1,\infty}^+ K_1(p^{u_p} M_{\mathcal{D}}^2 D_{\mathcal{K}})$ and that for $g = bk_\infty k_f$ in the support we have

$$\mathcal{F}'_{\mathcal{D}}(z,g) = (M_{\mathcal{D}}^2 D_{\mathcal{K}})^{\kappa/2-1} \tau_0 \bar{\lambda}(d_b d_{k_f}) |a_b/d_b|_{\mathbf{A}}^{z+1/2} J_1(k_\infty, i)^{2-\kappa}.$$

If $g \in \mathrm{SL}_2(\mathbf{R}) K_1(p^{u_p} M_{\mathcal{D}}^2 D_{\mathcal{K}})$ then $\mathcal{F}'_{\mathcal{D}}(z_\kappa, g) = (M_{\mathcal{D}}^2 D_{\mathcal{K}})^{\kappa/2-1} \tau_0 \bar{\lambda}(d_{g_f}) J_1(g_\infty, i)^{2-\kappa}$, and so

(11.9.1.b)

$$\begin{aligned}\mathcal{E}_{\mathfrak{D}}|_{\kappa-2}\alpha(w) &= B_{\mathfrak{D},1} (M_{\mathcal{D}}^2 D_{\mathcal{K}})^{\kappa/2-1} j(\gamma_\infty, i)^{\kappa-2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(p^{u_p} M_{\mathcal{D}}^2 D_{\mathcal{K}})} \chi\bar{\xi}'(d_\gamma) j(\gamma\gamma_\infty, i)^{2-\kappa} \\ &= B_{\mathfrak{D},1} (M_{\mathcal{D}}^2 D_{\mathcal{K}})^{\kappa/2-1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(p^{u_p} M_{\mathcal{D}}^2 D_{\mathcal{K}})} \chi\bar{\xi}'(d_\gamma) j(\gamma, w)^{2-\kappa},\end{aligned}$$

where $\Gamma_\infty := N_{B'}(\mathbf{Z})$. Put

$$\mathcal{E}'_{\mathfrak{D}} := \mathcal{E}_{\mathfrak{D}}|_{\kappa-2}\alpha.$$

We also have

(11.9.1.c)
$$\begin{aligned} f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}} & \end{pmatrix} \alpha(w) &= j(\gamma_\infty, i)^\kappa f_{\mathbf{A}}^c\left(\gamma_\infty \alpha_f^{-1} \begin{pmatrix} & p^{-r_p} M_{\mathcal{D}}^{-2} D_{\mathcal{K}}^{-1} \\ -1 & \end{pmatrix}_f\right) \\ &= j(\gamma_\infty)^\kappa f_{\mathbf{A}}^c(\gamma_\infty \rho_{r_p,f}^{-1}) \chi_p(-p^{-r_p})(-1)^\kappa \\ &= (-1)^\kappa \chi_p(-1) \chi_p(p)^{r_p} f^c|_\kappa \rho_{r_p}(w),\end{aligned}$$

(11.9.1.d)

$$\begin{aligned}\sum_{a \in \mathbf{Z}/p^{r_p - u_p}} f^c|_\kappa \rho_{r_p} \begin{pmatrix} 1 & \\ ap^{u_p} M_{\mathcal{D}}^2 D_{\mathcal{K}} & 1 \end{pmatrix}(w) &= \sum_{a \in \mathbf{Z}/p^{r_p - u_p}} f^c|_\kappa \begin{pmatrix} 1 & -ap^{u_p - r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}} \\ & 1 \end{pmatrix} \rho_{r_p}(w) \\ &= \sum_{a \in \mathbf{Z}/p^{r_p - u_p}} j(\rho_{r_p}, i)^{-\kappa} \det \rho_{r_p}^{\kappa/2} f^c(\rho_{r_p}(w) + aM_{\mathcal{D}}^2 D_{\mathcal{K}}/p^{r_p - u_p}) \\ &= (p^{1-\kappa/2} \chi_p(p) a(p,f))^{r_p - u_p} f^c|_\kappa \rho_{u_p}(w),\end{aligned}$$

and

(11.9.1.e)
$$\begin{aligned} f^c|_\kappa \rho_{u_p}(w) &= p^{(u_p - r)\kappa/2} f^c|_\kappa \rho_r(p^{u_p - r} w) \\ &= c(f) p^{(u_p - r)\kappa/2} f_1(p^{u_p - r} w),\end{aligned}$$

where f_1 is the newform associated with $f^c|_\kappa \rho_r$; since f^c is new of level $p^r M$ it follows that $f^c|_\kappa \rho_r = c(f) f_1$ for some non-zero constant $c(f)$ (note that $\pi(f_{1,\mathbf{A}}) = \pi(f_{\mathbf{A}}^c) \otimes \chi/\chi_0 = \pi(f_{\mathbf{A}}) \otimes \chi_0^{-1}$). The constant $c(f)$ can be expressed in terms of root numbers and characters, but we will have no need of this.

Let $h \in M_2(p^{u_p} M_{\mathcal{D}}^2 D_{\mathcal{K}}, \chi_0^{-2}\xi')$. Then by (11.9.1.b), (11.9.1.c), (11.9.1.d), and the usual unfolding

$$\begin{aligned}
< \mathcal{E}_{\mathfrak{D}} \cdot h|_2\alpha^{-1}, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}})} \\
&= (-1)^\kappa \chi_p(-1)\chi_p(p)^{-r_p} < \mathcal{E}_{\mathfrak{D}}|_{\kappa-2}\alpha \cdot h, f^c|_\kappa \rho_{r_p} >_{\Gamma_0(p^{r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}})} \\
&= (-1)^\kappa \chi_p(-1)\chi_p(p)^{-u_p}(p^{1-\kappa/2}a(p,f))^{r_p-u_p} \\
&\qquad \times < \mathcal{E}'_{\mathfrak{D}} \cdot h, f^c|_\kappa \rho_{u_p} >_{\Gamma_0(p^{u_p} M_{\mathcal{D}}^2 D_{\mathcal{K}})} \\
&= (-1)^\kappa \chi_p(-1) B_{\mathfrak{D},1}(M_{\mathcal{D}}^2 D_{\mathcal{K}})^{\kappa/2-1}(4\pi)^{1-\kappa}(\kappa-2)!\chi_p(p)^{-u_p}(p^{1-\kappa/2}a(p,f))^{r_p-u_p} \\
&\qquad \times D(f^c|_\kappa \rho_{u_p}, h; \kappa-1),
\end{aligned}$$

where

$$D(f^c|_\kappa \rho_{u_p}, h; s) = \sum_{n=1}^\infty \overline{a(n, f^c|_\kappa \rho_{u_p})} a(n,h) n^{-s}.$$

If h is a normalized eigenform (so $a(1,h) = 1$) then

$$D(f^c|_\kappa \rho_{u_p}, h; \kappa-1) = \overline{c(f)} p^{-(u_p-r)(\kappa/2-1)} a(p,h)^{u_p-r} L^\Sigma(\kappa-2, \bar{\chi}\xi')^{-1} L(f_1^c \times h, \kappa-1),$$

where $L(f_1^c \times h, s)$ is the usual product L-function associated with two eigenforms. This proves the following lemma.

Lemma 11.9.2. *Suppose $\mathfrak{D}$ satisfies* (11.8.1.a). *If $h \in M_2(p^{u_p} M_{\mathcal{D}}^2 D_{\mathcal{K}}, \chi_0^{-2}\xi')$ is a normalized eigenform (so $a(1,h) = 1$) then*

$$< \mathcal{E}_{\mathfrak{D}} \cdot h|_2\alpha^{-1}, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}}} = B_{\mathfrak{D},3} L(f_1^c \times h, \kappa-1)$$

where

$$B_{\mathfrak{D},3} := \frac{\overline{c(f)}(-1)^\kappa \chi_p(-p^{-u_p}) B_{\mathfrak{D},1}(M_{\mathcal{D}}^2 D_{\mathcal{K}})^{\kappa/2-1}(\kappa-2)! p^{(r_p-r)(1-\kappa/2)} a(p,f)^{r_p-u_p} a(p,h)^{u_p-r}}{(4\pi)^{\kappa-1} L^\Sigma(\kappa-2, \bar{\chi}\xi')}$$

and f_1 is the newform such that $\pi(f_{1,\mathbf{A}}) = \pi(f_{\mathbf{A}}) \otimes \chi_0^{-1}$. In particular, if $a(p,h) \neq 0$ and $\kappa \geq 3$ then this Petersson-product is non-zero.

The non-vanishing of the Rankin-Selberg product when $\kappa \geq 3$ is a simple consequence of the Ramanujan bounds on the eigenvalues of the eigenforms f_1 and h. The non-vanishing of the remaining factors when $\kappa \geq 3$ is clear from inspection.

Remark. Our subsequent use of this lemma will only invoke the non-vanishing. The expression for the constant $B_{\mathfrak{D},3}$ can be rewritten in a form more clearly related to the interpolating factors that appear in the various p-adic interpolation formulas for Rankin-Selberg products, but we have no need of this.

11.9.3. *Identifying sums of $\Theta_{\mathfrak{D},\beta}(h,w;u)$'s.* For simplicity we will assume

(11.9.3.a)
- χ_ℓ, ψ_ℓ, and ξ_ℓ are unramified if $\ell \neq p$;
- $\xi = \xi_1\xi_2$ with each ξ_i unramified at all $v \nmid p$, $\xi_{i,\infty}(z) = 1$, and $\xi_1|_{\mathcal{O}_p^\times} = (\xi_{p,1}, 1)$ and $\xi_2|_{\mathcal{O}_p^\times} = (1, \xi_{p,2})$.

Let $\gamma_0 \in \mathrm{GL}_2(\mathbf{A}_{\mathcal{K},f})$ be such that $\gamma_{0,p} = (1,\eta)$ and $\gamma_{0,\ell} = 1$ if $\ell \neq p$. For $i \leq i, j \leq h_{\mathcal{K}}$ we let $\beta_{ij} := \begin{pmatrix} q_j^{-1} & \\ & q_i q_j^{-1} \end{pmatrix}$ and $u_{ij} := \gamma_0 \begin{pmatrix} a_j & \\ & a_i^{-1} a_j \end{pmatrix}$. Then $\beta = \beta_{ij}$ and $u = u_{ij}$ satisfy (11.8.1.d).

Recall that

$$\text{(11.9.3.b)} \qquad \Phi_{\mathcal{D},\beta_{ij},u_{ij}} = \Phi_{\beta_{ij},\infty} \Phi_{\beta_{ij},\xi_p^c,x_p,\gamma_{0,p}} \prod_{\ell \in \Sigma \backslash \{p\}} \Phi_{\beta_{ij},1,M_{\mathcal{D}},1} \prod_{\ell \in \mathcal{Q}} \Phi_{0,u_{ij,\ell}} \prod_{\ell \notin \Sigma \cup \mathcal{Q}} \Phi_0.$$

Let $x := {}^t(x_1, x_2) \in V$ (so $x_1, x_2 \in \mathcal{K}$). We observe that if $g_\infty \in \mathrm{SL}_2(\mathbf{R})$ is such that $g_\infty(i) = D_{\mathcal{K}} w$ then

$$\omega_{\beta_{ij}}(g_\infty)\Phi_{\beta_{ij},\infty}(x) = e(\frac{\mathrm{Nm}(x_1)}{q_j} D_{\mathcal{K}} w) e(\frac{\mathrm{Nm}(x_2) q_i}{q_j} D_{\mathcal{K}} w) j(g_\infty, i)^{-2}.$$

By part (i) of Lemma 10.2.6 (with $\theta = \xi^c$, $\theta_1 = \xi_{p,2}$, and $\theta_2 = \xi_{1,p}$)

$$\Phi_{\beta_{ij},\xi_p^c,x_p,\gamma_{0,p}}(x) = \begin{cases} \bar{\xi}_{p,2}(q_j^{-1} x_1'') \mathfrak{g}(\xi_{p,2}) \bar{\xi}_{p,1}(q_i q_j^{-1} x_2') \mathfrak{g}(\xi_{p,1}) & x_1 = (x_1', x_1'') \in \mathbf{Z}_p \times \mathbf{Z}_p^\times, \\ & x_2 = (x_2', x_2'') \in \mathbf{Z}_p^\times \times \mathbf{Z}_p \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 10.2.5, if $\ell \in \Sigma \backslash \{p\}$ does not split in $\mathcal{K}$ then

$$\Phi_{\beta_{ij},1,M_{\mathcal{D}},1}(x) = D_\ell^{-1} |M_{\mathcal{D}}^2|_\ell^{-1} \begin{cases} 1 - 1/q_\ell & x_1 \in \frac{M_{\mathcal{D}}}{\delta_{\mathcal{K}}} \mathcal{O}_\ell, x_2 \in \frac{1}{\delta_{\mathcal{K}}} \mathcal{O}_\ell \\ -1/q_\ell & x_1 \in \frac{M_{\mathcal{D}}}{\delta_{\mathcal{K}} \varpi_\ell} \mathcal{O}_\ell^\times, x_2 \in \frac{1}{\delta_{\mathcal{K}}} \mathcal{O}_\ell \\ 0 & \text{otherwise,} \end{cases}$$

where ϖ_ℓ is a uniformizer at ℓ and q_ℓ is the order of the residue field $\mathcal{O}/\varpi_\ell \mathcal{O}$. Similarly, if $\ell \in \Sigma \backslash \{p\}$ and ℓ splits in $\mathcal{K}$ then

$$\Phi_{\beta_{ij},1,M_{\mathcal{D}},1}(x) = D_\ell^{-1} |M_{\mathcal{D}}^2|_\ell^{-1} \begin{cases} (1 - 1/\ell)^2 & x_1 \in M_{\mathcal{D}} \mathcal{O}_\ell, x_2 \in \mathcal{O}_\ell \\ -1/\ell(1 - 1/\ell) & x_2 \in \mathcal{O}_\ell, x_1 \in \frac{M_{\mathcal{D}}}{\ell} \mathbf{Z}_\ell^\times \times \mathbf{Z}_\ell^\times \\ -1/\ell(1 - 1/\ell) & x_2 \in \mathcal{O}_\ell, x_1 \in \mathbf{Z}_\ell^\times \times \frac{M_{\mathcal{D}}}{\ell} \mathbf{Z}_\ell^\times \\ 1/\ell^2 & x_2 \in \mathcal{O}_\ell, x_1 \in \frac{M_{\mathcal{D}}}{\ell} \mathbf{Z}_\ell^\times \times \frac{M_{\mathcal{D}}}{\ell} \mathbf{Z}_\ell^\times \\ 0 & \text{otherwise.} \end{cases}$$

If $\ell \notin \Sigma$ but $\ell \neq q_i, q_j$ then

$$\Phi_{0,u_{ij,\ell}}(x) = \Phi_0(x) = \begin{cases} 1 & x_1, x_2 \in \mathcal{O}_\ell \\ 0 & \text{otherwise,} \end{cases}$$

and if $\ell = q_i$ or q_j then

$$\Phi_{0,u_{ij,\ell}}(x) = \begin{cases} 1 & x_1 \in a_j \mathcal{O}_\ell, x_2 \in a_j a_i^{-1} \mathcal{O}_\ell \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Sigma_{\mathcal{K}}$ be the set of places of $\mathcal{K}$ dividing the primes in $\Sigma\backslash\{p\}$. For $v \in \Sigma_{\mathcal{K}}$ let ϖ_v be a uniformizer of $\mathcal{O}_v$ and let $q_v := \#(\mathcal{O}_v/\varpi_v\mathcal{O}_v)$. For $g \in U_1(\mathbf{R})$ and $x \in \mathcal{K}$ let

$$\Phi_j(g,x) = \sum_{\mathcal{S}\subseteq\Sigma_{\mathcal{K}}} \Phi_{j,\mathcal{S}}(g,x),$$

where

$$\Phi_{j,\mathcal{S}}(g,x) = j(g,i)^{-1}e(\frac{\mathrm{Nm}(x)}{q_j}w)M_{\mathcal{D}}^2 D_{\mathcal{K}}^{-1}\bar{\xi}_{p,2}(x)\mathfrak{g}(\xi_{p,2})\prod_{v\in\mathcal{S}}(-1/q_v)$$
$$\times \prod_{v\in\Sigma_{\mathcal{K}},v\notin\mathcal{S}}(1-1/q_v)\times\begin{cases}1 & x\in\frac{a_jM_{\mathcal{D}}}{\delta_{\mathcal{K}}\prod_{v\in\mathcal{S}}\varpi_v}\widehat{\mathcal{O}} \\ & x\in\mathbf{Z}_p\times\mathbf{Z}_p^\times \\ & x\in\delta_{\mathcal{K}}^{-1}\varpi_v^{-1}\mathcal{O}_v^\times, v\in\mathcal{S} \\ & x\in\mathcal{O}_v, v\in\Sigma_K/\mathcal{S} \\ 0 & \text{otherwise.}\end{cases}$$

Let

$$\Phi_{ij}(g,x) = j(g,i)^{-1}e(\frac{\mathrm{Nm}(x)q_i}{q_j}w)\bar{\xi}_{p,1}(x)\mathfrak{g}(\xi_{p,1})\begin{cases}1 & x\in\frac{a_j}{a_i\delta_{\mathcal{K}}}\widehat{\mathcal{O}}, x\in\mathbf{Z}_p^\times\times\mathbf{Z}_p \\ 0 & \text{otherwise.}\end{cases}$$

Putting everything together, we find that for $x = {}^t(x_1,x_2)\in V$

$$\xi_{p,1}(q_i)\xi_1(a_i)\bar{\xi}_p(q_j)\bar{\xi}(a_j)\omega_{\beta_{ij}}(g_\infty)\Phi_{\mathcal{D},\beta_{ij},u_{ij}}(x) = \bar{\xi}_2(a_j)\Phi_j(g_\infty, M_{\mathcal{D}}x_1)\bar{\xi}_1(a_j/a_i)\Phi_{ij}(g_\infty,x_2).$$

Let

(11.9.3.c) $$\beta_i := \begin{pmatrix}1 & \\ & q_i\end{pmatrix} \quad\text{and}\quad u_i := \gamma_0\begin{pmatrix}1 & \\ & a_i^{-1}\end{pmatrix}.$$

Let $\Theta_{\mathcal{D},\beta_i}(w) := \Theta_{\mathcal{D},\beta_i}(1,w;u_i)$. Then it follows that

(11.9.3.d)
$$\sum_{i=1}^{h_{\mathcal{K}}}\xi_{p,1}(q_i)\xi_1(a_i)\Theta_{\mathcal{D},\beta_i}(w)$$
$$= \sum_{i=1}^{h_{\mathcal{K}}}\xi_{p,1}(q_i)\xi_1(a_i)\sum_{j=1}^{h_{\mathcal{K}}}\bar{\xi}_p(q_j)\bar{\xi}(a_j)\Theta_{\mathcal{D},\beta_{ij},u_{ij}}(1,D_{\mathcal{K}}w)$$
$$= \sum_{j=1}^{h_{\mathcal{K}}}\bar{\xi}_2(a_j)j(g_\infty,i)\sum_{x_1\in\mathcal{K}}\Phi_j(g_\infty,x_1)\sum_{i=1}^{h_{\mathcal{K}}}\bar{\xi}_1(a_j/a_i)j(g_\infty,i)\sum_{x_2\in\mathcal{K}}\Phi_{ij}(g_\infty,x_2).$$

We have

$$\begin{aligned}
&\sum_{i=1}^{h_{\mathcal{K}}}\bar{\xi}_1(a_j/a_i)j(g_\infty,i)\sum_{x\in\mathcal{K}}\Phi_{ij}(g_\infty,x)\\
&\qquad=\mathfrak{g}(\xi_{p,1})\sum_{i=1}^{h_{\mathcal{K}}}\sum_{x\in\mathcal{K}\cap\frac{a_j}{a_i\delta_{\mathcal{K}}}\widehat{\mathcal{O}},x\in\mathbf{Z}_p^\times\times\mathbf{Z}_p}\bar{\xi}_{p,1}(x)e(\frac{\mathrm{Nm}(x)q_i}{q_j}D_{\mathcal{K}}w)\\
&\qquad=\mathfrak{g}(\xi_{p,1})\xi_{p,1}(\delta_{\mathcal{K}})\sum_{i=1}^{h_{\mathcal{K}}}\xi_1(a_i/a_j)\sum_{x\in K\cap\frac{a_j}{a_i}\widehat{\mathcal{O}},x\in\mathbf{Z}_p^\times\times\mathbf{Z}_p}\bar{\xi}_{p,1}(x)e(\frac{\mathrm{Nm}(x)q_i}{q_j}w)\\
&\qquad=\mathfrak{g}(\xi_{p,1})\xi_{p,1}(\delta_{\mathcal{K}})g_{\xi_1}(w),
\end{aligned}$$

where g_{ξ_1} is the usual newform associated with the idele class character ξ_1. Then the last line of (11.9.3.d) equals

$$\mathfrak{g}(\xi_{p,1})\xi_{p,1}(\delta_{\mathcal{K}})g_{\xi_1}(w)\sum_{j=1}^{h_{\mathcal{K}}}\bar{\xi}_2(a_j)j(g_\infty,i)\sum_{x\in\mathcal{K}}\Phi_j(g_\infty,x). \tag{11.9.3.e}$$

For $\mathcal{S}\subseteq\Sigma_{\mathcal{K}}$ let $M_{\mathcal{S}}:=\prod_{v\in\mathcal{S}}q_v$, $Q_{\mathcal{S}}:=M_{\mathcal{S}}^{-1}\prod_{v\in\Sigma_K/\mathcal{S}}(1-1/q_v)$ and $\varpi_{\mathcal{S}}:=(\varpi_v)_v\in\prod_{v\in\mathcal{S}}\mathcal{K}_v\subset\mathbf{A}_{\mathcal{K}}$. Let $X_{j,\mathcal{S}}:=\{x\in\mathcal{K}\cap\frac{a_j}{\varpi_{\mathcal{S}}}\widehat{\mathcal{O}}\;:\;x\in\mathbf{Z}_p\times\mathbf{Z}_p^\times\}$ and $Y_{j,\mathcal{S}}:=\{x\in X_{j,\mathcal{S}}\;:\;x\in\frac{1}{\varpi_v}\mathcal{O}_v^\times,v\in\mathcal{S}\}$. Then

$$\begin{aligned}
&\sum_{j=1}^{h_{\mathcal{K}}}\bar{\xi}_2(a_j)j(g_\infty,i)\sum_{x\in\mathcal{K}}\Phi_j(g_\infty,x)\\
&\quad=\mathfrak{g}(\xi_{p,2})M_{\mathcal{D}}^2D_{\mathcal{K}}^{-1}\sum_{\mathcal{S}\subseteq\Sigma_{\mathcal{K}}}(-1)^{\#\mathcal{S}}Q_{\mathcal{S}}\sum_{j=1}^{h_{\mathcal{K}}}\bar{\xi}_2(a_j)\sum_{x\in\frac{M_{\mathcal{D}}}{\delta_{\mathcal{K}}}Y_{j,\mathcal{S}}}\bar{\xi}_{p,2}(x)e(\frac{\mathrm{Nm}(x)}{q_j}D_{\mathcal{K}}w)\\
&\quad=\mathfrak{g}(\xi_{p,2})\xi_{p,2}(\delta_{\mathcal{K}}/M_{\mathcal{D}})M_{\mathcal{D}}^2D_{\mathcal{K}}^{-1}\sum_{\mathcal{S}\subseteq\Sigma_{\mathcal{K}}}(-1)^{\#\mathcal{S}}Q_{\mathcal{S}}\sum_{j=1}^{h_{\mathcal{K}}}\bar{\xi}_2(a_j)\sum_{x\in Y_{j,\mathcal{S}}}\bar{\xi}_{p,2}(x)e(\frac{\mathrm{Nm}(x)}{q_j}M_{\mathcal{D}}^2w),
\end{aligned}$$

and

$$\sum_{j=1}^{h_{\mathcal{K}}} \bar{\xi}_2(a_j) \sum_{x\in Y_{j,\mathcal{S}}} \bar{\xi}_{p,2}(x) e(\frac{\mathrm{Nm}(x)}{q_j} M_{\mathcal{D}}^2 w)$$

$$= \sum_{j=1}^{h_{\mathcal{K}}} \bar{\xi}_2(a_j) \sum_{\mathcal{S}'\subseteq\mathcal{S}} (-1)^{\#\mathcal{S}/\mathcal{S}'} \sum_{x\in X_{j,\mathcal{S}'}} \bar{\xi}_{p,2}(x) e(\frac{\mathrm{Nm}(x)}{q_j} M_{\mathcal{D}}^2 w)$$

$$= \sum_{\mathcal{S}'\subseteq\mathcal{S}} (-1)^{\#\mathcal{S}/\mathcal{S}'} \xi_2(\varpi_{\mathcal{S}'}) \sum_{j=1}^{h_{\mathcal{K}}} \bar{\xi}_2(a_j\varpi_{\mathcal{S}'}) \sum_{x\in X_{j,\mathcal{S}'}} \bar{\xi}_2(x) e(\frac{M_{\mathcal{D}}^2 \mathrm{Nm}(x)}{q_j} w)$$

$$= \sum_{\mathcal{S}'\subseteq\mathcal{S}} (-1)^{\#\mathcal{S}/\mathcal{S}'} \xi_2(\varpi_{\mathcal{S}'}) g_{\xi_2}(\frac{M_{\mathcal{D}}^2}{M_{\mathcal{S}'}} w),$$

where g_{ξ_2} is the newform associated with the idele class character ξ_2. It follows that

$$\sum_{j=1}^{h_{\mathcal{K}}} \bar{\xi}_2(a_j) j(g_\infty, i) \sum_{x\in\mathcal{K}} \Phi_j(g_\infty, x)$$

$$= \mathfrak{g}(\xi_{p,2}) \xi_{p,2}(\delta_{\mathcal{K}}/M_{\mathcal{D}}) M_{\mathcal{D}}^2 D_{\mathcal{K}}^{-1} \sum_{\mathcal{S}\subseteq\Sigma_{\mathcal{K}}} (-1)^{\#\mathcal{S}} Q_{\mathcal{S}} \sum_{\mathcal{S}'\subseteq\mathcal{S}} (-1)^{\#\mathcal{S}/\mathcal{S}'} \xi_2(\varpi_{\mathcal{S}'}) g_{\xi_2}(\frac{M_{\mathcal{D}}^2}{M_{\mathcal{S}'}} w).$$

Comparing this with (11.9.3.e) and (11.9.3.d) yields

$$\sum_{i=1}^{h_{\mathcal{K}}} \xi_{p,1}(q_i) \xi_1(a_i) \Theta_{\mathcal{D},\beta_i}(w)$$

$$= \mathfrak{g}(\xi_{p,1}) \mathfrak{g}(\xi_{p,2}) \xi_p(\delta_{\mathcal{K}}) \xi_{p,2}(M_{\mathcal{D}}) M_{\mathcal{D}}^2 D_{\mathcal{K}}^{-1}$$

$$\times g_{\xi_1}(w) \sum_{\mathcal{S}\subseteq\Sigma_{\mathcal{K}}} Q_{\mathcal{S}} \sum_{\mathcal{S}'\subseteq\mathcal{S}} (-1)^{\#\mathcal{S}'} \xi_2(\varpi_{\mathcal{S}'}) g_{\xi_2}(\frac{M_{\mathcal{D}}^2}{M_{\mathcal{S}'}} w).$$

Since $\Theta_{\mathfrak{D},\beta_i}(1, w; u_i) = B_{\mathfrak{D},2} \Theta_{\mathcal{D},\beta_i}(w)$ we therefore have

(11.9.3.f)

$$\sum_{i=1}^{h_{\mathcal{K}}} \xi_{p,1}(q_i) \xi_1(a_i) \Theta_{\mathfrak{D},\beta_i}(1, w; u_i) = B_{\mathfrak{D},4} \sum_{\mathcal{S}\subseteq\Sigma_{\mathcal{K}}} Q_{\mathcal{S}} \sum_{\mathcal{S}'\subseteq\mathcal{S}} (-1)^{\#\mathcal{S}'} \xi_1(\varpi_{\mathcal{S}'}) g_{\xi_1}(w) g_{\xi_2}(\frac{M_{\mathcal{D}}^2}{M_{\mathcal{S}'}} w),$$

where

$$B_{\mathfrak{D},4} := 2^3 i^{-1} D_{\mathcal{K}}^{-1/2} (D_{\mathcal{K}} M_{\mathcal{D}} M_1)^{\kappa/2} \chi_p(M_{\mathcal{D}})^2 M_{\mathcal{D}}^{2-2\kappa} \xi_p(\delta_{\mathcal{K}}).$$

As $g_{\xi_i} \in S_1(p^{r_p}, \chi_{\mathcal{K}}\xi_i')$, the following lemma is a consequence of (11.9.3.f).

Lemma 11.9.4. *If $h \in S_2(p^{r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}}, \xi')$ is new at all $\ell | M_{\mathcal{D}}^2 D_{\mathcal{K}}$ then*

$$< \sum_{i=1}^{h_{\mathcal{K}}} \xi_{p,1}(q_i) \xi_1(a_i) \Theta_{\mathfrak{D},\beta_i}(1, -; u_i), h >_{\Gamma_0(p^{r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}})}$$

$$= B_{\mathfrak{D},4} < g_{\xi_1}(-) g_{\xi_2}(M_{\mathcal{D}}^2(-)), h >_{\Gamma_0(p^{r_p} M_{\mathcal{D}}^2 D_{\mathcal{K}})} .$$

12. p-ADIC INTERPOLATIONS

In this section we combine the explicit formulas from 11.9 with general constructions involving p-adic families of modular forms to construct the p-adic L-functions and the p-adic Eisenstein series used in the proof of Theorem 3.6.1 and the other results in 3.6.

12.1. **p-adic families of Eisenstein data.** Recall (see 6.5) that a p-adic Eisenstein datum is a 6-tuple $\mathbf{D} = (A, \mathbb{I}, \mathbf{f}, \psi, \xi, \Sigma)$ consisting of

- the ring of integers A of a finite extension of $\mathbf{Q}_p$;
- a domain $\mathbb{I}$ that is a finite integral extension of $\Lambda_{W,A}$;
- an ordinary $\mathbb{I}$-adic newform $\mathbf{f}$ of some tame level M with associated A-valued Dirichlet character $\chi_{\mathbf{f}}$;
- a finite order A-valued idele class character ψ of $\mathbf{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ such that $\psi|_{\mathbf{A}^\times} = \chi_{\mathbf{f}}$;
- a finite order A-valued idele class character ξ of $\mathbf{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$;
- a finite set Σ of primes containing those that divide $MpD_{\mathcal{K}}$ as well as those ℓ such that ψ_ℓ or ξ_ℓ is ramified.

Recall also that $\Lambda_{\mathbf{D}} := \mathbb{I}[\![\Gamma_{\mathcal{K}}^- \times \Gamma_{\mathcal{K}}]\!] = \mathbb{I}_{\mathcal{K}}[\![\Gamma_{\mathcal{K}}^-]\!]$. Let $\mathcal{X}_{\mathbf{D}}^a \subset \mathcal{X}_{\Lambda_{\mathbf{D}},A}$ comprise those ϕ such that $\phi|\mathbb{I}_{\mathcal{K}} \in \mathcal{X}_{\mathbb{I}_{\mathcal{K}},A}^a$ (see 3.4.5) and $\phi|\Gamma_{\mathcal{K}}^-$ is a finite character. Following our conventions for $\mathcal{X}_{\mathbb{I}_{\mathcal{K}},A}^a$, we let κ_ϕ be the weight of $\phi|\mathbb{I}$, which we also call the weight of ϕ, and let $t_\phi := t_{\phi|\mathbb{I}}$ and $\chi_\phi := \chi_{\phi|\mathbb{I}}$. We also write $\mathbf{g}_\phi$ for $\mathbf{g}_{\phi|\mathbb{I}}$ for any $\mathbb{I}$-adic form $\mathbf{g}$. Note that our conventions for ϕ and $\phi|\mathbb{I}_{\mathcal{K}}$ are compatible.

In 6.5 we defined homomorphisms

$$\alpha : A[\![\Gamma_{\mathcal{K}}]\!] \to \mathbb{I}[\![\Gamma_{\mathcal{K}}^-]\!], \quad \alpha(\gamma_+) = (1+p)(1+W)^{1/2},\ \alpha(\gamma_-) = (1+p)(1+W)^{1/2}\gamma_-,$$

$$\beta : A[\![\Gamma_{\mathcal{K}}]\!] \to \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!], \quad \beta(\gamma_+) = (1+W)^{-1}\gamma_+, \quad \beta(\gamma_-) = \gamma_-.$$

These define a homomorphism

$$\alpha \otimes \beta : A[\![\Gamma_{\mathcal{K}} \times \Gamma_{\mathcal{K}}]\!] \to \mathbb{I}[\![\Gamma_{\mathcal{K}}^-]\!]\hat{\otimes}_{\mathbb{I}}\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!] = \Lambda_{\mathbf{D}}.$$

Let

$$\boldsymbol{\psi} := \alpha \circ \omega^{-1}\psi\Psi_{\mathcal{K}}^{-1} \quad \text{and} \quad \boldsymbol{\xi} := \beta \circ \chi_{\mathbf{f}}\xi\Psi_{\mathcal{K}}.$$

For $\phi \in \mathcal{X}_{\mathbf{D}}^a$ we define idele class characters of $\mathbf{A}_{\mathcal{K}}^\times$ by

$$\psi_\phi(x) := x_\infty^{-\kappa_\phi} x_{v_0}^{\kappa_\phi}(\phi \circ \boldsymbol{\psi})(x) \quad \text{and} \quad \xi_\phi := \phi \circ \boldsymbol{\xi}.$$

Then ξ_ϕ is a finite idele class character, and ψ_ϕ has infinity type $z^{-\kappa_\phi}$ and satisfies

$$\psi_\phi|_{\mathbf{A}^\times} = \chi_{\mathbf{f}_\phi}|\cdot|_{\mathbf{Q}}^{-\kappa_\phi}.$$

For each $\phi \in \mathcal{X}_{\mathbf{D}}^a$ we let

$$\mathfrak{D}_\phi := (\mathbf{f}_\phi, \psi_\phi, \xi_\phi, \Sigma).$$

This is a classical datum in the sense of 9.4. In this way the elements of $\mathcal{X}_{\mathbf{D}}^a$ correspond to certain classical Eisenstein data.

For reasons having to do with the hypotheses in force in the formulas in 11.7 and 11.9, we distinguish three subsets $\mathcal{X}_{\mathbf{D}}^{gen} \subseteq \mathcal{X}''_{\mathbf{D}} \subseteq \mathcal{X}'_{\mathbf{D}} \subseteq \mathcal{X}_{\mathbf{D}}^{a}$. These are defined as

$$\mathcal{X}'_{\mathbf{D}} := \{\phi \in \mathcal{X}_{\mathbf{D}}^{a} \ : \ p | \mathfrak{f}_{\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi}, p^{t_\phi} | \mathrm{Nm}(\mathfrak{f}_{\xi_\phi}), \mathfrak{f}_{\xi_\phi^c}\mathcal{O}_p \subseteq \mathfrak{f}_{\psi_\phi}\mathcal{O}_p\},$$

$\mathcal{X}''_{\mathbf{D}} := \{\phi \in \mathcal{X}'_{\mathbf{D}} \ : \ \kappa_\phi > 6\}$, and by defining $\mathcal{X}_{\mathbf{D}}^{gen}$ to comprise those $\phi \in \mathcal{X}''_{\mathbf{D}}$ such that $t_\phi \geq 2$ and such that the Eisenstein datum associated with $\mathfrak{D}_\phi$ has p-constituents as in the Generic Case of 9.2.5. It is readily checked that $\mathcal{X}_{\mathbf{D}}^{gen}$, and hence $\mathcal{X}''_{\mathbf{D}}$, $\mathcal{X}'_{\mathbf{D}}$, and $\mathcal{X}_{\mathbf{D}}^{a}$, are Zariski-dense subsets of $\mathrm{Spec}\,\Lambda_{\mathbf{D}}(\overline{\mathbf{Q}}_p)$.

For a p-adic datum $\mathbf{D}$ we choose $M_{\mathcal{D}}$ as in 11.5 for any Eisenstein datum $\mathcal{D}$ associated with some $\mathfrak{D}_\phi$, $\phi \in \mathcal{X}_{\mathbf{D}}^{a}$; we let $M_{\mathbf{D}} := M_{\mathcal{D}}$.

12.2. **Key facts, lemmas, and interpolations.** Here we collect the key ingredients that make our interpolations possible.

12.2.1. *Ordinary eigenforms and projectors.* Let $R \subset \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Z}_p$ and let $S_\kappa^{\mathrm{ord}}(Mp^r, \chi; R)$ and $M_\kappa^{\mathrm{ord}}(Mp^r, \chi; R)$ the the submodules of ordinary forms in $S_\kappa(Mp^r, \chi; R)$ and $M_\kappa(Mp^r, \chi; R)$, respectively. Let $\mathbf{T}_\kappa^{\mathrm{ord}}(Mp^r, \chi; R)$ be the R-subalgebra of $\mathrm{End}_R(M_\kappa^{\mathrm{ord}}(Mp^r, \chi; R))$ generated by the Hecke operators T_ℓ, ℓ a prime. In what follows we always assume $\kappa \geq 2$. Suppose $f \in S_\kappa^{\mathrm{ord}}(Mp^r, \chi; R)$ is a p-stabilized eigenform; either f is new or $r = 1$ and f is old at p but new at M. Let F be the field of fractions of R. Then $\mathbf{T}_\kappa^{\mathrm{ord}}(Mp^r, \chi; R) \otimes_R F \simeq \mathbb{T}' \times F$ where projection onto the second factor sends a Hecke operator to the eigenvalue of its action on f. We let $1_f \in \mathbf{T}_\kappa^{\mathrm{ord}}(Mp^r, \chi; R) \otimes_R F$ be the idempotent associated with the second factor. We let e denote Hida's ordinary projector (projecting modular forms onto the ordinary subspace). If $g \in M_\kappa(Mp^t, \chi; F)$, $t \geq r$, then $eg \in M_\kappa^{\mathrm{ord}}(Mp^r, \chi; F)$, so $1_f eg = cf$ for some $c \in F$; in particular, Hida has shown [Hi85, Prop. 4.5] that

$$1_f eg = \frac{< g, f^c|_\kappa \begin{pmatrix} & -1 \\ p^tM & \end{pmatrix} >_{\Gamma_0(p^tM)}}{< f, f^c_\kappa \begin{pmatrix} & -1 \\ p^tM & \end{pmatrix} >_{\Gamma_0(p^tM)}} f. \tag{12.2.1.a}$$

Let $\mathfrak{m} = \mathfrak{m}_f$ be the maximal ideal of $\mathbf{T}_\kappa^{\mathrm{ord}}(N, \chi; R)$ associated with f. If R is integrally closed or $\mathbf{T}_\kappa^{\mathrm{ord}}(N, \chi; R)_\mathfrak{m}$ is a Gorenstein R-algebra (a finite R-algebra T is a Gorenstein R-algebra if $\mathrm{Hom}_R(T, R)$ is a free T-module of rank one) then the intersection $\mathbf{T}^{\mathrm{ord}}(N, \chi; R) \cap (0 \times F)$ in $\mathbb{T}' \times F$ is a free R-module of rank one. In this case we let ℓ_f be an R-generator (this is well-defined only up to an element of $R^\times$). Then $\ell_f = \eta_f 1_f$ for some $\eta_f \in R$, and if $g \in M_\kappa^{\mathrm{ord}}(Mp^r, \chi; R)$ then $\ell_f g = cf$ for some $c \in R$.

Lemma 12.2.2. *Let R be a finite integral extension of $\mathbf{Z}_p$ and $f \in M_\kappa^{\mathrm{ord}}(Mp^r, \chi; R)$ be a p-stabilized newform. If $\bar{\rho}_f$ satisfies* (**irred**) *and* (**dist**) *then $\mathbf{T}^{\mathrm{ord}}(Mp^r, \chi; R)_{\mathfrak{m}_f}$ is a Gorenstein R-algebra and*

$$\eta_f = u_f \frac{2^{-3}(2i)^{\kappa+1} S(f) < f, f^c|_\kappa \begin{pmatrix} & -1 \\ p^rM & \end{pmatrix} >_{\Gamma_0(p^rM)}}{\Omega_f^+ \Omega_f^-}, \quad u_f \in R^\times,$$

where $S(f)$ *is as in* (11.7.3.b).

This is [Hi88c, Thm. 0.1] (combined with (4.6a,b) and (4.7) of *loc. cit.*

Let $\mathbb{I}$ be a domain that is a finite integral extension of $\Lambda_{W,\mathbf{Z}_p[\chi]}$, χ a Dirichlet character modulo some integer Mp with $p \nmid M$. We define $\mathcal{M}(M,\chi;\mathbb{I})$ and $\mathcal{M}^{\mathrm{ord}}(M,\chi;\mathbb{I})$ to be the $\mathbb{I}$-modules of $\mathbb{I}$-adic modular forms and ordinary $\mathbb{I}$-adic modular forms, respectively. Recall that the Hecke operators T_ℓ also act on $\mathcal{M}^{\mathrm{ord}}(M,\chi;\mathbb{I})$ in a manner that commutes with specialization (this action can be defined through the usual actions on q-expansions). Let $\mathbb{T}^{\mathrm{ord}}(M,\chi;\mathbb{I})$ be the $\mathbb{I}$-subalgebra of $\mathrm{End}_{\mathbb{I}}(\mathcal{M}^{\mathrm{ord}}(M,\chi;\mathbb{I}))$ generated by the Hecke operators T_ℓ.

Suppose $\mathbf{f} \in \mathcal{M}^{\mathrm{ord}}(M,\chi;\mathbb{I})$ is an ordinary $\mathbb{I}$-adic cuspidal newform. Then $\mathbb{T}^{\mathrm{ord}}(M,\chi;\mathbb{I})\otimes F_{\mathbb{I}} \cong \mathbb{T}' \times F_{\mathbb{I}}$, $F_{\mathbb{I}}$ being the fraction field of $\mathbb{I}$, where projection onto the second factor gives the eigenvalues for the actions on $\mathbf{f}$. Let $1_{\mathbf{f}}$ be the idempotent corresponding to the second factor. Then for any $\mathbf{g} \in \mathcal{M}^{\mathrm{ord}}(M,\chi;\mathbb{I}) \otimes_{\mathbb{I}} F_{\mathbb{I}}$, $1_{\mathbf{f}}\mathbf{g} = c\mathbf{f}$ for some $c \in F_{\mathbb{I}}$.

An element $c \in F_{\mathbb{I}}$ defines a rational function on $\mathcal{X}_{\mathbb{I}} = \mathrm{Spec}\,\mathbb{I}(\bar{\mathbf{Q}}_p)$; we denote the value of c at a ϕ where c is finite by $\phi(c)$ (the notation is consistent if $c \in \mathbb{I}$). As Hida has demonstrated (cf. [Hi88a]), for any $\mathbf{g} \in \mathcal{M}^{\mathrm{ord}}(M,\chi;\mathbb{I})$ the rational function $\mathbf{a}(1,1_{\mathbf{f}}\mathbf{g}) \in F_{\mathbb{I}}$ (the first Fourier coefficient of $1_{\mathbf{f}}\mathbf{g}$) is finite at each $\phi \in \mathcal{X}^a_{\mathbb{I},\mathbf{Z}_p[\chi]}$ and satisfies

$$\phi(\mathbf{a}(1,1_{\mathbf{f}}\mathbf{g})) = a(1,1_{\mathbf{f}_\phi}\mathbf{g}_\phi).$$

Let $\mathfrak{m}_{\mathbf{f}}$ be the maximal ideal of $\mathbb{T}^{\mathrm{ord}}(M,\chi;\mathbb{I})$ associated with $\mathbf{f}$. If $(\mathbf{irred})_{\mathbf{f}}$ and $(\mathbf{dist})_{\mathbf{f}}$ hold for $\mathbf{f}$ (equivalently, (**irred**) and (**dist**) hold for $\bar{\rho}_{\mathbf{f}_\phi}$ for one, and so all, $\phi \in \mathcal{X}_{\mathbb{I},\mathbf{Z}_p[\chi]}$) then $\mathbb{T}_{\mathrm{ord}}(M,\chi_f;\mathbb{I})_{\mathfrak{m}_{\mathbf{f}}}$ is a Gorenstein $\mathbb{I}$-algebra (this follows from [Wi95, Cor. 2, p. 482]) and so $\mathbb{T}_{\mathrm{ord}}(M,\chi_f;\mathbb{I}) \cap (0 \times F_{\mathbb{I}})$ is a rank one $\mathbb{I}$-module. We let $\ell_{\mathbf{f}}$ be a generator; so $\ell_{\mathbf{f}} = \eta_{\mathbf{f}} 1_{\mathbf{f}}$ for some $\eta_{\mathbf{f}} \in \mathbb{I}$. Since the assumption on $\mathbf{f}$ implies the same properties of each $\bar{\rho}_{\mathbf{f}_\phi}$ for any $\phi \in \mathcal{X}^a_{\mathbb{I},\mathbf{Z}_p[\chi]}$, for such ϕ both $\ell_{\mathbf{f}_\phi}$ and $\eta_{\mathbf{f}_\phi}$ are defined and may be chosen so that

$$\ell_{\mathbf{f}_\phi}\mathbf{g}_\phi = (\ell_{\mathbf{f}}\mathbf{g})_\phi \quad \text{and} \quad \eta_{\mathbf{f}_\phi} = \phi(\eta_{\mathbf{f}}).$$

Henceforth we will always assume that such choices have been made.

12.2.3. $\Lambda_{\mathbf{D}}$-*adic forms.* Let $\mathcal{X} \subset \mathcal{X}^a_{\mathbf{D}}$ be a Zariski-dense subset of $\mathrm{Spec}\,\Lambda_{\mathbf{D}}(\overline{\mathbf{Q}}_p)$. Let B be any integer coprime to p and let θ be an A-valued Dirichlet character modulo Bp. For technical reasons having to do with the hypotheses in the results from 11.7, we introduce the $\Lambda_{\mathbf{D}}$-modules $\mathcal{M}_{\mathcal{X}}(B,\theta;\Lambda_{\mathbf{D}})$ comprising those formal q-expansions $\mathbf{g} = \sum_{n=0}^\infty \mathbf{c}(n)q^n$, $\mathbf{c}(n) \in \Lambda_{\mathbf{D}}$, such that for each $\phi \in \mathcal{X}$

$$\mathbf{g}_\phi := \sum_{n=0}^\infty \phi(\mathbf{c}(n))q^n \in M_{\kappa_\phi}(Bp^{r_\phi(\mathbf{g})}, \theta\omega^{\kappa_\phi - 2}\chi_\phi; \phi(\Lambda_{\mathbf{D}})),$$

where $r_\phi(\mathbf{g}) > 0$ depends on ϕ and $\mathbf{g}$. An important example of such a formal q-expansion is given in proof of Proposition 12.2.5 below.

Lemma 12.2.4. *There exists an idempotent* $e \in \mathrm{End}_{\Lambda_{\mathbf{D}}}(\mathcal{M}_{\mathcal{X}}(B,\theta;\Lambda_{\mathbf{D}}))$ *such that*

(i) *for any* $\mathbf{g} \in \mathcal{M}_{\mathcal{X}}(B,\theta;\Lambda_{\mathbf{D}})$, $(e\mathbf{g})_\phi = e\mathbf{g}_\phi \in M^{\mathrm{ord}}_{\kappa_\phi}(Bp^{t_\phi}, \theta\omega^{\kappa_\phi - 2}\chi_\phi; \phi(\Lambda_{\mathbf{D}}))$ *for all* $\phi \in \mathcal{X}$;
(ii) $e\mathcal{M}_{\mathcal{X}}(B,\theta;\Lambda_{\mathbf{D}}) = \mathcal{M}^{\mathrm{ord}}(B,\theta;\mathbb{I}) \otimes_{\mathbb{I}} \Lambda_{\mathbf{D}}$.

Proof. Let $\mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots$ be a filtration of $\mathcal{X}$ by finite sets (so $\mathcal{X} = \cup_{n=1}^\infty \mathcal{X}_n$). Let $K_n := \cap_{\phi\in\mathcal{X}_n} \ker\phi$.

Let $\mathbf{g} = \sum_{n=0}^\infty \mathbf{c}(n)q^n \in \mathcal{M}_{\mathcal{X}}(B,\theta;\Lambda_{\mathbf{D}})$. For $\phi \in \mathcal{X}$ the sequence $\{\phi(\mathbf{c}(p^r n))\}_{0\le r<\infty}$ has a unique limit (equal to the nth coefficient of $e\mathbf{g}_\phi$). Therefore for each $m = 1, 2, \ldots$, the sequence $\{\mathbf{c}(p^r n)\}_{0\le r<\infty}$ has a unique limit, say $\mathbf{a}^{(m)}(n)$, in the finite A-module $\Lambda_{\mathbf{D}}/K_m$ and $\sum_{n=0}^\infty \phi(\mathbf{a}^{(m)}(n))q^n = e\mathbf{g}_\phi$ for $\phi \in \mathcal{X}_m$. By uniqueness, the $\mathbf{a}^{(m)}(n)$'s are compatible via the projections $\Lambda_{\mathbf{D}}/K_m \to \Lambda_{\mathbf{D}}/K_{m'}$, $m \ge m'$. So $\{\mathbf{c}(p^r n)\}_{0\le r\le\infty}$ has a unique limit $\mathbf{a}(n)$ in $\Lambda_{\mathbf{D}}$. We define e by $e\mathbf{g} = \sum_{n=0}^\infty \mathbf{a}(n)q^n$; this is independent of the chosen filtration and clearly has the desired properties. This proves part (i).

The proof of part (ii) is fairly standard. Let $\Lambda_{\mathbf{D}}[\![q^{\mathbf{Z}_{\ge 0}}]\!]$ denote the ring of formal q-expansions $\sum_{n=0}^\infty \mathbf{c}(n)q^n$, $\mathbf{c}(n) \in \Lambda_{\mathbf{D}}$. A standard argument from Hida theory (essentially appealing to the mod p q-expansion principle) shows that the cokernel of the inclusion $\mathcal{M}^{\mathrm{ord}}(B,\theta;\mathbb{I}) \otimes_{\mathbb{I}} \Lambda_{\mathbf{D}} \hookrightarrow \Lambda_{\mathbf{D}}[\![q^{\mathbf{Z}_{\ge 0}}]\!]$ is $\Lambda_{\mathbf{D}}$-torsion-free. On the other hand, another standard argument (cf. [Ur04, Prop. 2.4.22]) shows that the cokernel of the inclusion $\mathcal{M}^{\mathrm{ord}}(B,\theta;\mathbb{I}) \otimes_{\mathbb{I}} \Lambda_{\mathbf{D}} \hookrightarrow e\mathcal{M}_{\mathcal{X}}(B,\theta;\Lambda_{\mathbf{D}})$ is $\Lambda_{\mathbf{D}}$-torsion. Combining these two facts yields part (ii). ∎

Proposition 12.2.5. *Let* $\mathbf{D} = (A, \mathbb{I}, \mathbf{f}, \psi, \xi, \Sigma)$ *be a p-adic Eisenstein datum. Let* $m = 1$ *or* 2 *and let* $y := \mathrm{diag}(u, \bar{u}^{-1})$, $u \in \mathrm{GL}_2(\mathbf{A}^\Sigma_{\mathcal{K},f})$. *Suppose A contains i,* $D_{\mathcal{K}}^{1/2}$, *and the following local Gauss sums for each* $\ell \in \Sigma\backslash\{p\}$:

- $\mathfrak{g}(\xi^c_\ell, a_\ell\delta_{\mathcal{K}})$, $(a_\ell) = \mathrm{cond}(\xi^c_\ell)$,
- $\mathfrak{g}(\bar{\chi}_{\mathbf{f},\ell}\xi^c_\ell, b_\ell\delta_{\mathcal{K}})$, $(b_\ell) = \mathrm{cond}(\bar{\chi}_{\mathbf{f},\ell}\xi^c_\ell)$,
- $\mathfrak{g}(\chi_{\mathbf{f},\ell}\bar{\xi}'_\ell)$.

Then for $\beta \in S_m(\mathbf{Q})$, $\beta \ge 0$, *there exists* $\mathbf{f}^{(m)}_{\mathbf{D},\beta,y} \in \mathcal{M}^{\mathrm{ord}}(M_{\mathbf{D}}, \chi_{\mathbf{f}}; \mathbb{I}) \otimes_{\mathbb{I}} \Lambda_{\mathbf{D}}$ *such that*

$$\mathbf{f}^{(m)}_{\mathbf{D},\beta,y,\phi} = f^{(m)}_{\mathfrak{D}_\phi,\beta,y}, \quad \phi \in \mathcal{X}'_{\mathbf{D}},$$

where $f^{(m)}_{\mathfrak{D}_\phi,\beta,y}$ *is as in 11.7.*

Proof. Let $a_1, \ldots, a_{h_{\mathcal{K}}} \in \mathbf{A}^\times_{\mathcal{K},f}$ be representatives for the class group of $\mathcal{K}$ as in 11.6 and 11.7. Let v_i and $L^{(m)}_{v_i}(\beta, n)$ be as in Lemma 11.7.3. Let $\beta \in S_m(\mathbf{Q})$, $\beta \ge 0$, and $n \in \mathbf{Z}$,

$n \geq 0$. Let $T \in L_{v_i}^{(m)}(\beta, n)$ and define $\mathbf{R}_{\mathbf{D},T}^{(m)} \in \Lambda_{\mathbf{D}}$ by

$$\begin{aligned}
\mathbf{R}_{\mathbf{D},T}^{(m)} := & (\det T|\det T|_p)^{-m-1}\boldsymbol{\xi}_p^{-c}(T_p^*)\prod_{\ell\neq p}\boldsymbol{\xi}_\ell^{-1}\boldsymbol{\psi}_\ell(\det T) \\
& \times \psi(-1)(\chi_{\mathbf{f}}^{-1}\Psi_W^+\boldsymbol{\xi})_p(M_{\mathbf{D}}^{-1}) \\
& \times \prod_{\ell\in\Sigma\setminus\{p\}} |x_\ell|_{\mathcal{K}} \begin{cases} \boldsymbol{\xi}_\ell^c(T_\ell^*)^{-1}\mathfrak{g}(\chi_{\mathbf{f},\ell}\xi_\ell^c, x_\ell\delta_{\mathcal{K}}) & (x_\ell)\neq\mathcal{O}_\ell, T_\ell^*\in\mathcal{O}_\ell^\times \\ 0 & (x_\ell)\neq\mathcal{O}_\ell, T_\ell^*\notin\mathcal{O}_\ell^\times \\ 1 & (x_\ell)=\mathcal{O}_\ell \end{cases} \\
& \times \prod_{\ell\in\Sigma\setminus\{p\}} \Psi_{W,\ell}^{-1}\chi_{\mathbf{f}}\boldsymbol{\xi}_\ell^{-c}(y_\ell\delta_{\mathcal{K}})\mathfrak{g}(\xi_\ell^c, y_\ell\delta_{\mathcal{K}}) \\
& \times \chi_{\mathbf{f},0}^{-1}\boldsymbol{\xi}^{-1}\boldsymbol{\psi}(a_i^m\det u)|a_i^m\det u|_{\mathcal{K}}^{m+1}\prod_{\ell\notin\Sigma} h_{\ell,q_i\bar{u}_\ell Tu_\ell}(\boldsymbol{\psi}_\ell^{-1}\boldsymbol{\xi}_\ell(\ell)) \\
& \times \begin{cases} \prod_{\ell\in\Sigma\{p\}} \frac{\Phi_{W,\ell}^{-1}(\ell^{e_\ell})}{\bar{\chi}_{\mathbf{f},\ell}\boldsymbol{\xi}'_\ell(\ell^{e_\ell})\mathfrak{g}(\chi_{\mathbf{f},\ell}\bar{\xi}'_\ell)} & m=2 \\ 1 & m=1. \end{cases}
\end{aligned}$$

Here $x_\ell := \mathrm{cond}(\chi_{\mathbf{f},\ell}\xi_\ell^c)$, $y_\ell := \mathrm{cond}(\xi_\ell^c)$, and $(\ell^{e_\ell}) := \mathrm{cond}(\chi_{\mathbf{f},\ell}\xi'_\ell)$. It is easily checked that for $\phi \in \mathcal{X}'_{\mathbf{D}}$,

$$\phi(\mathbf{R}_{\mathbf{D},T}^{(m)}) = R_{\mathfrak{D}_\phi,T}^{(m)}$$

where the right-hand side is as in Lemma 11.7.3. We leave this simple verification to the reader.

Let

$$\begin{aligned}
\mathbf{r}_{\mathbf{D},\beta}^{(m)}(n,y) := & -i2^{m(m+1)-1}D_{\mathcal{K}}^{-m(m+1)/4}M^{-1} \\
& \times \sum_{i=1}^{h_{\mathcal{K}}} \boldsymbol{\psi}_p^{-m}\boldsymbol{\xi}_p^{m-1}(q_i)\chi_{\mathbf{f},0}\boldsymbol{\psi}^{-m}\boldsymbol{\xi}^{m-1}(a_i) \sum_{T\in L_{v_i}^{(m)}(\beta, M_{\mathbf{D}}^2 n/M)} \mathbf{R}_{\mathbf{D},T}^{(m)} \in \Lambda_{\mathbf{D}}.
\end{aligned}$$

Then for $\phi \in \mathcal{X}'_{\mathbf{D}}$, $\phi(\mathbf{r}_{\mathbf{D},\beta}^{(m)}(n,y)) = \rho_{\mathfrak{D}_\phi,\beta}^{(m)}(n,y)$ where again the right-hand side is as in Lemma 11.7.3. It then follows from this same lemma that $\mathbf{g} := \sum_{n=0}^\infty \mathbf{r}_{\mathbf{D},\beta}^{(m)}(n,y)q^n$ is such that for $\phi \in \mathcal{X}'_{\mathbf{D}}$, $\mathbf{g}_\phi = f_{\mathfrak{D}_\phi,\beta,y}^{(m)}$. So $\mathbf{g} \in \mathcal{M}_{\mathcal{X}'_{\mathbf{D}}}(M, \chi_{\mathbf{f}}; \Lambda_{\mathbf{D}})$. By Lemma 12.2.4 we may then take $\mathbf{f}_{\mathfrak{D},\beta,y}^{(m)} := e\mathbf{g}$. ∎

12.2.6. *The key interpolation lemmas.* The following lemmas are our main tools for constructing three- and two-variable p-adic L-functions and interpolating the Eisenstein series $E_{\mathfrak{D}}$.

Lemma 12.2.7. *Let $\mathbf{f} \in \mathcal{M}^{\mathrm{ord}}(M, \chi_{\mathbf{f}}; \mathbb{I})$ be an ordinary newform. Let R be any integral extension of $\mathbb{I}$. Let $\mathbf{g} \in \mathcal{M}^{\mathrm{ord}}(M, \chi; \mathbb{I}) \otimes_{\mathbb{I}} R$.*

(i) *There exists* $\mathcal{I}_{\mathbf{g}} \in F_{\mathbb{I}} \otimes_{\mathbb{I}} R$ *that is finite at each* $\phi \in \mathcal{X}_{R,\mathbf{Z}_p[\chi_{\mathbf{f}}]}$ *for which* $\phi|\mathbb{I}$ *is arithmetic with* $\kappa_{\phi|\mathbb{I}} \geq 2$*, and for such* ϕ

$$\phi(\mathcal{I}_{\mathbf{g}}) = a(1, 1_{\mathbf{f}_\phi}\mathbf{g}_\phi),$$

where if $\mathbf{g} = \sum \mathbf{g}_i \otimes c_i$ *then* $\mathbf{g}_\phi := \sum \phi(c_i)\mathbf{g}_{i,\phi|\mathbb{I}}$*. Furthermore, if* $a \in \mathbb{I}$ *is such that* $a1_{\mathbf{f}} \in \mathbb{T}^{\mathrm{ord}}(M, \chi_{\mathbf{f}}; \mathbb{I})$ *then* $a\mathcal{I}_{\mathfrak{g}} \in \Lambda_{\mathbf{D}}$*.*

(ii) *If* $(\mathbf{irred})_{\mathbf{f}}$ *and* $(\mathbf{dist})_{\mathbf{f}}$ *hold, then there exists an element* $\mathcal{N}_{\mathbf{g}} \in R$ *such that for any* $\phi \in \mathcal{X}_{R,\mathbf{Z}_p[\chi_{\mathbf{f}}]}$ *for which* $\phi|\mathbb{I}$ *is arithmetic with* $\kappa_{\phi|\mathbb{I}} \geq 2$*,*

$$\phi(\mathcal{N}_{\mathbf{g}}) = a(1, \ell_{\mathbf{f}_\phi}\mathbf{g}_\phi),$$

with $\mathbf{g}_\phi$ *as in part (i).*

Any element $c \in F_{\mathbb{I}} \otimes_{\mathbb{I}} R$ defines a rational function on $\operatorname{Spec} R$ and hence a notion of c being finite at ϕ; we denote the value of c at such a ϕ by $\phi(c)$.

Proof. Write $\mathbf{g} = \sum \mathbf{g}_i \otimes c_i$, $\mathbf{g}_i \in \mathcal{M}^{\mathrm{ord}}(M, \chi; \mathbb{I})$ and $c_i \in \Lambda_{\mathbf{D}}$. We may then take $\mathcal{I}_{\mathbf{g}} := a(1, \sum c_i 1_{\mathbf{f}}\mathbf{g}_i)$ for part (i) and $\mathcal{N}_{\mathbf{g}} := a(1, \sum c_i \ell_{\mathbf{f}}\mathbf{g}_i)$ for part (ii).∎

Lemma 12.2.8. *Let* $f \in M_\kappa^{\mathrm{ord}}(N, \chi; A)$*,* $\kappa \geq 2$ *and* $A \subset \overline{\mathbf{Q}}_p$ *a finite integral extension of* $\mathbf{Z}_p$*, be a* p*-stabilized newform. Let* R *be an integral extension of* A*. Let* $g \in M_\kappa^{\mathrm{ord}}(N, \chi; A) \otimes_A R$*.*

(i) *There exists* $\mathcal{I}_g \in F_A \otimes_A R$*,* F_A *the field of fractions of* A*, such that for* $\phi \in \mathcal{X}_{R,A}$

$$\phi(\mathcal{I}_g) = a(1, 1_f g_\phi),$$

where $\phi(\mathcal{I}_g)$ *is the image of* $\mathcal{I}_g$ *under the canonical extension of* ϕ *to* $F_A \otimes_A R$ *and* $g_\phi \in M_\kappa^{\mathrm{ord}}(N, \chi; \overline{\mathbf{Q}}_p)$ *is the image of* g *under* $id \otimes \phi : M_\kappa^{\mathrm{ord}}(N, \chi; A) \otimes_A R \to M_\kappa^{\mathrm{ord}}(N, \chi; A) \otimes_A \overline{\mathbf{Q}}_p = M_\kappa^{\mathrm{ord}}(N, \chi; \overline{\mathbf{Q}}_p)$*.*

(ii) *If* $(\mathbf{irred})$ *and* $(\mathbf{dist})$ *hold for* $\bar{\rho}_f$ *then there exists* $\mathcal{N}_g \in R$ *such that for* $\phi \in \mathcal{X}_{R,A}$

$$\phi(\mathcal{N}_g) = a(1, \ell_f g_\phi),$$

with g_ϕ *as in part (i).*

The proof is the same as for the preceding lemma.

12.3. **Application I: p-adic L-functions.** As our first application of the key interpolation lemmas we construct the three- and two-variable p-adic L-functions that feature in our main theorems.

Let A, $\mathbb{I}$, $\mathbf{f}$, ξ and Σ be such that

- A is the ring of integers of a finite extension of $\mathbf{Q}_p$;
- $\mathbb{I}$ is a domain and a finite $\Lambda_{W,A}$-algebra;
- $\mathbf{f} \in \mathcal{M}^{\mathrm{ord}}(M, \chi_{\mathbf{f}}, \mathbb{I})$ is an ordinary newform with $\chi_{\mathbf{f}}$ taking values in A;
- ξ is a finite A-valued idele class character of $\mathcal{K}$;

- Σ is a finite set of primes containing all primes $\ell|MpD_{\mathcal{K}}$ and all primes ℓ such that ξ_ℓ is ramified.

Let

$$\mathcal{X}_{\mathbb{I}_{\mathcal{K}},A}(\mathbf{f},\psi,\xi) := \{\phi \in \mathcal{X}^a_{\mathbb{I}_{\mathcal{K}},A} \ : \ \mathfrak{p}|\mathfrak{f}_{\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi}, p^{t_\phi}|\mathrm{Nm}(\mathfrak{f}_{\xi_\phi}), \mathfrak{f}_{\xi^c_\phi}\mathcal{O}_p \subseteq \mathfrak{f}_\psi\mathcal{O}_p\}.$$

Theorem 12.3.1. *Let A, $\mathbb{I}$, $\mathbf{f}$, ξ, and Σ be as above. Suppose that A satisfies the hypotheses of Proposition 12.2.5 and that there exists a finite A-valued idele class character ψ of $\mathbf{A}^\times_{\mathcal{K}}$ such that $\psi|_{\mathbf{A}^\times} = \chi_{\mathbf{f}}$ and ψ is unramified outside Σ.*

(i) *There exists $\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi} \in F_{\mathbb{I}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{K}}$ such that for any $\phi \in \mathcal{X}_{\mathbb{I}_{\mathcal{K}},A}(\mathbf{f},\psi,\xi)$, $\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi}$ is finite at ϕ and*

$$\phi(\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi}) = a(p,\mathbf{f}_\phi)^{-\mathrm{ord}_p(\mathrm{Nm}(\bar{\chi}_{\mathbf{f}_\phi}\mathfrak{f}_{\xi_\phi}))}\frac{((\kappa_\phi-2)!)^2\mathfrak{g}(\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi)\mathrm{Nm}(\mathfrak{f}_{\xi_\phi}\mathfrak{d})^{\kappa_\phi-2}L^\Sigma_{\mathcal{K}}(\mathbf{f}_\phi,\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi,\kappa_\phi-1)}{(-2\pi i)^{2\kappa_\phi-2}2^{-3}(2i)^{\kappa_\phi+1}S(\mathbf{f}_\phi) < \mathbf{f}_\phi, \mathbf{f}^c_\phi|_{\kappa_\phi}\begin{pmatrix} & -1 \\ N_\phi & \end{pmatrix} >_{\Gamma_0(N_\phi)}}.$$

(ii) *If $(\mathbf{irred})_{\mathbf{f}}$ and $(\mathbf{dist})_{\mathbf{f}}$ hold, then there exists $\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi} \in \mathbb{I}_{\mathcal{K}}$ such that for $\phi \in \mathcal{X}_{\mathbb{I}_{\mathcal{K}},A}(\mathbf{f},\psi,\xi)$,*

$$\phi(\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi}) = u_{\mathbf{f}_\phi}a(p,\mathbf{f}_\phi)^{-\mathrm{ord}_p(\mathrm{Nm}(\mathfrak{f}_{\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi}))}\frac{((\kappa_\phi-2)!)^2\mathfrak{g}(\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi)\mathrm{Nm}(\mathfrak{f}_{\xi_\phi}\mathfrak{d})^{\kappa_\phi-2}L^\Sigma_{\mathcal{K}}(\mathbf{f}_\phi,\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi,\kappa_\phi-1)}{(-2\pi i)^{2\kappa_\phi-2}\Omega^+_{\mathbf{f}_\phi}\Omega^-_{\mathbf{f}_\phi}}$$

where $u_{\mathbf{f}_\phi}$ is a p-adic unit depending only on $\mathbf{f}_\phi$.

Remark. It is very easy to add conditions that ensure the existence of a ψ as in the theorem; this is not necessary for our application.

Proof. The hypotheses ensure that $\mathbf{D} := (A,\mathbb{I},\mathbf{f},\psi,\xi,\Sigma)$ is a p-adic Eisenstein datum. Let $\mathbf{g} := \mathbf{f}^{(1)}_{\mathbf{D},1/M,1}$ be as in Proposition 12.2.5. Then applying Lemma 12.2.7 to $\mathbf{g}$ with $R = \Lambda_{\mathbf{D}}$ yields $\mathcal{I}_{\mathbf{g}} \in F_{\mathbb{I}} \otimes_{\mathbb{I}} \Lambda_{\mathbf{D}}$ such that for $\phi \in \mathcal{X}'_{\mathbf{D}}$

$$\phi(\mathcal{I}_{\mathbf{g}}) = a(1, 1_{\mathbf{f}_\phi}ef^{(1)}_{\mathfrak{D}_\phi,1/M,1}) = \frac{< f^{(1)}_{\mathfrak{D}_\phi,1/M,1}, \mathbf{f}^c_\phi|_\kappa\begin{pmatrix} & -1 \\ p^{r_\phi}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}_\phi}}}{< \mathbf{f}_\phi, \mathbf{f}^c_\phi|_\kappa\begin{pmatrix} & -1 \\ p^{r_\phi}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}_\phi}}},$$

where r_ϕ is such that $\Gamma_{\mathfrak{D}_\phi} = \Gamma_0(Mp^{r_\phi})$. Let $\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi} := \mathcal{I}_{\mathbf{g}}$. It then follows from Corollary 11.7.5 that if $\phi \in \mathcal{X}'_{\mathbf{D}}$, then $\phi(\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi})$ equals the expression in part (i). Since $\mathcal{X}'_{\mathbf{D}}$ is Zariski dense in $\mathrm{Spec}\,\Lambda_{\mathbf{D}}(\overline{\mathbf{Q}}_p)$ and since these values are independent of $\phi|\Gamma^-_{\mathcal{K}}$, $\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi}$ belongs to the subring $F_{\mathbb{I}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{K}}$ of $F_{\mathbb{I}} \otimes_{\mathbb{I}} \Lambda_{\mathbf{D}} = F_{\mathbb{I}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{K}}[\![\Gamma^-_{\mathcal{K}}]\!]$. Part (i) follows upon noting that any $\phi \in \mathcal{X}_{\mathbb{I}_{\mathcal{K}},A}(\mathbf{f},\psi,\xi)$ is the restriction to $\mathbb{I}_{\mathcal{K}}$ of some element of $\mathcal{X}'_{\mathbf{D}}$. For part (ii), take $\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi} := \mathcal{N}_{\mathbf{g}}$, with $\mathcal{N}_{\mathbf{g}}$ as in part (ii) of Lemma 12.2.7. ∎

The next theorem is a two-variable version of the preceding theorem. Let A, ξ and Σ be as before and let $f \in S_\kappa^{\mathrm{ord}}(Mp^t,\chi;A)$ with $\kappa \geq 2$ be a p-stabilized newform. Write $\chi = \chi_1\chi_2$ with χ_1 a character modulo Mp and χ_2 a character modulo p^t of p-power order. Let $A_{\mathcal{K}} := A[\![\Gamma_{\mathcal{K}}]\!]$.

Theorem 12.3.2. *Suppose that A satisfies the hypotheses of Proposition 12.2.5 (but with $\chi_{\mathbf{f}}$ replaced by $\chi_1\omega^{2-\kappa}$) and that there exists a finite A-valued idele class character ψ of $\mathbf{A}_{\mathcal{K}}^\times$ such that $\psi|_{\mathbf{A}^\times} = \chi_1\omega^{2-\kappa}$ and ψ is unramified outside Σ.*

(i) *There exists $\mathcal{L}^\Sigma_{f,\mathcal{K},\xi} \in F_A \otimes_A A_{\mathcal{K}}$ such that for any $\phi \in \mathcal{X}_{A_{\mathcal{K}},A}$ with $\phi(\gamma_+) = \zeta_+(1+p)^{\kappa-2}$, ζ_+ a p-power root of unity, $\phi|\Gamma_{\mathcal{K}}^-$ of finite order, $p|\mathfrak{f}_{\bar{\chi}\xi_\phi}$, $p^t|\mathrm{Nm}(\mathfrak{f}_{\xi_\phi^c})$, and $\mathfrak{f}_{\xi_\phi^c}\mathcal{O}_p \subseteq \mathfrak{f}_\psi\mathcal{O}_p$,*

$$\phi(\mathcal{L}^\Sigma_{f,\mathcal{K},\xi}) = a(p,f)^{-\mathrm{ord}_p(\mathrm{Nm}(\mathfrak{f}_{\bar{\chi}\xi_\phi}))}\frac{((\kappa-2)!)^2\mathfrak{g}(\bar{\chi}\xi_\phi)\mathrm{Nm}(\mathfrak{f}_{\xi_\phi}\mathfrak{d})^{\kappa-2}L^\Sigma_{\mathcal{K}}(f,\bar{\chi}\xi_\phi,\kappa-1)}{(-2\pi i)^{2\kappa-2}\Omega_f^+\Omega_f^-},$$

where for $x \in F_A \otimes_A A_{\mathcal{K}}$, $\phi(x)$ is the image of x under the canonical extension of ϕ to $F_A \otimes_A A_{\mathcal{K}}$.

(ii) *If* (**irred**) *and* (**dist**) *hold for $\bar{\rho}_f$, then $\mathcal{L}^\Sigma_{f,\mathcal{K},\xi} \in A_{\mathcal{K}}$. Furthermore, if $f = \mathbf{f}_{\phi_0}$ for some ordinary newform $\mathbf{f} \in \mathcal{M}^{\mathrm{ord}}(M,\chi_1\omega^{2-\kappa};\mathbb{I})$, $\phi_0(\mathbb{I}) \subseteq A$, then $\mathcal{L}^\Sigma_{f,\mathcal{K},\xi}$ is the product of a unit in A and the image of $\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi}$ under the projection $\mathbb{I}_{\mathcal{K}} \to A_{\mathcal{K}}$ induced by ϕ_0.*

Here by ξ_ϕ we mean $\phi \circ \boldsymbol{\xi}$ with $\boldsymbol{\xi}$ as before except with $\chi_{\mathbf{f}}$ replaced with $\chi_1\omega^{2-\kappa}$.

Proof. It is easy to see that there exists an ordinary newform $\mathbf{f} \in \mathcal{M}^{\mathrm{ord}}(M,\chi_1\omega^{2-\kappa};\mathbb{I})$, $\mathbb{I}$ a domain and a finite integral extension of $\Lambda_{W,A}$, and an arithmetic prime $\phi_0 \in \mathcal{X}_{\mathbb{I},A}$ such that $f = \mathbf{f}_{\phi_0}$ and $A = \phi_0(\mathbb{I})$. Let $\mathbf{D} := (A,\mathbb{I},\mathbf{f},\psi,\chi_2\xi,\Sigma)$; this is a p-adic datum. Let $\mathbf{g}$ be as in the proof of Theorem 12.3.1. Let g be the image of $\mathbf{g}$ under the map $\mathcal{M}^{\mathrm{ord}}(M,\chi_1\omega^{2-\kappa};\mathbb{I}) \otimes_{\mathbb{I}} \Lambda_{\mathbf{D}} \to M_\kappa^{\mathrm{ord}}(Mp^t,\chi;A) \otimes_A A[\![\Gamma_{\mathcal{K}} \times \Gamma_{\mathcal{K}}^-]\!]$ induced by ϕ_0. Then let

$$\mathcal{L}^\Sigma_{f,\mathcal{K},\xi} := \frac{2^{-3}(2i)^{\kappa+1}S(f) < f, f^c|\begin{pmatrix} & -1 \\ Mp^t & \end{pmatrix} >_{\Gamma_0(Mp^t)}}{\Omega_f^+\Omega_f^-}\mathcal{I}_g,$$

where $\mathcal{I}_g$ is as in Lemma 12.2.8. That the factor in front of $\mathcal{I}_g$ in the displayed equation belongs to F_A follows from [Hi81], and by Lemma 12.2.2 it equals η_f up to a unit if (**irred**) and (**dist**) hold for $\bar{\rho}_f$. The arguments proving Theorem 12.3.1 are now easily adapted to prove this theorem. ∎

12.3.3. *Connections with cyclotomic p-adic L-functions.* Let $f \in S_\kappa^{\mathrm{ord}}(Mp^r,\chi;A)$ be an ordinary p-stabilized newform, A being the ring of integers of some finite extension of $\mathbf{Q}_p$. Given a primitive A-valued Dirichlet character ψ of conductor C prime to p and any finite set Σ of primes, let $\mathcal{L}^\Sigma_{f,\psi} \in \Lambda_{\mathbf{Q},A}$ be the p-adic L-function constructed by Amice-Vélu [AV75] and Vishik [Vi76] (see also [MTT86]) and recalled in 3.4.4.

Let $\alpha : \Lambda_{\mathcal{K},A} \to \Lambda_{\mathbf{Q},A}$ be induced by the canonical projection $\Gamma_{\mathcal{K}} \to \Gamma_{\mathbf{Q}}$. The following is an immediate consequence of Theorem 12.3.2 and the specialization properties of $\mathcal{L}^{\Sigma}_{f,\mathcal{K},\chi\omega^{2-\kappa}}$.

Proposition 12.3.4. *Suppose the hypotheses of Theorem 12.3.2 hold with $\xi = \chi\omega^{2-\kappa}$. Let Σ be a set of primes containing all those that divide $MpD_{\mathcal{K}}$. Then*

$$\alpha(\mathcal{L}^{\Sigma}_{f,\mathcal{K},\chi\omega^{2-\kappa}}) \sim \mathcal{L}^{\Sigma}_{f}\mathcal{L}^{\Sigma}_{f,\chi_{\mathcal{K}}} \sim \mathcal{L}^{\Sigma}_{f}\mathcal{L}^{\Sigma}_{f\otimes\chi_{\mathcal{K}}}.$$

Here '$\sim$' denotes equality up to a unit in $\Lambda_{\mathbf{Q},A}$.

12.3.5. *Connections with anticyclotomic p-adic L-functions.* Let $\beta : \Lambda_{\mathcal{K},A} \to \Lambda^{-}_{\mathcal{K},A}$ be the homomorphism induced by the canonical projection $\Gamma_{\mathcal{K}} \to \Gamma^{-}_{\mathcal{K}}$. For A reduced, β of course extends to $F_A \otimes_A \Lambda_{\mathcal{K},A} \to F_A \otimes_A \Lambda^{-}_{\mathcal{K},A}$, F_A the ring of fractions of A.

If A, $f \in S_2^{\mathrm{ord}}(Mp^t, \chi; A)$, and ξ are as in Theorem 12.3.2 and assuming the hypotheses of this theorem hold, then $\mathcal{L}^{\Sigma,-}_{f,\mathcal{K},\xi} := \beta(\mathcal{L}^{\Sigma}_{f,\mathcal{K},\xi}) \in F_{\mathbf{A}} \otimes_A \Lambda^{-}_{\mathcal{K},A}$ is an anticyclotomic L-function in the sense that if $\phi \in \mathcal{X}_{\Lambda^{-}_{\mathcal{K},A},A}$ is such that $\xi_\phi = \xi_{\phi\circ\beta}$ is a finite character such that $p|\mathfrak{n}_\phi$, $\mathfrak{n}_\phi := \mathrm{cond}(\omega^{\kappa-2}\xi_\phi$, then

$$\phi(\mathcal{L}^{\Sigma,-}_{f,\mathcal{K},\xi}) = a(p,f)^{-\mathrm{ord}_p(\mathrm{Nm}(\mathfrak{f}_{\bar{\chi}\xi_\phi}))}\frac{\mathfrak{g}(\bar{\chi}\xi_\phi)L^{\Sigma}_{\mathcal{K}}(f,\bar{\chi}\xi_\phi,1)}{(-2\pi i)^2\Omega_f^{+}\Omega_f^{-}}$$

(so interpolates values of an L-function twisted by characters of $\Gamma^{-}_{\mathcal{K}}$, the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension of $\mathcal{K}$). If (**irred**) and (**dist**) hold for $\bar{\rho}_f$ then $\mathcal{L}^{\Sigma,-}_{f,\mathcal{K},\xi} \in \Lambda^{-}_{\mathcal{K},A}$.

Suppose that

- $(M, D_{\mathcal{K}}) = 1, \xi = \bar{\xi}^c,\ \mathrm{cond}(\xi)|(p)$;
- $\chi = 1$;

(12.3.5.a)

- $M = M^{+}M^{-}$ with M^{+} divisible only by primes that split in $\mathcal{K}$ and M^{-} is divisible only by primes that are inert in $\mathcal{K}$ and that M^{-} is square-free with an odd number of prime factors.

Then as explained in [Va03], Perrin-Riou [PR88] and Bertolini and Darmon [BD96] have independently constructed[11] a p-adic L-function $L(f,\xi,\gamma_{-}-1) \in \Lambda^{-}_{\mathcal{K},A}$ closely connected with $\mathcal{L}^{\Sigma,-}_{f,\mathcal{K},\xi}$ (we are essentially following the notation in [Va03] where $\gamma_{-}-1$ is denoted by T). When (**irred**) and (**dist**) hold for $\bar{\rho}_f$, comparing the interpolation formulae for these p-adic L-functions shows that

(12.3.5.b) $$\mathcal{L}^{\Sigma,-}_{f,\mathcal{K},\xi} = L(f,\xi,\gamma_{-}-1)\prod_{\ell\in\Sigma\setminus\{p\}}\prod_{w|\ell}\det\left(1-\ell^{-1}\mathrm{trace}(\rho_f\otimes\sigma_{\boldsymbol{\xi}^{-}}(\mathrm{frob}_w))^{I_w}\right),$$

[11]The construction in [PR88] is a specialization of one of the measures constructed by Hida in [Hi88a].

where w is a place of $\mathcal{K}$, ρ_f is the p-adic Galois representation associated with ρ, $\boldsymbol{\xi}^- := \beta \circ \boldsymbol{\xi}$, $\sigma_{\boldsymbol{\xi}^-} := \beta \circ \boldsymbol{\varepsilon}_{\mathcal{K}}$, and the superscript I_w denotes restriction to the subspace fixed by I_w. It follows that the Iwasawa-theoretic μ-invariants of the left and right-hand sides of (12.3.5.b) are the same. If we further assume that

(12.3.5.c) • if $\ell | M^-$ then $\bar{\rho}_f$ is ramified at ℓ,

then by Theorem 1.1 of [Va03] these μ-invariants are zero. This allows us to deduce the following proposition about the $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}$'s.

Proposition 12.3.6. *Let A, $\mathbb{I}$, $\mathbf{f}$, ξ, and Σ be as in Theorem 12.3.1 and assume that the hypotheses of Theorem 12.3.1 hold and that* $(\mathbf{irred})_{\mathbf{f}}$ *and* $(\mathbf{dist})_{\mathbf{f}}$ *hold. Assume also that*

- $\chi_{\mathbf{f}} = 1$;
- $\xi = \bar{\xi}^c$, $\mathrm{cond}(\xi)|(p)$;
- $(M, D_{\mathcal{K}}) = 1$;
- *$M = M^+M^-$ with M^+ divisible only by primes that split in $\mathcal{K}$ and M^- divisible only by primes that are inert in $\mathcal{K}$ and that M^- is square-free with an odd number of prime divisors;*
- *if $\ell | M^-$ then $\bar{\rho}_{\mathbf{f}}$ is ramified at ℓ.*

Then $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}$ is not contained in any prime of $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ of the form $Q\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ for some height one prime $Q \subset \mathbb{I}[\![\Gamma^+_{\mathcal{K}}]\!]$.

Proof. We have

$$\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi} = a_0 + a_1(\gamma_- - 1) + a_2(\gamma_- - 1)^2 + \cdots, \quad a_i \in \mathbb{I}[\![\Gamma^+_{\mathcal{K}}]\!],$$

and the condition that $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi} \in Q\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ for some height one prime $Q \subset \mathbb{I}[\![\Gamma^+_{\mathcal{K}}]\!]$ is equivalent to each a_i being an element of Q.

Let $\phi \in \mathcal{X}_{\mathbb{I},A}$ be an arithmetic weight 2 prime such that $f = \mathbf{f}_\phi$. Let $\phi \in \mathcal{X}_{\mathbb{I}[\![\Gamma^+_{\mathcal{K}}]\!],A}$ be the extension of ϕ such that $\phi(\gamma^+) = 1$. Then for some $u \in A^\times$

$$u\mathcal{L}^{\Sigma,-}_{f,\mathcal{K},\xi} = \phi(a_0) + \phi(a_1)(\gamma^- - 1) + \phi(a_2)(\gamma^- - 1)^2 + \cdots.$$

The hypotheses of the proposition ensure that (12.3.5.a) and (12.3.5.c) hold, so, as noted above, the μ-invariant of $\mathcal{L}^{\Sigma,-}_{f,\mathcal{K},\xi}$ is zero. That is, some $\phi(a_i) \in A^\times$. However, if Q is a prime of $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ then $\phi(Q)$ is contained in the maximal ideal of A, hence not every a_i can belong to Q. ∎

12.4. **Application II: p-adic Eisenstein series.** Our second application of the interpolation lemmas is the construction of the Eisenstein series used in the proof of Theorem 6.5.4.

Theorem 12.4.1. *Let $\mathbf{D} = (A, \mathbb{I}, \mathbf{f}, \psi, \xi, \Sigma)$ be a p-adic Eisenstein datum. Suppose that* $(\mathbf{irred})_{\mathbf{f}}$ *and* $(\mathbf{dist})_{\mathbf{f}}$ *hold. Suppose also that A satisfies the hypotheses of Proposition 12.2.5. Then for each $x = \mathrm{diag}(u, {}^t\bar{u}^{-1}) \in G(\mathbf{A}^{\Sigma}_f)$ there exists a formal q-expansion*

$\mathbf{E_D}(x) := \sum_{\beta\in S(\mathbf{Q}),\beta\geq 0}\mathbf{c_D}(\beta,x)q^\beta$, $\mathbf{c_D}(\beta,z)\in\Lambda_\mathbf{D}$, *with the property that for each* $\phi\in\mathcal{X}''_\mathbf{D}$ *and* $Z\in\mathbf{H}$

$$\mathbf{E}_{\mathbf{D},\phi}(Z,x) := \sum_{\beta\in S(\mathbf{Q}),\beta\geq 0}\phi(\mathbf{c_D}(\beta,x))e(\mathrm{Tr}\,\beta Z) = u_\phi\frac{G_{\mathfrak{D}_\phi}(Z,x)}{\Omega^+_{\mathbf{f}_\phi}\Omega^-_{\mathbf{f}_\phi}},\quad |u_\phi|_p = 1,$$

with $G_{\mathfrak{D}_\phi}(Z,x)$ *being as in* (11.7.5.a).

Proof. Let $\mathbf{g} := \mathbf{f}^{(2)}_{\mathbf{D},\beta,x}\in\mathcal{M}^{\mathrm{ord}}(M,\chi_\mathbf{f};\mathbb{I})\otimes_\mathbb{I}\Lambda_\mathbf{D}$ be as in Proposition 12.2.5, and let $\mathbf{c_D}(\beta,x) := \mathcal{N}_\mathbf{g}$, where $\mathcal{N}_\mathbf{g}$ is as in part (ii) of Lemma 12.2.7. Then for $\phi\in\mathcal{X}'_\mathbf{D}$

$$\begin{aligned}\phi(\mathbf{c_D}(\beta,x)) &= a(1,\ell_{\mathbf{f}_\phi}\mathbf{g}_\phi) = \eta_{\mathbf{f}_\phi}a(1,1_{\mathbf{f}_\phi}ef^{(2)}_{\mathfrak{D}_\phi,\beta,x})\\ &= \eta_{\mathbf{f}_\phi}\frac{< f^{(2)}_{\mathfrak{D},\beta,x}, f^c|_\kappa\begin{pmatrix} & -1\\ p^{r_p}M & \end{pmatrix} >_{\Gamma_\mathfrak{D}}}{< f, f^c|_\kappa\begin{pmatrix} & -1\\ p^{r_p}M & \end{pmatrix} >_{\Gamma_\mathfrak{D}}},\quad \Gamma_\mathfrak{D} = \Gamma_0(p^{r_p}M)\\ &= u_\phi\frac{C_{\mathfrak{D}_\phi}(\beta,x)}{\Omega^+_{\mathbf{f}_\phi}\Omega^-_{\mathbf{f}_\phi}},\qquad |u_\phi|_p = 1.\end{aligned}$$

The last equality follows from Corollary 11.7.6. The theorem then follows from the definition of the $C_{\mathfrak{D}_\phi}(\beta,x)$'s. ∎

Let $K^p_\mathbf{D} := \prod_{\ell\neq p}K_{\mathfrak{D}_\phi,\ell}$ and $\nu_\mathbf{D} := \prod_{\ell\neq p}\nu_{\mathfrak{D}_\phi,\ell}$ for any $\phi\in\mathcal{X}^a_\mathbf{D}$ (these are defined with respect to the fixed choice $M_{\mathfrak{D}_\phi} := M_\mathbf{D}$ and are independent of the choice of ϕ). Let $K'_\mathbf{D} := \ker\nu_\mathbf{D}$. For each $\phi\in\mathcal{X}^a_\mathbf{D}$ let $\nu_{\phi,p}: T(\mathbf{Q}_p)\to\overline{\mathbf{Q}}_p^\times$ be the character defined by by $\nu_{\phi,p}(\mathrm{diag}(a,b,c,d)) = \bar\chi_{\mathbf{f},0}\psi_\phi(cd)\bar\xi_\phi(d)$. We view this as a character of $(\mathbf{Z}_p^\times)^4$ via the identification of the latter with the diagonal torus of $\mathrm{GL}_4(\mathbf{Z}_p) = U(\mathbf{Z}_p)$. We let $\underline{a}\in(\mathbf{Z}/(p-1)\mathbf{Z})^4$ be such that for each $\phi\in\mathcal{X}^a_\mathbf{D}$ the restriction of $\nu_{\phi,p}$ to $\boldsymbol{\mu}^4_{p-1}\subset(\mathbf{Z}_p^\times)^4$ is $\omega^{\underline{a}-\underline{\kappa}_\phi}$, where $\underline{\kappa}_\phi = (0,0,\kappa_\phi,\kappa_\phi)$ and $\omega^{\underline{b}}$ is as in 6.3.

Theorem 12.4.2. *Let* $\mathbf{D} = (A,\mathbb{I},\mathbf{f},\psi,\xi,\Sigma)$ *be a p-adic Eisenstein datum such that*

(12.4.2.a) $$\mathrm{cond}(\psi_p),\mathrm{cond}(\xi_p)|p.$$

Suppose that $(\mathbf{irred})_\mathbf{f}$ *and* $(\mathbf{dist})_\mathbf{f}$ *hold and that* A *satisfies the hypotheses of Proposition 12.2.5. Let* $K'\subseteq K'_\mathbf{D}$ *be an open compact subgroup such that* $K'K_p$ *is neat.*

(i) *There exists* $\mathbf{E_D}\in\mathcal{M}_{\underline{a},\mathrm{ord}}(K';\Lambda_\mathbf{D})$ *such that for all* $\phi\in\mathcal{X}''_\mathbf{D}$

$$\mathbf{E}_{\mathbf{D},\phi} = u_\phi\frac{G_{\mathfrak{D}_\phi}}{\Omega^+_{\mathbf{f}_\phi}\Omega^-_{\mathbf{f}_\phi}},\quad |u_\phi|_p = 1.$$

(ii) *For* $x\in G(\mathbf{A}_f)$, $x_p\in Q(\mathbf{Z}_p)$, *let* $\sum_{\beta\in S(\mathbf{Q}),\beta\geq 0}\mathbf{c_D}(\beta,x)q^\beta$ *be the q-expansion of* $\mathbf{E_D}$ *at* x. *If* $\det\beta = 0$ *then*

$$\mathbf{c_D}(\beta,x)\in\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi}\mathcal{L}^\Sigma_{\bar\chi_\mathbf{f}\bar\chi'}\in\Lambda_\mathbf{D}.$$

Proof. We first note that each $G_{\mathfrak{D}_\phi}$ is ordinary by Proposition 9.6.2, so the claim of part (i) makes sense. Let $a_1, ..., a_{h_\mathcal{K}} \in \mathbf{A}_{\mathcal{K},}^{\Sigma,\times}$ be representatives for the class group of $\mathcal{K}$ and let $t_i := \mathrm{diag}(1, a_i, 1, \bar{a}_i^{-1})$. Then

$$G(\mathbf{A}_f) = \sqcup_{i=1}^{h_\mathcal{K}} G(\mathbf{Q}) t_i K'_\mathbf{D} K_p.$$

It then follows from (12.4.2.a) and Lemma 6.3.7 that it suffices to exhibit a formal q-expansion $E_\mathbf{D}(x) = \sum_{\beta\in S(\mathbf{Q}),\beta\geq 0} \mathbf{c}(\beta, x) q^\beta$ for each $x = t_i k$, $k \in K_\mathbf{D}^p$ that specializes to $u_\phi G_{\mathfrak{D}_\phi}(Z, x)/\Omega^+_{\mathbf{f}_\phi}\Omega^-_{\mathbf{f}_\phi}$ at each $\phi \in \mathcal{X}''_\mathbf{D}$. Theorem 12.4.1 provides such a $E_\mathbf{D}(t_i)$ for each $i = 1, ..., h_\mathcal{K}$, and for a general $x = t_i k$ we take $E_\mathbf{D}(x) := \nu_\mathbf{D}(k) E_\mathbf{D}(t_i)$. This proves part (i).

We now prove part (ii). We note that by part (i), for $\phi \in \mathcal{X}''_\mathbf{D}$ we have $\phi(\mathbf{c}_\mathbf{D}(\beta, x)) = u_\phi C_{\mathfrak{D}_\phi}(\beta, x)/\Omega^+_{\mathbf{f}_\phi}\Omega^-_{\mathbf{f}_\phi}$. Let $\beta \in S(\mathbf{Q})$, $\beta \geq 0$ and $\det\beta = 0$. Since $x_p \in Q(\mathbf{Z}_p)$, $\mathbf{c}_\mathbf{D}(\beta, x) = 0$ unless $\beta \in M_2(\mathcal{O}_p)$ (this can be deduced from the same property of each $C_{\mathfrak{D}_\phi}(\beta, x)$). We assume then that $\beta \in M_2(\mathcal{O}_p)$. It follows that there exists $\zeta \in \mathrm{SL}_2(\mathcal{K})$, $\zeta \in \mathrm{SL}_2(\mathcal{O}_p)$, such that $\beta = {}^t\bar{\zeta} \left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right) \zeta$, $n \geq 0$. Then $\mathbf{c}_\mathbf{D}(\beta, x) = \mathbf{c}_\mathbf{D}(\left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right), \mathrm{diag}(\zeta, {}^t\bar{\zeta}^{-1})x)$ (this too can be deduced from the corresponding equality for the $C_{\mathfrak{D}_\phi}(\beta, x)$'s). We may therefore assume $\beta = \left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right)$. As $\mathcal{X}^{gen}_\mathbf{D} \subseteq \mathcal{X}''_\mathbf{D}$ is also Zariski dense in $\mathrm{Spec}\,\Lambda_D(\overline{\mathbf{Q}}_p)$, we deduce from parts (i) and (ii) of Lemma 9.4.1 that $\mathbf{c}_\mathbf{D}(\beta, x) = 0$ unless $x^p \in P(\mathbf{A}^p_{\mathcal{K},f}) w^{(M)} K^p_\mathbf{D}$ and (since x_p is assumed to be in $Q(\mathbf{Z}_p)$) $x_p \in B(\mathbf{Z}_p)\{\begin{pmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & \\ & & * & 1 \end{pmatrix} \in M_Q(\mathbf{Z}_p)\}$. For such an x the inclusion claimed in part (ii) is an easy consequence of part (iii) of Lemma 9.4.1, the interpolation property of the Kubota-Leopoldt p-adic L-functions, and parts (ii) and (iii) of Proposition 11.7.4. The key point is the existence of an $\mathbb{I}$-adic form $\mathbf{f}_\mathbf{D}$ such that $\mathbf{f}_{\mathbf{D},\phi} = f_{\mathfrak{D}_\phi}$; this follows easily from (9.4.1.a). The existence of $\mathbf{f}_\mathbf{D}$ provides for each $A \in \mathrm{GL}_2(\mathbf{A}^p_f)$ a q-expansion $\sum_{n>0} \mathbf{a}_\mathbf{D}(n, A) q^n$ such that $\phi(\mathbf{a}_\mathbf{D}(n, A)) = a(n, f_{\mathfrak{D}_\phi}(-, A))$. ■

Remark. Note that while $\mathbf{E}_\mathbf{D}$ depends on $M_\mathbf{D}$ it does not, in fact, depend on K'. If K' and K'' both satisfy the hypotheses of the theorem, then the corresponding $\Lambda_\mathbf{D}$-forms are the same in $\mathcal{M}_{\underline{a},\mathrm{ord}}(K' \cap K''; \Lambda_\mathbf{D})$.

Suppose K' as in the preceding theorem satisfies $K' = K'_\Sigma K^\Sigma$ with $K^\Sigma = \prod_{\ell\notin\Sigma} K_\ell$. Since each $G_{\mathfrak{D}_\phi}$ is an eigenform for the u_t-operators and for the local Hecke algebras $\mathcal{H}'_{K_\ell}$ at primes away from Σ (see 9.6), the form $\mathbf{E}_\mathbf{D}$ from the preceding theorem is an eigenform for the universal ordinary Hecke algebra $\mathbf{h}_\mathbf{D}(K') := \mathbf{h}^{\Sigma,2}(K'; \Lambda_\mathbf{D})$ (see 6.4; by definition this is the $\Lambda_\mathbf{D}$-algebra generated by the image of the abstract Hecke algebra $\mathcal{U}_p \otimes \mathcal{H}^\Sigma$ in $\mathrm{End}_{\Lambda_\mathbf{D}}(\mathcal{M}_{\underline{a},\mathrm{ord}}(K'; \Lambda_\mathbf{D})))$. In particular, there is a $\Lambda_\mathbf{D}$-homomorphism $\lambda_{\mathbf{D},K'} : \mathbf{h}_\mathbf{D} \to \Lambda_\mathbf{D}$ such that for $h \in \mathbf{h}_\mathbf{D}$, $h.\mathbf{E}_\mathbf{D} = \lambda_{\mathbf{D},K'}(h)\mathbf{E}_\mathbf{D}$. This, of course, extends to a homomorphism of polynomial rings $\mathbf{h}_\mathbf{D}[X] \to \Lambda_\mathbf{D}[X]$, which we also denote by $\lambda_{\mathbf{D},K'}$. The following proposition follows from Propositions 9.6.1 and 9.6.2.

Proposition 12.4.3. *Let $\mathbf{D} = (A, \mathbb{I}, \mathbf{f}, \psi, \xi, \Sigma)$ be a p-adic Eisenstein datum as in Theorem 12.4.2. Suppose $K' \subseteq K'_{\mathbf{D}}$ is such that $K'K_p$ is neat and $K' = K'_\Sigma K^\Sigma$ with $K^\Sigma = \prod_{\ell \notin \Sigma} K_\ell$. Under the hypotheses of Theorem 12.4.2*

(i) *for $t = \mathrm{diag}(p^{a_1}, p^{a_2}, p^{a_4}, p^{a_3})$ with $a_1 \geq a_2 \geq a_3 \geq a_4$,*

$$\lambda_{\mathbf{D},K'}(u_t) = \prod_{i=1}^{4} \boldsymbol{\beta}_i^{a_i},$$

where

$$(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4) = (a(p,\mathbf{f})\boldsymbol{\psi}^{-1}(\varpi^c), \chi_{\mathbf{f},0}\boldsymbol{\psi}^{-1}\boldsymbol{\xi}^{-1}(\varpi^c), \chi_{\mathbf{f},0}^{-1}\boldsymbol{\psi}\boldsymbol{\xi}(\varpi), a(p,\mathbf{f})^{-1}\boldsymbol{\psi}(\varpi))$$

with $\chi_{\mathbf{f},0}$ the Dirichlet character constructed from $\chi_{\mathbf{f}}$ as in 9.4 and $\varpi \in \mathcal{K}_{v_0}$ (resp. $\varpi^c \in \mathcal{K}_{\bar{v}_0}$) the uniformizer identified with p via the identification $K_{v_0} = \mathbf{Q}_p$ (resp. $K_{\bar{v}_0} = \mathbf{Q}_p$);

(ii) *for v a finite place of $\mathcal{K}$ not dividing a prime in Σ and for any $\phi \in \mathcal{X}^a_{\mathbf{D}}$*

$$(\phi \circ \lambda_{\mathbf{D},K'})(Q_v) = \det(1 - X\rho_{\mathfrak{D}_\phi}(\mathrm{frob}_v))$$

where

$$\rho_{\mathfrak{D}_\phi} := \sigma_{\bar{\chi}_{\mathbf{f},0}\psi_\phi^c}\epsilon^{-3} \oplus (\rho_{\mathbf{f}_\phi} \otimes \sigma_{\bar{\chi}_{\mathbf{f},0}\bar{\xi}^c_\phi\psi^c_\phi}\epsilon^{-2}) \oplus \sigma_{\bar{\chi}_{\mathbf{f},0}\psi^c_\phi\bar{\xi}'_\phi}\epsilon^{-1}\det\rho_{\mathbf{f}};$$

(iii) *for $\ell \notin \Sigma$*

$$\lambda_{\mathbf{D},K'}(Z_{\ell,0}) = \chi_{\mathbf{f},0,\ell}^{-2}\boldsymbol{\psi}_\ell^2\boldsymbol{\xi}_\ell^{-1}(\ell),$$

and if ℓ splits in $\mathcal{K}$ then $\lambda_{\mathbf{D},K'}(Z^{(i)}_{\ell,0}) = \chi^{-1}_{\mathbf{f},0,\ell}\boldsymbol{\psi}_{\ell,i}\boldsymbol{\xi}^{-1}_{\ell,i}(\ell)$.

In part (iii), $Z_{\ell,0}$ and $Z^{(i)}_{\ell,0}$ are the images of the Hecke operators in $\mathcal{H}'_{K_\ell}$ denoted Z_0 and $Z_0^{(i)}$ in 9.5.2.

Proof. Part (i) is immediate from Proposition 9.6.2 and the Zariski-density of $\mathcal{X}''_{\mathbf{D}}$. Furthermore, if $\phi \in \mathcal{X}''_{\mathbf{D}}$ then $\phi \circ \lambda_{\mathbf{D},K'}|_{\mathcal{H}^\Sigma} = \lambda_{\mathfrak{D}_\phi}$, so part (ii) follows from 9.6.1 and the Zariski-density of $\mathcal{X}''_{\mathbf{D}}$. Part (iii) is obvious. ∎

13. p-adic properties of Fourier coefficients of $\mathbf{E}_{\mathbf{D}}$

In this section we prove (under certain hypotheses) that for a p-adic Eisenstein datum $\mathbf{D} = (A, \mathbb{I}, \mathbf{f}, \psi, \xi, \Sigma)$, given any height one prime divisor P of the the p-adic L-function $\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi}$, there is a q-expansion coefficient of $\mathbf{E}_{\mathbf{D}}$ that is not divisible by P. This is a key input into the proof of Theorem 6.5.4 which establishes that the length at such a P of the quotient by the Eisenstein ideal is at least the order at P of the p-adic L-function. Our proof of this indivisibility result involves identifying various combinations of q-expansion coefficients as interpolating Rankin-Selberg convolutions of the $\mathbf{f}_\phi$'s with theta-lifts of forms from definite unitary groups. This makes use of the explicit formulas from 11.9. It also involves some small input from the theory of automorphic forms on definite unitary groups (developed *ad hoc* here).

13.1. **Automorphic forms on some definite unitary groups.**

13.1.1. *Generalities.* Let $\beta \in S_2(\mathbf{Q})$, $\beta > 0$. As explained in 10.1, β defines a definite Hermitian $\mathcal{K}$-pairing on the two-dimensional $\mathcal{K}$-space V of column vectors: $(x,y)_\beta = {}^t\bar{x}\beta y$. We let H_β denote the unitary group of this pairing (an algebraic group over $\mathbf{Q}$), writing H for H_β when β is unimportant or understood.

For an open compact subgroup $U \subset H(\mathbf{A}_f)$ and any $\mathbf{Z}$-algebra R we let

$$\mathcal{A}_H(U;R) := \{f : H(\mathbf{A}) \to R \ : \ f(\gamma hku) = f(h), \gamma \in H(\mathbf{Q}), k \in H(\mathbf{R}), u \in U\}.$$

By restriction to $H(\mathbf{A}_f)$, $\mathcal{A}_H(U;R)$ is identified with the set of functions $f : H(\mathbf{A}_f) \to R$ such that $f(\gamma hu) = f(h)$ for all $\gamma \in H(\mathbf{Q})$ and $u \in U$. For any subgroup $K \subset H(\mathbf{A}_f)$ We let

$$\mathcal{A}_H(K;R) := \varinjlim_{U \supseteq K} \mathcal{A}_H(U;R),$$

where the transition map for $U' \subseteq U$ is the inclusion $\mathcal{A}_H(U;R) \subseteq \mathcal{A}_H(U';R)$. When $R = \mathbf{C}$ we may drop it from our notation as we do the K when $K = \{1\}$ (so $\mathcal{A}_H = \mathcal{A}_H(\{1\};\mathbf{C})$).

Examples.

(a) Let $\Phi = \Phi_{\beta,\infty} \otimes_{\ell\nmid\infty} \Phi_\ell \in \mathcal{S}(V \otimes \mathbf{A})$. For each $g \in U_1(\mathbf{A})$, as a function of $h \in H_\beta(\mathbf{A})$ the theta function $\Theta_\beta(h,g;\Phi)$ belongs to $\mathcal{A}_{H_\beta}$. Therefore, so does $\Theta_\beta(h,w;\Phi) := j(g_\infty,i)^2\Theta_\beta(w,g_\infty;\Phi)$, $g_\infty \in U_1(\mathbf{R})$ such that $w = g_\infty(i)$.

(b) Let $\mathcal{D} = (\Sigma,\varphi,\psi,\tau)$ be an Eisenstein datum. For any $x \in G(\mathbf{A}_f)$ it follows from (9.3.4.c) that the function $c_{\mathcal{D}}(\beta,x;-) : H_\beta(\mathbf{A}_f) \to \mathbf{C}$ defined by

$$c_{\mathcal{D}}(\beta,x;h) := \bar{\tau}(\det h) c_{\mathcal{D}}(\beta, \mathrm{diag}(h, {}^t\bar{h}^{-1})x)$$

belongs to $\mathcal{A}_{H_\beta}$.

(c) Let $\mathfrak{D}$ be a classical datum and let $\mathcal{D} = (\Sigma,\varphi,\psi_0,\tau_0)$ be its associated Eisenstein datum. Let $C_{\mathfrak{D}}(\beta,x;h) := \bar{\tau}_0(\det h)C_{\mathfrak{D}}(\beta,\mathrm{diag}(h,{}^t\bar{h}^{-1})x)$, where $C_{\mathfrak{D}}(\beta,y)$ is the β-Fourier coefficient of $G_{\mathfrak{D}}(Z,y)$, the latter being as in (11.7.5.a). It follows from (b) that $C_{\mathfrak{D}}(\beta,x;-) \in \mathcal{A}_{H_\beta}$.

13.1.2. *Hecke operators.* Let $U, U' \subset H(\mathbf{A}_f)$ be open compact subgroups and let $h \in H(\mathbf{A}_f)$. We define a Hecke operator $[U'hU] : \mathcal{A}_H(U;R) \to \mathcal{A}_H(U';R)$ by

$$[U'hU]f(x) = \sum f(xh_i), \quad U'hU = \sqcup_i h_i U.$$

These operators are clearly functorial in R.

We will be concerned with two special cases.

<u>The unramified case</u>. Suppose ℓ splits in $\mathcal{K}$. The identification $\mathrm{GL}_2(\mathcal{K}_\ell) = \mathrm{GL}_2(\mathbf{Q}_\ell) \times \mathrm{GL}_2(\mathbf{Q}_\ell)$ identifies $H(\mathbf{Q}_\ell)$ with $\mathrm{GL}_2(\mathbf{Q}_\ell)$ via projection onto the first factor: $H(\mathbf{Q}_\ell) = \{(A, \beta^{-1\,t}A^{-1}\beta) \in \mathrm{GL}_2(\mathcal{K}_\ell)\}$. We let $H_\ell \subset H(\mathbf{Q}_\ell)$ be the subgroup identified with $\mathrm{GL}_2(\mathbf{Z}_\ell)$.

For $U = H_\ell U'$, $U' \subset H(\mathbf{A}_f^\ell)$, we write T_ℓ^H for the Hecke operator $[Uh_\ell U]$, $h_\ell := \begin{pmatrix} \ell & \\ & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_\ell) = H(\mathbf{Q}_\ell)$. Clearly the action of T_ℓ^H respects variation in U': if $f \in \mathcal{A}_H(U; R)$ and $U'' \subseteq U'$ then $[Uh_\ell U]f = [U_1 h_\ell U_1]f$, $U_1 = H_\ell \times U''$. The T_ℓ^H's commute with each other.

Hecke operators at p. For a positive integer n we let $I_n \subset H_p$ be the subgroup identified with the set of $g \in \mathrm{GL}_2(\mathbf{Z}_p)$ such that g modulo p^n belongs to $N_{B'}(\mathbf{Z}/p^n\mathbf{Z}_p)$. For $U = I_n U'$, $U' \subset H(\mathbf{A}_f^{\{p\}})$, we write U_p^H for the Hecke operator $[Uh_pU]$. This operator respects variation in n and U' and commutes with the T_ℓ^H's for $\ell \neq p$.

13.1.3. *The (nearly) ordinary projector.* Let R be either a p-adic ring or of the form $R = R_0 \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ with R_0 a p-adic ring. Then for $U = I_n U'$, $U' \subset H(\mathbf{A}_f^{\{p\}})$,

$$e_H := \varinjlim_m (U_p^H)^{m!} \in \mathrm{End}_R(\mathcal{A}_H(U; R))$$

exists and is an idempotent. The idempotent e_H clearly respects variation in n and U'. Furthermore, since we have fixed an isomorphism $\mathbf{C}_p \cong \mathbf{C}$, e_H is defined on $\mathcal{A}'_H := \lim_{n\to\infty} \mathcal{A}_H(I_n)$.

13.2. **Applications to Fourier coefficients.**

13.2.1. *Forms on $H \times U_1$.* If ℓ splits in $\mathcal{K}$ then we view representations of $H(\mathbf{Q}_\ell)$ and $U_1(\mathbf{Q}_\ell)$ as representations of $\mathrm{GL}_2(\mathbf{Q}_\ell)$ via the respective identifications of these groups with $\mathrm{GL}_2(\mathbf{Q}_\ell)$ (projection onto the first factor of $\mathrm{GL}_2(\mathcal{K}_\ell) = \mathrm{GL}_2(\mathbf{Q}_\ell) \times \mathrm{GL}_2(\mathbf{Q}_\ell)$).

Let λ be a character of $\mathbf{A}_\mathcal{K}^\times/\mathcal{K}^\times$ such that $\lambda_\infty(z) = (z/|z|)^{-2}$ and $\lambda|_{\mathbf{A}^\times} = 1$. Let $(\pi, \mathcal{V})$, $\mathcal{V} \subseteq \mathcal{A}_H$, be an irreducible representation of $H(\mathbf{A}_f)$ and let $(\sigma, \mathcal{W})$, $\mathcal{W} \subseteq \mathcal{A}(U_1)$, be an irreducible representation of $U_1(\mathbf{A}_f)$. Let χ_π and χ_σ be their respective central characters. We assume that

(13.2.1.a)
- $\chi_\sigma = \lambda\chi_\pi^{-1}$;
- if ℓ splits in $\mathcal{K}$ then $\sigma_\ell \simeq \tilde{\pi}_\ell \otimes \lambda_{\ell,1}$ as representations of $\mathrm{GL}_2(\mathbf{Q}_\ell)$.

We also assume that we are given

(13.2.1.b)
- a finite set S of primes outside of which λ is unramified;
- a finite order character θ of $\mathbf{A}_\mathcal{K}^\times/\mathcal{K}^\times$ extending χ_π^{-1} and unramified outside S

Let $\varphi \in \mathcal{V} \otimes \mathcal{W}$. We assume that

(13.2.1.c)
- if $\ell \notin S$ then $\varphi(hu, g) = \varphi(h, g)$ for $u \in H_\ell$;
- there exists an integer N divisible only by primes in S such that $\varphi(h, gk) = \lambda\theta(d_k)\varphi(h, g)$ for all $k \in U_1(\widehat{\mathbf{Z}})$ satisfying $N|c_k$.

For $w \in \mathfrak{h}$ we let

$$\varphi(h, w) := j(g_\infty, i)^2\varphi(h, g_\infty)$$

for some $g_\infty \in \mathrm{SL}_2(\mathbf{R})$ such that $g_\infty(i) = w$. We will assume that $\varphi(h,w)$ does not depend on the choice of g_∞ and that

$$\varphi(h,w) \in M_2(N,\theta'). \tag{13.2.1.d}$$

Lemma 13.2.2. *If* (13.2.1.a)-(13.2.1.d) *hold, then for any $\ell \notin S$ that splits in $\mathcal{K}$*

$$\theta_{\ell,1}(\ell)T_\ell^H\varphi(h,w) = T_\ell\varphi(h,w),$$

where $\theta_\ell = (\theta_{\ell,1},\theta_{\ell,2})$ as a character of $\mathcal{K}_\ell^\times = \mathbf{Q}_\ell^\times \times \mathbf{Q}_\ell^\times$.

Proof. By (13.2.1.c) π_ℓ is unramified, say $\pi_\ell \cong \pi(\mu_1,\mu_2)$ with μ_1 and μ_2 unramified, and $\varphi(h,w)$ is the newvector for π_ℓ. Hence

$$T_\ell^H\varphi(h,w) = \ell^{1/2}(\mu_1(\ell)+\mu_2(\ell))\varphi(h,w). \tag{13.2.2.a}$$

Noting that $G_1(\mathbf{A}) = G_1(\mathbf{Q})U_1(\mathbf{A})\mathbf{A}_\mathcal{K}^\times K_1(N)$, for a fixed $h \in H(\mathbf{A})$ we extend $\varphi(h,g)$ to a function $\tilde\varphi(h,g)$ on $G_1(\mathbf{A})$ by setting $\tilde\varphi(h,g) := \theta(a)\theta(d_k)\varphi(h,g')$ for $g = \gamma g' a k$, $\gamma \in G_1(\mathbf{Q})$, $g' \in U_1(\mathbf{A})$, $a \in \mathbf{A}_\mathcal{K}^\times$, $k \in K_1(N)$. Clearly $\tilde\varphi(h,g)$ is the form on $G_1(\mathbf{A})$ defined by the pair $\varphi(h,w)_\mathbf{A}$ and θ. Then $\tilde\varphi(h,w) := j(g_\infty,i)^2\tilde\varphi(h,g_\infty) = \varphi(g,w)$. Let σ' be an irreducible constituent of the $\mathrm{GL}_2(\mathbf{A})$-representation generated by $\tilde\varphi(h,g)$. Then $\sigma'_\ell \cong \pi(\mu_1,\mu_2)\otimes\theta_{\ell,1}$. It follows that

$$\begin{aligned} T_\ell\varphi(h,w) = T_\ell\tilde\varphi(h,w) &= \theta_{\ell,1}(\ell)\ell^{1/2}(\mu_1(\ell)+\mu_2(\ell))\tilde\varphi(h,w) \\ &= \theta_{\ell,1}(\ell)\ell^{1/2}(\mu_1(\ell)+\mu_2(\ell))\varphi(h,w). \end{aligned}$$

Comparing this with (13.2.2.a) yields the lemma. ∎

Suppose additionally that

(13.2.2.b)
- λ_p is unramified;
- $\mathrm{cond}(\theta_{p,1}) = (p^r)$, $\mathrm{cond}(\theta_{p,2}) = (p^s)$, $r > s$;
- $p^r || N$;
- $\varphi(hk,g) = \theta_{p,1}^{-1}(a_{k_1})\theta_{p,2}(a_{k_1})\varphi(h,g)$ for $k = (k_1,k_2) \in H_p$, $p^r | c_{k_1}$.

Note that this implies $\varphi \in \mathcal{V}^{I_r}\otimes\mathcal{W}$.

Lemma 13.2.3. *Assume* (13.2.1.a)-(13.2.1.d) *and* (13.2.2.b) *hold. Then*

$$e_H\varphi(h,w) = e\varphi(h,w).$$

Here, e is the usual ordinary idempotent acting on $\varphi(h,-) \in M_2(N,\theta')$.

Proof. Since $\mathrm{cond}(\theta_1\theta_2) = \mathrm{cond}(\theta_1)$, the vector $v := \varphi(-,g) \in \mathcal{V}^{I_r}$ corresponds to the new vector for $\pi\otimes\theta_1$. It follows that $U_p^H v \neq 0$ if and only if $\pi \cong \pi(\mu_1,\mu_2)$ with $\mu_1\theta_1$ unramified and $\mu_2\theta_1$ ramified with $\mathrm{cond}(\mu_2\theta_1) = (p^r)$, in which case $U_p^H v = p^{1/2}\mu_1(p)$. So $e^H v \neq 0$ if and only if $p^{1/2}\mu_1(p)$ is a p-adic unit, in which case $e^H v = v$.

Let $v = \tilde\varphi(h,-) \in \pi\otimes\theta_1$ be as in the proof of Lemma 13.2.2. Then v is the new vector of $\pi\otimes\theta_1$ and arguing as above shows that $U_p v \neq 0$ if and only if $\pi \cong \pi(\mu_1,\mu_2)$

with $\mu_1 \otimes \theta_1$ unramfied and $\mu_2 \otimes \theta_1$ ramified with $\mathrm{cond}(\mu_2 \otimes \theta_1) = (p^r)$, in which case $U_p v = p^{1/2}\mu_1\theta_1(p)$. It follows that $e\varphi(h,w) \neq 0$ if and only if $p^{1/2}\mu_1\theta_1(p)$ is a p-adic unit, which happens if and only if $p^{1/2}\mu_1(p)$ is a p-adic unit since θ_1 has finite order, in which case $e\varphi(h,w) = \varphi(h,w)$. Combining this with the preceding observations about $e^H\varphi(h,w)$ yields the lemma. ∎

13.2.4. *Consequences for Fourier coefficients.* We return to the notation and setup of 11.8.1 and 11.9. In particular, $\mathfrak{D} = (f, \psi, \xi, \Sigma)$ is a classical datum with associated Eisenstein datum $\mathcal{D} = (\varphi, \psi_0, \tau_0, \Sigma)$ such that (11.8.1.a) holds and $M_{\mathcal{D}}$ is as in 11.8.1. We assume that (11.9.3.a) holds. Recall that for $1 \leq i, j \leq h_{\mathcal{K}}$, $\beta_{ij} = \begin{pmatrix} q_j^{-1} & \\ & q_i q_j^{-1} \end{pmatrix}$, so for fixed i the unitary groups $H_{\beta_{ij}}$ are independent of j; we denote $H_{\beta_{ij}}$ by H_i.

Let $\Theta_{ij}(h,g) := \Theta_{\beta_{ij}}(h, g \begin{pmatrix} \delta_{\mathcal{K}} & \\ & \bar{\delta}_{\mathcal{K}}^{-1} \end{pmatrix}_f ; \Phi_{\mathcal{D},\beta_{ij},u_{ij}})$. From the definition of $\Phi_{\mathcal{D},\beta_{ij},u_{ij}}$ (see (11.9.3.b)) and Lemmas 10.2.4, 10.2.5, and 10.2.6 it follows that

(13.2.4.a)

- if $\ell \notin \Sigma \cup \mathcal{Q}$ then $\Theta_{ij}(hu, gk) = \Theta_{ij}(h,g)$ for $u \in H_{i,\ell}$ and $k \in U_1(\mathbf{Z}_\ell)$;
- if $\ell \in \Sigma \cup \mathcal{Q}, \ell \neq p$ then $\Theta_{ij}(h, gk) = \lambda(d_k)\Theta_{ij}(h,g)$ for $k \in K_{1,\ell}(M_{\mathcal{D}}^2 D_{\mathcal{K}} \prod_{q \in \mathcal{Q}} q)$;
- $\Theta_{ij}(hu, gk) = \xi_{p,2}^{-1}(a_{u_1})\xi_{p,1}(d_{u_1})\lambda\xi^c(d_k)\Theta_{ij}(h,g)$ for $u = (u_1, u_2) \in H_{i,p}$ with $p^{u_p} | c_{u_1}$ and $k \in U_1(\mathbf{Z}_p)$ with $p^{u_p} | c_k$ (recall that $(p^{u_p}) := (x_p) \cap \mathbf{Z}_p$).

Here λ is the character fixed for the definition of the Weil representations used to define the theta functions.

For $w \in \mathfrak{h}$ let $\Theta_{ij}(h,w) := j(g_\infty)^2\Theta_{ij}(h, g_\infty)$ for any $g_\infty \in \mathrm{SL}_2(\mathbf{R})$ such that $g_\infty(i) = w$. Note that $\Theta_{ij}(h,w) = \bar{\delta}_{\mathcal{K}}^{-2}\Theta_{\mathcal{D},\beta_i/q_j,u_i a_j}(h, D_{\mathcal{K}} w)$. We decompose each $\Theta_{ij}(h,g)$ with respect to irreducible automorphic representations π_H of $H_i(\mathbf{A}_f)$:

$$\Theta_{ij}(h,g) = \sum_{\pi_{\mathbf{H}}} \varphi_{\pi_H}^{(ij)}(h,g).$$

(For each $g \in U_1(\mathbf{A})$, $\varphi^{(ij)}(-,g)$ generaters π_H.) The hypotheses ensure that each $\Theta_{ij}(h,g)$ is orthogonal to all finite-dimensional representations of $H_i(\mathbf{A}_f)$. Hence we may assume that the preceding decomposition of $\Theta_{ij}(h,g)$ is over infinite-dimensional representations of $H_i(\mathbf{A}_f)$. It is then a general consequence of the theta correspondence (in the split case) that the irreducible constituents σ of the representation of $U_1(\mathbf{A})$ generated by $\phi_{\pi_H}^{(ij)}$ satisfy

(13.2.4.b) $\sigma_\ell \simeq \pi_{H,\ell} \otimes \lambda_{\ell,1}$ as representations of $\mathrm{GL}_2(\mathbf{Q}_\ell)$ for all ℓ split in $\mathcal{K}$.

We may then decompose $\Theta_{ij}(h,g)$ as

$$\Theta_{ij}(h,g)=\sum_{(\pi_H,\sigma)}\varphi^{(ij)}_{(\pi_H,\sigma)}(h,g),\quad \varphi^{(ij)}_{(\pi_H,\sigma)}\in\pi_H\otimes\sigma.$$

with each pair (π_H,σ) consisting of an irreducible automorphic representation π_H of $H_i(\mathbf{A}_f)$ and an irreducible automorphic representation σ of $U_1(\mathbf{A})$ satisfying (13.2.4.b) and such that π_H is orthogonal to all finite-dimensional representations of $H_i(\mathbf{A}_f)$.

Suppose that the central character χ_{π_H} of π_H satisfies $\chi_{\pi_H}=\xi(=\psi_0/\tau_0)$. Then by (13.2.4.a) and (13.2.4.b), each $\varphi^{(ij)}_{(\pi_H,\sigma)}(h,g)$ satisfies (13.2.1.a)-(13.2.1.d) and (13.2.2.b) for $S=\Sigma\cup\mathcal{Q}$, $\theta=\xi^c$, and $N=p^{u_p}M_{\mathcal{D}}^2D_{\mathcal{K}}\prod_{q\in\mathcal{Q}}q$. So Lemmas 13.2.2 and 13.2.3 hold for the $\varphi^{(ij)}_{(\pi_H,\sigma)}(h,w)$'s.

For $i=1,\dots,h_{\mathcal{K}}$ let

$$C_{\mathfrak{D},i}(h):=\bar\tau_0(\det h)C_{\mathfrak{D}}(\beta_i,\mathrm{diag}(u_i,{}^t\bar u_i^{-1});h)\in\mathcal{A}_{H_i},$$

(notation as in Example (c) of 13.1.1).

Proposition 13.2.5. *Let $\mathcal{L}=\{\ell_1,\dots,\ell_m\}$ be a set of primes that split in $\mathcal{K}$ and do not belong to $\Sigma\cup\mathcal{Q}$. Let $P\in\mathbf{C}[X_1,\dots,X_m]$. Let $P_{H_i}:=P(\xi_{\ell_1,1}(\ell_1)T^{H_i}_{\ell_1},\dots,\xi_{\ell_m,1}(\ell_m)T^{H_i}_{\ell_m})$ and $P_1:=P(T_{\ell_1},\dots,T_{\ell_m})$. Then*

$$\begin{aligned}&\frac{e_{H_i}P_{H_i}C_{\mathfrak{D},i}(h)}{2^{-3}(2i)^{\kappa+1}S(f)<f,f^c|_\kappa\begin{pmatrix} & -1\\ N & \end{pmatrix}>_{\Gamma_0(N)}}\\ &\qquad=\bar\tau_0(\det h)B_{\mathfrak{D}}(\beta_i,h,u_i)\\ &\qquad\qquad\times\frac{<\mathcal{E}_{\mathfrak{D}}\cdot eP_1\Theta_{\mathfrak{D},\beta_i}(h,-;u_i),f^c|_\kappa\begin{pmatrix} & -1\\ p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}} & \end{pmatrix}>_{\Gamma_0(p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}})}}{<f,f^c|_\kappa\begin{pmatrix} & -1\\ p^{r_p}M & \end{pmatrix}>_{\Gamma_{\mathfrak{D}}}},\end{aligned}$$

where $B_{\mathfrak{D}}(\beta_i,h,u_i)$ is as in Proposition 11.8.2.

Proof. Recall that $\Theta_{\mathfrak{D},\beta_i}(-,-;u_i)$ is a linear combination of the Θ_{ij}'s, say $\Theta_{\mathfrak{D},\beta_i}(h,w;u_i) = \sum_{j=1}^{h_{\mathcal{K}}} c_{ij}\Theta_{ij}(h,w)$. Then by Proposition 11.8.2

$$\frac{\tau_0(\det h)e_{H_i}P_{H_i}C_{\mathfrak{D},i}(h)}{2^{-3}(2i)^{\kappa+1}S(f) < f, f^c|_\kappa \left(\begin{smallmatrix} & -1 \\ N & \end{smallmatrix}\right) >_{\Gamma_0(N)}}$$

$$= B_{\mathfrak{D}}(\beta_i,h,u_i)\frac{< \mathcal{E}_{\mathfrak{D}}\cdot e_{H_i}P_{H_i}\Theta_{\mathfrak{D},\beta_i}(h,-;u_i), f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}})}}{< f, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}}}$$

$$= B_{\mathfrak{D}}(\beta_i,h,u_i)\sum_{j=1}^{h_{\mathcal{K}}} c_{ij}\frac{< \mathcal{E}_{\mathfrak{D}}\cdot e_{H_i}P_{H_i}\Theta_{ij}, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}})}}{< f, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}}}$$

$$= B_{\mathfrak{D}}(\beta_i,h,u_i)\sum_{j=1}^{h_{\mathcal{K}}} c_{ij}\sum_{(\pi_H,\sigma)}\frac{< \mathcal{E}_{\mathfrak{D}}\cdot e_{H_i}P_{H_i}\varphi^{(ij)}_{(\pi_H,\sigma)}, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_p}M_{\mathcal{D}}^2D_{\mathcal{K}})}}{< f, f^c|_\kappa \begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}}}}.$$

Since $C_{\mathfrak{D},i}(ha) = \xi(a)C_{\mathfrak{D},i}(h)$ for $a \in \mathbf{A}_{\mathcal{K}_f}^\times \cap H(\mathbf{A}_f)$, it follows that the inner-product in the last line is zero unless $\chi_{\pi_H} = \xi$. For such π_H, as earlier noted, Lemmas 13.2.2 and 13.2.3 hold for $\varphi^{(ij)}_{(\pi_H,\sigma)}$. In particular, $e_{H_i}P_{H_i}\varphi^{(ij)}_{(\pi,\sigma)}(h,w) = eP_1\varphi^{(ij)}_{(\pi,\sigma)}(h,w)$. The proposition follows. ∎

13.3. **p-adic properties of the $\mathbf{c_D}(\beta,x)$'s.** Let $\mathbf{D} = (A,\mathbb{I},\mathbf{f},\psi,\xi,\Sigma)$ be p-adic Eisenstein datum as in §12. We assume that (12.4.2.a) holds and that $(\mathbf{irred})_{\mathbf{f}}$ and $(\mathbf{dist})_{\mathbf{f}}$ hold. We let $\mathbf{E_D} \in \mathcal{M}_{\underline{a},\mathrm{ord}}(K'_{\mathbf{D}};\Lambda_{\mathbf{D}})$ be as in Theorem 12.4.2 (the definition of $\mathbf{E_D}$ depends on a choice of $M_{\mathbf{D}}$). For $x \in G(\mathbf{A}_f)$ with $x_p \in Q(\mathbf{Z}_p)$ we let $\mathbf{c_D}(\beta,x) \in \Lambda_{\mathbf{D}}$ be the β-Fourier coefficient of $\mathbf{E_D}$ at x. So for $\phi \in \mathcal{X}^a_{\mathbf{D}}$, $c_{\mathbf{D},\phi}(\beta,x) := \phi(\mathbf{c_D}(\beta,x))$ is the β-coefficient of the Fourier expansion at x of a holomorphic Hermitian modular form $\mathbf{E}_{\mathbf{D},\phi}(Z,x)$. For *any* $x \in G(\mathbf{A}_f)$ we define $c_{\mathbf{D},\phi}(\beta,x)$ to be the β-coefficient of the Fourier expansion of $\mathbf{E}_{\mathbf{D},\phi}(Z,x)$. Note that if $\phi \in \mathcal{X}'_{\mathbf{D}}$ then $\mathbf{E}_{\mathbf{D},\phi}(Z,x) = u_\phi G_{\mathfrak{D}_\phi}(Z,x)/\Omega^+_{\mathbf{f}_\phi}\Omega^-_{\mathbf{f}_\phi}$ and $c_{\mathbf{D},\phi}(\beta,x) = u_\phi C_{\mathfrak{D}_\phi}(\beta,x)/\Omega^+_{\mathbf{f}_\phi}\Omega^-_{\mathbf{f}_\phi}$.

Since $c_{\mathbf{D},\phi}(\beta,x)$ is the β-Fourier coefficient of $\mathbf{E}_{\mathbf{D},\phi}(Z,x)$, the $\phi(\Lambda_{\mathbf{D}})$-valued function on $\mathrm{GL}_2(\mathbf{A}_{\mathcal{K},f})$ defined by

$$\varphi_{\mathbf{D},\beta,x,\phi}(h) := \chi_{\mathbf{f}}\chi_{\mathbf{f},0}\psi_\phi^{-1}\xi_\phi(\det h)c_{\mathbf{D},\phi}(\beta, \left(\begin{smallmatrix} h & \\ & {}^t\bar{h}^{-1}\end{smallmatrix}\right) x)$$

belongs to $\mathcal{A}_{H_\beta}(\phi(\Lambda_{\mathbf{D}}))$ when restricted to $H_\beta(\mathbf{A}_f)$ (see Example (c) of 13.1.1).

13.3.1. *Interpolating Hecke operators.* For $i = 1,...,h_{\mathcal{K}}$ recall that $\beta_i = \left(\begin{smallmatrix} 1 & \\ & q_i\end{smallmatrix}\right)$ and $u_i = \gamma_0\begin{pmatrix} 1 & \\ & a_i^{-1}\end{pmatrix}$ (see 11.9, especially for the definition of γ_0). For $h \in \mathrm{GL}_2(\mathbf{A}_{\mathcal{K},f})$ with $h_p \in \mathrm{GL}_2(\mathcal{O}_p)$ let

$$\varphi_{\mathbf{D},i} := \chi_{\mathbf{f}}\chi_{\mathbf{f},0}\boldsymbol{\psi}^{-1}\boldsymbol{\xi}^{-1}(\det h)\mathbf{c_D}(\beta_i, \begin{pmatrix} hu_i & \\ & {}^t\bar{h}^{-1}{}^t\bar{u}_i\end{pmatrix}) \in \Lambda_{\mathbf{D}}.$$

Similary, for $\phi \in \mathcal{X}_{\mathbf{D}}^a$ and $h \in \mathrm{GL}_2(\mathbf{A}_{\mathcal{K},f})$ let

$$\varphi_{\mathbf{D},i,\phi}(h) := \varphi_{\mathbf{D},\beta_i,\begin{pmatrix} u_i & \\ & {}^t\bar{u}_i^{-1}\end{pmatrix},\phi}(h).$$

If $h_p \in \mathrm{GL}_2(\mathcal{O}_p)$, then $\phi(\varphi_{\mathbf{D},i}(h)) = \varphi_{\mathbf{D},i,\phi}(h)$.

Let $H_i := H_{\beta_i}$ and note that $H_{i,p} \subset \mathrm{GL}_2(\mathcal{O}_p)$. There exists an open compact subgroup $U_i = \prod_{\ell\neq p} U_{i,\ell} \subset H_i(\mathbf{A}_f^{\{p\}})$, $U_{i,\ell} \subseteq H_{i,\ell}$ if $\ell \notin \Sigma \cup \mathcal{Q}$, such that for $\phi \in \mathcal{X}_{\mathbf{D}}$ there exists $n_\phi > 0$ so that $\varphi_{\mathbf{D},i,\phi} \in \mathcal{A}_{H_i}(I_{n_\phi}U_i; \phi(\Lambda_{\mathbf{D}}))$.

Lemma 13.3.2. *Let $\mathcal{L} = \{\ell_1, ...\ell_m\}$ be a finite set of primes that split in $\mathcal{K}$ and do not belong to $\Sigma \cup \mathcal{Q}$. Let $P \in \Lambda_{\mathbf{D}}[X_1, ..., X_m]$. For $h \in H_i(\mathbf{A}_f)$ with $h_p \in H_{i,p}$, there exists $\varphi_{\mathbf{D},i}(\mathcal{L}, P; h) \in \Lambda_{\mathbf{D}}$ such that*

(a) *for all $\phi \in \mathcal{X}_{\mathbf{D}}^a$,*

$$\phi(\varphi_{\mathbf{D},i}(\mathcal{L}, P; h)) = P_\phi(\xi_{\phi,\ell_1,2}(\ell_1)T_{\ell_1}^{H_i}, ..., \xi_{\phi,\ell_m,2}(\ell_m)T_{\ell_m}^{H_i})e_{H_i}\varphi_{\mathbf{D},i,\phi}(h),$$

where P_ϕ is the polynomial obtained by applying ϕ to the coefficients of P;

(b) *if $M \subseteq \Lambda_{\mathbf{D}}$ is a closed $\Lambda_{\mathbf{D}}$-submodule and $\varphi_{\mathbf{D},i}(h) \in M$ for all h with $h_p \in H_{i,p}$, then $\varphi_{\mathbf{D},i}(\mathcal{L}, P; h) \in M$.*

Proof. Let $\mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots$, $\mathcal{X}_{\mathbf{D}}^a = \cup_{i=1}^\infty \mathcal{X}_i$, be a filtration of $\mathcal{X}_{\mathbf{D}}^a$ by finite subsets. Let $n_r := \max\{n_\phi \ : \ \phi \in \mathcal{X}_r\}$. Let $I_{n_r}h_p^{n_r!}I_{n_r} = \sqcup h_{rj}I_{n_r}$. Write $hh_{rj} = \gamma_{rj}k_{rj}x_{rj}u_{rj}$ with $\gamma_{rj} \in H_i(\mathbf{Q})$, $k_{rj} \in H_i(\mathbf{R})$, $x_{rj} \in H_i(\mathbf{A}_f^p)$, and $u_{rj} \in I_{n_r}U_i$. Put $\varphi_{\mathbf{D},i}^{(r)}(h) := \sum_j \varphi_{\mathbf{D},i}(x_{rj}) \in M$. Then $\phi(\varphi_{\mathbf{D},i}^{(r)}(h)) = U_p^{n_r!}\varphi_{\mathbf{D},i,\phi}(h)$ for $\phi \in \mathcal{X}_r$. Let $\tilde\varphi_{\mathbf{D},i}(h) := \lim_{r\to\infty} \varphi_{\mathbf{D},i}^{(r)} \in M$ (the limit belongs to M as M is closed in $\Lambda_{\mathbf{D}}$). Then $\phi(\tilde\varphi_{\mathbf{D},i}(h)) = e_{H_i}\varphi_{\mathbf{D},i,\phi}(h)$ for all $\phi \in \mathcal{X}_{\mathbf{D}}^a$, proving the lemma for P a constant polynomial.

For $1 \leq i_1, ..., i_m \leq m$ (not necessarily distinct) we let

$$\varphi_{\mathbf{D},i}(\mathcal{L}, X_{i_1}\cdots X_{i_m}; h) := \prod_{a=1}^m \boldsymbol{\xi}_{\ell_{i_a},1}(\ell_{i_a}) \sum_{j_1} \cdots \sum_{j_m} \tilde\varphi_{\mathbf{D},i}(hh_{j_1}\cdots h_{j_m}) \in M,$$

$$H_{\ell_{i_a}}h_{\ell_{i_a}}H_{\ell_{i_a}} = \sqcup_{j_a} h_{j_a}H_{\ell_{i_a}}.$$

Then for any $\phi \in \mathcal{X}_{\mathbf{D}}^a$

$$\phi(\varphi_{\mathbf{D},i}(\mathcal{L}, X_{i_1}\cdots X_{i_m}; h)) = T_{\ell_{i_1}}^{H_i}\cdots T_{\ell_{i_m}}^{H_i} e_{H_i}\varphi_{\mathbf{D},i,\phi}(h).$$

This proves the lemma for P a monomial; the general case follows by linear extension. ∎

13.3.3. *Another interpolation formula.* Suppose

(13.3.3.a)

- $M_{\mathbf{D}}$ satisfies (11.5.0.c);
- $\chi_{\mathbf{f}} = 1$;
- ψ and ξ satisfy (11.9.3.a).

These assumptions ensure that $\chi_{\mathbf{f}_\phi}$, ξ_ϕ, and ψ_ϕ also satisfy (11.9.3.a) for all $\phi \in \mathcal{X}^a_{\mathbf{D}}$. The assumption $\chi_{\mathbf{f}} = 1$, while essential to later arguments, is only made at this point to simplify subsequent formulas.

Let $\mathfrak{p}_1 := \mathfrak{p}$ and $\mathfrak{p}_2 := \bar{\mathfrak{p}}$, so $p\mathcal{O} = \mathfrak{p}_1\mathfrak{p}_2$. Write $\Gamma_{\mathcal{K}} = \Gamma^{(1)}_{\mathcal{K}} \times \Gamma^{(2)}_{\mathcal{K}}$ where $\Gamma^{(i)}_{\mathcal{K}}$ is the unique rank one $\mathbf{Z}_p$-summand containing $\Psi_{\mathcal{K}}(O^\times_{\mathfrak{p}_i})$. Let $\Psi^{(i)}_{\mathcal{K}}$ be composition of $\Psi_{\mathcal{K}}$ with the canonical projection $\Gamma_{\mathcal{K}} \to \Gamma^{(i)}_{\mathcal{K}}$. Then $\Psi^{(i)}_{\mathcal{K}}$ is ramified at $\mathfrak{p}_i$ and unramified at $\bar{\mathfrak{p}}_i$, and $\Psi_{\mathcal{K}} = \Psi^{(1)}_{\mathcal{K}}\Psi^{(2)}_{\mathcal{K}}$. Let

$$\boldsymbol{\xi}_i := \beta \circ \xi_i \Phi^{(i)}_{\mathcal{K}}.$$

Then $\boldsymbol{\xi}_i$ is ramified at $\mathfrak{p}_i$ and unramified at $\bar{\mathfrak{p}}_i$, and

$$\boldsymbol{\xi} = \boldsymbol{\xi}_1\boldsymbol{\xi}_2.$$

Furthermore, for $\phi \in \mathcal{X}^a_{\mathbf{D}}$, ξ_ϕ and $\xi_{i,\phi} := \phi \circ \boldsymbol{\xi}_i$ satisfy (11.9.3.a).

Let $R^+ := \mathbb{I}[\![\Gamma^+_{\mathcal{K}}]\!] \subset \Lambda_{\mathbf{D}}$. We give R^+ a new $\Lambda_{W,A}$-algebra structure via $(1+W) \mapsto (1+W)^{-1}\gamma_+$. Then for any $\phi \in \mathcal{X}^a_{\mathbf{D}}$, $\phi|R^+$ is an arithmetic homomorphism of weight 2 (in the sense that its pullback to $\Lambda_{W,A}$ under the new structure map is arithmetic of weight 2). Let $\mathbf{g} \in \mathcal{M}^{\mathrm{ord}}(M^2_{\mathbf{D}}D_{\mathcal{K}}, \xi'; \Lambda_{W,A})$ be an ordinary cuspidal newform such that

(13.3.3.b)
- $\mathbf{g} \otimes \chi_{\mathcal{K}} = \mathbf{g}$, where $\mathbf{g} \otimes \chi_{\mathcal{K}}$ is the ordinary newform associated with the twist of $\mathbf{g}$ by $\chi_{\mathcal{K}}$.

(So $(\mathbf{g} \otimes \chi_{\mathcal{K}})_\phi = \mathbf{g}_\phi \otimes \chi_{\mathcal{K}}$ for an arithmetic $\phi \in \mathcal{X}_{R^+,A}$.) We consider $\mathbf{g}$ as an element of $\mathcal{M}^{\mathrm{ord}}(M^2_{\mathbf{D}}D_{\mathcal{K}}, \xi; R^+)$. Thus for any $\phi \in \mathcal{X}^a_{\mathbf{D}}$, $\mathbf{g}_\phi$ is a weight 2 ordinary (p-stabilized) newform. Assume $(\mathbf{irred})_{\mathbf{g}}$ and $(\mathbf{dist})_{\mathbf{g}}$ hold. The hypothesis (13.3.3.b) ensures that there exists a finite set $\mathcal{L} = \{\ell_1, ..., \ell_m\}$ of primes that split in $\mathcal{K}$ and are disjoint from $\Sigma \cup \mathcal{Q}$ and a polynomial $P \in R^+[X_1, ..., X_m]$ such that $P_{\mathbf{g}} := P(T_{\ell_1}, ..., T_{\ell_m}) \in \mathbb{T}^{\mathrm{ord}}(M^2_{\mathbf{D}}D_{\mathcal{K}}, \xi'; R^+)$ is a non-zero R^+-multiple of $\ell_{\mathbf{g}}$, say $P_{\mathbf{g}} = a_{\mathbf{g}}\ell_{\mathbf{g}}$ with $0 \neq a_{\mathbf{g}} \in R^+$.

Proposition 13.3.4. *Under the above hypotheses,*

$$\sum_{i=1}^{h_{\mathcal{K}}} \boldsymbol{\xi}^{-1}\boldsymbol{\xi}_1(a_i)\boldsymbol{\xi}_{1,p}(q_i)\varphi_{\mathbf{D},i}(\mathcal{L}, P_{\mathbf{g}}; 1) = \mathcal{A}_{\mathbf{D},\mathbf{g}}\mathcal{B}_{\mathbf{D},\mathbf{g}}$$

with $\mathcal{A}_{\mathbf{D},\mathbf{g}} \in \mathbb{I}[\![\Gamma^+_{\mathcal{K}}]\!]$ *and* $\mathcal{B}_{\mathbf{D},\mathbf{g}} \in \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ *such that for* $\phi \in \mathcal{X}'_{\mathbf{D}}$

$$\phi(\mathcal{A}_{\mathbf{D},\mathbf{g}}) = \phi(a_{\mathbf{g}})\eta_{\mathbf{f}_\phi}B_{\mathfrak{D}_\phi,4}\frac{< \mathcal{E}_{\mathfrak{D}_\phi}\mathbf{g}_\phi, \mathbf{f}^c_\phi|_{\kappa_\phi}\begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathbf{D}}D_{\mathcal{K}})}}{< \mathbf{f}, \mathbf{f}^c_\phi|_{\kappa_\phi}\begin{pmatrix} & -1 \\ p^{r_\phi}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}_\phi}}},$$

where $B_{\mathfrak{D}_\phi,4}$ *is as in Lemma 11.9.4, and for* $\phi \in \mathcal{X}^a_{\mathbf{D}}$

$$\phi(\mathcal{B}_{\mathbf{D},\mathbf{g}}) = \eta_{\mathbf{g}_\phi}\frac{< g_{\xi_{1,\phi}}(-)g_{\xi_{2,\phi}}(M^2_{\mathbf{D}}(-)), \mathbf{g}^c_\phi|_2\begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathbf{D}}D_{\mathcal{K}})}}{< \mathbf{g}_\phi, \mathbf{g}^c_\phi|_2\begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathbf{D}}D_{\mathcal{K}})}}.$$

Furthermore, $\mathcal{A}_{\mathbf{D},\mathbf{g}} \neq 0$.

Proof. Let $\phi \in \mathcal{X}^a_{\mathbf{D}}$. Then by Lemma 13.3.2

$$\begin{aligned}\phi(\sum_{i=1}^{h_\mathcal{K}} \boldsymbol{\xi}_1\boldsymbol{\xi}^{-1}(a_i)\boldsymbol{\xi}_{1,p}(q_i)\varphi_{\mathbf{D},i}(P_{\mathbf{g}};1)) & \\ &= \sum_{i=1}^{h_\mathcal{K}} \xi_\phi^{-1}\xi_{1,\phi}(a_i)\xi_{1,\phi,p}(q_i)P_{\mathbf{g},H_i,\phi}e_{H_i}\varphi_{\mathbf{D},i,\phi}(1),\end{aligned} \tag{13.3.4.a}$$

with $P_{\mathbf{g},H_i,\phi} := P_{\mathbf{g},\phi}(\xi_{\ell_1,1}(\ell_1)T^{H_i}_{\ell_1},...,\xi_{\ell_m,1}(\ell_m)T^{H_i}_{\ell_m})$.

If $\phi \in \mathcal{X}'_{\mathbf{D}}$ then $\varphi_{\mathbf{D},i,\phi}(h) = u_\phi C_{\mathfrak{D}_\phi,i}(h)/\Omega^+_{\mathbf{f}_\phi}\Omega^-_{\mathbf{f}_\phi}$, where $C_{\mathfrak{D}_\phi,i}$ is as in Proposition 13.2.5 and u_ϕ is the p-adic unit from Theorem 12.4.2 (so $u_\phi = u_{\mathbf{f}_\phi}$ with $u_{\mathbf{f}_\phi}$ as in Lemma 12.2.2). It then follows from Proposition 13.2.5 and Lemma 12.2.2 that the right-hand side of (13.3.4.a) equals

$$\begin{aligned}\sum_{i=1}^{h_\mathcal{K}} \xi_\phi^{-1}\xi_{1,\phi}(a_i)\xi_{1,\phi,p}(q_i)\eta_{\mathbf{f}_\phi}B_{\mathfrak{D}_\phi}(\beta_i,1,u_i)\phi(a_{\mathbf{g}}) & \\ \times \frac{< \mathcal{E}_{\mathfrak{D}_\phi}\cdot \ell_{\mathbf{g}_\phi}(\Theta_{\mathfrak{D}_\phi,\beta_i}(1,-;u_i)), \mathbf{f}^c_\phi|_{\kappa_\phi}\begin{pmatrix} & -1 \\ p^{r_\phi}M_{\mathbf{D}}^2D_\mathcal{K} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K})}}{< \mathbf{f},\mathbf{f}^c_\phi|_{\kappa_\phi}\begin{pmatrix} & -1 \\ p^{r_\phi}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}_\phi}}}&.\end{aligned} \tag{13.3.4.b}$$

By our hypotheses on $\chi_{\mathbf{f}}$, ψ, and ξ, $B_{\mathfrak{D}_\phi}(\beta_i,1,u_i) = \xi_\phi(a_i)$ so

$$\begin{aligned}&\sum_{i=1}^{h_\mathcal{K}}\xi_\phi^{-1}\xi_{1,\phi}(a_i)\xi_{1,\phi,p}(q_i)B_{\mathfrak{D}_\phi}(\beta_i,1,u_i)\ell_{\mathbf{g}_\phi}(\Theta_{\mathfrak{D}_\phi,\beta_i}(1,-;u_i)) \\ &\quad = \sum_{i=1}^{h_\mathcal{K}} \xi_{1,\phi}(a_i)\xi_{1,\phi,p}(q_i)\ell_{\mathbf{g}_\phi}(\Theta_{\mathfrak{D}_\phi,\beta_i}(1,-;u_i)) \\ &\quad = \eta_{\mathbf{g}_\phi}\frac{< \sum_{i=1}^{h_\mathcal{K}}\xi_{1,\phi}(a_i)\xi_{1,\phi,p}(q_i)\Theta_{\mathfrak{D}_\phi,\beta_i}(1,-;u_i), \mathbf{g}^c_\phi|_2\begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathcal{D}}D_\mathcal{K})}}{< \mathbf{g}_\phi, \mathbf{g}^c_\phi|_2\begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathcal{D}}D_\mathcal{K})}} \\ &\quad = \eta_{\mathbf{g}_\phi}B_{\mathfrak{D}_\phi,4}\frac{< g_{\xi_{1,\phi}}(-)g_{\xi_{2,\phi}}(M^2_{\mathbf{D}}(-)), \mathbf{g}^c_\phi|_2\begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K})}}{< \mathbf{g}_\phi, \mathbf{g}^c_\phi|_2\begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K})}},\end{aligned}$$

the last equality following from Lemma 11.9.4. It follows that (13.3.4.b) equals

$$\begin{aligned}&\phi(a_{\mathbf{g}})\eta_{\mathbf{f}_\phi}B_{\mathfrak{D}_\phi,4}\frac{< \mathcal{E}_{\mathfrak{D}_\phi}\mathbf{g}_\phi, \mathbf{f}^c_\phi|_{\kappa_\phi}\begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K})}}{< \mathbf{f},\mathbf{f}^c_\phi|_{\kappa_\phi}\begin{pmatrix} & -1 \\ p^{r_\phi}M & \end{pmatrix} >_{\Gamma_{\mathfrak{D}_\phi}}} \\ &\qquad\qquad \times \eta_{\mathbf{g}_\phi}\frac{< g_{\xi_{1,\phi}}(-)g_{\xi_{2,\phi}}(M^2_{\mathbf{D}}(-)), \mathbf{g}^c_\phi|_2\begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K})}}{< \mathbf{g}_\phi, \mathbf{g}^c_\phi|_2\begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K})}}.\end{aligned}$$

By the Zariski density of $\mathcal{X}'_{\mathbf{D}}$ in $\operatorname{Spec}\Lambda_{\mathbf{D}}(\overline{\mathbf{Q}}_p)$, it follows that to complete the proof of the proposition it suffices to prove the existence of $\mathcal{A}_{\mathbf{D},\mathbf{g}} \in \mathbb{I}[\![\Gamma_{\mathcal{K}}^+]\!]$ and $\mathcal{B}_{\mathbf{D},\mathbf{g}} \in \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ with the specialization properties as in the statement of the proposition; the non-vanishing of $\mathcal{A}_{\mathbf{D},\mathbf{g}}$ is a consequence of Lemma 11.9.2.

We first construct $\mathcal{A}_{\mathbf{D},\mathbf{g}}$. Note that

$$
\begin{aligned}
&< \mathcal{E}_{\mathfrak{D}_\phi}\mathbf{g}_\phi, \mathbf{f}_\phi^c|_{\kappa_\phi}\begin{pmatrix} & -1 \\ p^{r_\phi}M_{\mathbf{D}}^2D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M_{\mathbf{D}}^2D_{\mathcal{K}})} \\
&\qquad = (M_{\mathbf{D}}M_1D_{\mathcal{K}})^{-\kappa_\phi/2} < \sum_{a\in\mathbf{Z}/M_{\mathbf{D}}M_1D_{\mathcal{K}}} \mathcal{E}_{\mathfrak{D}_\phi}\mathbf{g}_\phi\Big(\frac{w+a}{M_{\mathbf{D}}M_1D_{\mathcal{K}}}\Big), \mathbf{f}_\phi^c|_{\kappa_\phi}\begin{pmatrix} & -1 \\ p^{r_\phi}M & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\phi(a_{\mathbf{g}})\eta_{\mathbf{f}_\phi}B_{\mathfrak{D}_\phi,4} < \mathcal{E}_{\mathfrak{D}_\phi}\mathbf{g}_\phi, \mathbf{f}_\phi^c|_{\kappa_\phi}\begin{pmatrix} & -1 \\ p^{r_\phi}M_{\mathbf{D}}^2D_{\mathcal{K}} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M_{\mathbf{D}}^2D_{\mathcal{K}})} \\
&\qquad = 2^3i^{-1}\xi_{\phi,p}(\delta_{\mathcal{K}})D_{\mathcal{K}}^{-1/2}\omega_p^{\kappa_\phi-2}\chi_{\phi,p}(M_{\mathbf{D}})^2M_{\mathbf{D}}^{2-2\kappa_\phi}\phi(a_{\mathbf{g}})\eta_{\mathbf{f}_\phi} \\
&\qquad\quad \times < \sum_{a\in\mathbf{Z}/M_{\mathbf{D}}M_1D_{\mathcal{K}}} \mathcal{E}_{\mathfrak{D}_\phi}\mathbf{g}_\phi\Big(\frac{w+a}{M_{\mathbf{D}}M_1D_{\mathcal{K}}}\Big), \mathbf{f}_\phi^c|_{\kappa_\phi}\begin{pmatrix} & -1 \\ p^{r_\phi}M & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M)}
\end{aligned}
\tag{13.3.4.c}
$$

Let $h_{0,\phi}(w) := \mathcal{E}_{\mathfrak{D}_\phi}\mathbf{g}_\phi(w)$ and $h_\phi(w) := \sum_{a\in\mathbf{Z}/M_{\mathbf{D}}M_1D_{\mathcal{K}}}\mathcal{E}_{\mathfrak{D}_\phi}\mathbf{g}_\phi(\frac{w+a}{M_{\mathbf{D}}M_1D_{\mathcal{K}}})$. Write $\mathcal{E}_{\mathfrak{D}_\phi}(w) = \sum_{n=1}^\infty a_\phi(n)q^n$ and $\mathbf{g}_\phi(w) = \sum_{n=1}^\infty b_\phi(n)q^n$. Then $h_{0,\phi}(w) = \sum_{n=1}^\infty c_{0,\phi}(n)$ where

$$
c_{0,\phi}(n) := \sum_{\substack{m_1+m_2=n\\ m_1,m_2>0}} a_\phi(m_1)b_\phi(m_2),
$$

and $h_\phi(w) = \sum_{n=1}^\infty c_\phi(n)q^n$ with

$$
c_\phi(n) := M_{\mathbf{D}}M_1D_{\mathcal{K}}c_{0,\phi}(nM_{\mathbf{D}}M_1D_{\mathcal{K}}).
$$

Let

$$
\mathbf{a}(n) := \sum_{\substack{d|n,\, p\nmid d\\ (\ell,d)=1\forall\ell\in\Sigma\setminus\{p\}}} d^{-1}\Phi_{W,p}\boldsymbol{\xi}_p(d) \in \mathbb{I}[\![\Gamma_{\mathcal{K}}^+]\!].
$$

Then for $\phi\in\mathcal{X}'_{\mathbf{D}}$ it follows from (11.9.1.a) and the hypothesis that ξ is unramified at $\ell\neq p$ that $\phi(\mathbf{a}(n)) = a_\phi(n)$. Let $\mathbf{g} = \sum_{n=1}^\infty \mathbf{b}(n)q^n$, $\mathbf{b}(n)\in A[\![\Gamma_{\mathcal{K}}^+]\!]$, be the q-expansion of $\mathbf{g}$. Put

$$
\mathbf{c}_0(n) := \sum_{\substack{m_1+m_2=n\\ m_1,m_2>0}} \mathbf{a}(m_1)\mathbf{b}(m_2) \quad\text{and}\quad \mathbf{c}(n) := M_{\mathbf{D}}M_1D_{\mathcal{K}}\mathbf{c}_0(nM_{\mathbf{D}}M_1D_{\mathcal{K}}).
$$

Then for $\phi\in\mathcal{X}'_{\mathbf{D}}$ we have $\phi(\mathbf{c}_0(n)) = c_{0,\phi}(n)$ and $\phi(\mathbf{c}(n)) = c_\phi(n)$. Let $\mathbf{h} := \sum_{n=1}^\infty \mathbf{c}(n)q^n$. Then for $\phi\in\mathcal{X}'_{\mathbf{D}}$,

$$
\mathbf{h}_\phi(w) = \sum_{n=1}^\infty \phi(\mathbf{c}(n))q^n = \sum_{n=1}^\infty c_\phi(n)q^n = h_\phi(w).
$$

As $\{\phi|_{\mathbb{I}[\![\Gamma_{\mathcal{K}}^+]\!]} : \phi\in\mathcal{X}'_{\mathbf{D}}\}$ is Zariski dense in $\operatorname{Spec}\mathbb{I}[\![\Gamma_{\mathcal{K}}^+]\!](\overline{\mathbf{Q}}_p)$, the proof of Lemma 12.2.4 is easily adapted to show that there exists $\mathbf{k}\in\mathcal{M}^{\mathrm{ord}}(M,\mathbf{1};\mathbb{I})\otimes_{\mathbb{I}}\mathbb{I}[\![\Gamma_{\mathcal{K}}^+]\!]$ such that $\mathbf{k}_\phi = e\mathbf{h}_\phi$

for all $\phi \in \mathcal{X}'_{\mathbf{D}}$ (replace $\Lambda_{\mathbf{D}}$ with $\mathbb{I}[\![\Gamma^+_{\mathcal{K}}]\!]$ in the proof of Lemma 12.2.4 and take $\mathcal{X} := \{\phi|_{\mathbb{I}[\![\Gamma^+_{\mathcal{K}}]\!]} \ : \ \phi \in \mathcal{X}'_{\mathbf{D}}\}$). Write $\mathbf{k} = \sum a_i \mathbf{k}_i$ with $a_i \in \mathbb{I}[\![\Gamma^+_{\mathcal{K}}]\!]$ and $\mathbf{k}_i \in \mathcal{M}^{\mathrm{ord}}(M, \mathbf{1}; \mathbb{I})$. Let

$$\mathcal{A}^0_{\mathbf{D},\mathbf{g}} := \sum a_i a(1, \ell_{\mathbf{f}}(\mathbf{k}_i)) \in I[\![\Gamma^+_{\mathcal{K}}]\!].$$

Then for $\phi \in \mathcal{X}'_{\mathbf{D}}$,

$$\begin{aligned} \phi(\mathcal{A}^0_{\mathbf{D},\mathbf{g}}) &= \sum \phi(a_i) a(1, \ell_{\mathbf{f}_\phi} \mathbf{k}_{i,\phi}) = a(1, \ell_{\mathbf{f}_\phi} e \mathbf{h}_\phi) \\ &= \eta_{\mathbf{f}_\phi} \frac{< \mathbf{h}_\phi, \mathbf{f}^c_\phi|_{\kappa_\phi} \begin{pmatrix} & -1 \\ p^{r_\phi} M & \end{pmatrix} >_{\Gamma_0(p^{r_\phi} M)}}{< \mathbf{f}_\phi, \mathbf{f}^c_\phi|_{\kappa_\phi} \begin{pmatrix} & -1 \\ p^{r_\phi} M & \end{pmatrix} >_{\Gamma_0(p^{r_\phi} M)}}, \end{aligned} \tag{13.3.4.d}$$

the last equality following from (12.2.1.a). Putting

$$\mathcal{A}_{\mathbf{D},\mathbf{g}} := 2^3 i^{-1} D_{\mathcal{K}}^{-1/2} M^2_{\mathbf{D}} \boldsymbol{\xi}_p(\delta_{\mathcal{K}}) \Phi^{-1}_{W,p}(M^2_{\mathbf{D}}) a_{\mathbf{g}} \mathcal{A}^0_{\mathbf{D},\mathbf{g}}$$

and comparing (13.3.4.d) with (13.3.4.c) shows that $\mathcal{A}_{\mathbf{D},\mathbf{g}}$ has the sought-for interpolation properties.

Let $\mathbf{a}_1(n), \mathbf{a}_2(n) \in A[\![\Gamma_{\mathcal{K}}]\!]$ be defined by

$$\sum_{n=1}^\infty \mathbf{a}_1(n) q^n = \sum_{j=1}^\infty \boldsymbol{\xi}_1^{-1}(a_j) \sum_{\substack{x \in K \cap a_j \widehat{\mathcal{O}} \\ x \in \mathbf{Z}_p^\times \times \mathbf{Z}_p}} \boldsymbol{\xi}^{-1}_{1,p}(x) q^{\mathrm{Nm}(x)/q_j}$$

and

$$\sum_{n=1}^\infty \mathbf{a}_2(n) q^n = \sum_{j=1}^\infty \boldsymbol{\xi}_2^{-1}(a_j) \sum_{\substack{x \in K \cap a_j \widehat{\mathcal{O}} \\ x \in \mathbf{Z}_p \times \mathbf{Z}_p^\times}} \boldsymbol{\xi}^{-1}_{2,p}(x) q^{M^2_{\mathbf{D}} \mathrm{Nm}(x)/q_j}.$$

Then for all $\phi \in \mathcal{X}^a_{\mathbf{D}}$, $\mathbf{h}_i := \sum_{n=1}^\infty \mathbf{a}_i(n) q^n$ satisfies

$$\mathbf{h}_{i,\phi} = \sum_{n=1}^\infty \phi(\mathbf{a}_i(n)) q^n = \begin{cases} g_{\xi_{1,\phi}}(w) & i = 1 \\ g_{\xi_{2,\phi}}(M^2_{\mathbf{D}} w) & i = 2. \end{cases}$$

Note that

$$\mathbf{h}_{i,\phi} = \begin{cases} E'(w) & i = 1 \\ E'(M^2_{\mathbf{D}} w) & i = 2. \end{cases}, \quad \text{if } \phi|_{\Gamma_{\mathcal{K}}} = 1, \tag{13.3.4.e}$$

$$E'(w) = E(\chi_{\mathcal{K}}; w) - E(\chi_{\mathcal{K}}; pw),$$

where $E(\chi_{\mathcal{K}}; w)$ is the Eisenstein series of weight 1 with L-function $\zeta(s) L(s, \chi_{\mathcal{K}})$.

Let $\mathbf{h} := \sum_{n=1}^\infty \mathbf{c}(n) q^n$ with

$$\mathbf{c}(n) := \sum_{\substack{m_1 + m_2 = n \\ m_1, m_2 > 0}} \mathbf{a}_1(n) \mathbf{a}_2(n),$$

so $\mathbf{h}_\phi = \mathbf{h}_{1,\phi}\mathbf{h}_{2,\phi}$ for all $\phi \in \mathcal{X}_{\mathbf{D}}$. As in the construction of $\mathcal{A}_{\mathbf{D},\mathbf{g}}$, the proof of Lemma 12.2.4 is easily adapted to prove the existence of $\mathbf{k} \in \mathcal{M}^{\mathrm{ord}}(M, \mathbf{1}; R^+) \otimes_{R^+} \mathbb{I}[\![\Gamma_\mathcal{K}]\!]$ such that $\mathbf{k}_\phi = e\mathbf{h}_\phi$ for $\phi \in \mathcal{X}^a_{\mathbf{D}}$. Put

$$\mathcal{B}_{\mathbf{D},\mathbf{g}} := a(1, \ell_{\mathbf{g}}\mathbf{k}) \in \mathbb{I}[\![\Gamma_\mathcal{K}]\!].$$

For $\phi \in \mathcal{X}^a_{\mathbf{D}}$ we then have

$$\begin{aligned}\phi(\mathcal{B}_{\mathbf{D},\mathbf{g}}) &= a(1, \ell_{\mathbf{g}_\phi}\mathbf{k}_\phi) = a(1, \ell_{\mathbf{g}_\phi}(g_{\xi_{1,\phi}}(w)g_{\xi_{2,\phi}}(M^2_{\mathbf{D}}w))) \\ &= \eta_{\mathbf{g}_\phi} \frac{< g_{\xi_{1,\phi}}(-)g_{\xi_{2,\phi}}(M^2_{\mathbf{D}}(-)), \mathbf{g}^c_\phi|_2 \begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K})}}{< \mathbf{g}_\phi, \mathbf{g}^c_\phi|_2 \begin{pmatrix} & -1 \\ p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K} & \end{pmatrix} >_{\Gamma_0(p^{r_\phi}M^2_{\mathbf{D}}D_\mathcal{K})}},\end{aligned}$$

again by (12.2.1.a). ∎

13.4. **Independence of constant terms and non-singular Fourier coefficients.** Proposition 13.4.1 below is the key ingredient in deducing the main result - Theorem 3.6.1 - from Theorem 6.5.4. It establishes the existence of certain suitable p-adic Eisenstein data $\mathbf{D}$ such that given a prime divisor of the p-adic L-function $\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K}}$ showing up in the singular Fourier coefficients of the p-adic Eisenstein series $\mathbf{E}_{\mathbf{D}}$, there is some non-singular Fourier coefficient $\mathbf{c}_{\mathbf{D}}(\beta_i, x)$, for some β_i as in (11.9.3.c), that is not divisible by this prime. This is essentially done by showing that a suitable $\Lambda_{\mathbf{D}}$-combination of the $\mathbf{c}_{\mathbf{D}}(\beta_i, x)$'s - the combination in Proposition 13.3.4 for a good choice of $\mathbf{g}$ - is not divisible by the given prime. While it follows from this that some $\mathbf{c}_{\mathbf{D}}(\beta_i, x)$ is not divisible by the given prime divisor of the p-adic L-function, this does determine one such coefficient.

Proposition 13.4.1. *Let A be the ring of integers of a finite extension of $\mathbf{Q}_p$, $\mathbb{I}$ a domain and a finite $\Lambda_{W,A}$-algebra, and $\mathbf{f} \in \mathcal{M}^{\mathrm{ord}}(M, \mathbf{1}; \mathbb{I})$ an $\mathbb{I}$-adic newform such that* $(\mathbf{irred})_{\mathbf{f}}$ *and* $(\mathbf{dist})_{\mathbf{f}}$ *hold. Suppose A satisfies the hypotheses of Proposition 12.2.5. Let Σ_0 be any finite set of primes containing those that divide $pMD_\mathcal{K}$. Then there exists a finite set of primes $\Sigma \supset \Sigma_0$ such that for the p-adic Eisenstein datum $\mathbf{D} = (A, \mathbb{I}, \mathbf{f}, 1, 1, \Sigma)$ there is an associated $\Lambda_{\mathbf{D}}$-adic Eisenstein series $\mathbf{E}_{\mathbf{D}}$ with the following holding for the set $\mathcal{C}_{\mathbf{D}} = \{\mathbf{c}_{\mathbf{D}}(\beta_i, x) \ : \ i = 1, ..., h_\mathcal{K}, x \in G(\mathbf{A}_f) \cap Q(\mathbf{Z}_p)\}$ of Fourier coefficients of $\mathbf{E}_{\mathbf{D}}$.*

(i) *If $R \subseteq \Lambda_{\mathbf{D}}$ is any height-one prime containing $\mathcal{C}_{\mathbf{D}}$, then $R = P\Lambda_{\mathbf{D}}$ for some height-one prime $P \subset \mathbb{I}[\![\Gamma^+_\mathcal{K}]\!]$.*

(ii) *If $M = M^+M^-$ with M^+ divisible only by primes that split in $\mathcal{K}$ and M^- divisible only by primes that are inert in $\mathcal{K}$, M^- is square-free and has an odd number of prime divisors, and $\bar{\rho}_{\mathbf{f}}$ is ramified at all $\ell | M^-$, then there are no height-one primes of $\Lambda_{\mathbf{D}}$ containing $\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},1}$ and $\mathcal{C}_{\mathbf{D}}$.*

Recall that for a given $\mathbf{D}$ we make a choice[12] of $M_{\mathbf{D}}$ in the definition of $\mathbf{E}_{\mathbf{D}}$.

[12] This is mostly a technical point. This choice has no effect on the properties of the Fourier coefficients we are interested in.

Proof. Part (ii) follows from part (i) and Proposition 12.3.6, so we need only prove part (i).

Clearly, to prove part (i) we may assume A is sufficiently large; replacing A with the integer ring A' of a larger extension of $\mathbf{Q}_p$ and $\mathbb{I}$ with any irreducible component $\mathbb{I}'$ of $\mathbb{I}\otimes_A A'$ does not change $\mathbf{E_D}$, and if the conclusions of (i) hold in $\Lambda_{\mathbf{D}}\otimes_{\mathbb{I}}\mathbb{I}'$ then they hold in $\Lambda_{\mathbf{D}}$. Suppose then we can find $\Sigma\supset\Sigma_0$ and $M_{\mathbf{D}}$ so that there exists an R^+-adic newform $\mathbf{g}\in\mathcal{M}^{\mathrm{ord}}(M_{\mathbf{D}}^2D_{\mathcal{K}},\mathbf{1};R^+)$ satisfying (13.3.3.b) (recall that $R^+=\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$). Let $R\subset\Lambda_{\mathbf{D}}$ be a height-one prime containing $\mathcal{C}_{\mathbf{D}}$. By Lemma 13.3.2, if R contains $\mathcal{C}_{\mathbf{D}}$ then R also contains each $\varphi_{\mathcal{D},i}(\mathcal{L},P_{\mathbf{g}};1)$, where $\mathcal{L}$ and $P_{\mathbf{g}}$ are as in Proposition 13.3.4. Then it also follows from Proposition 13.3.4 that R contains $\mathcal{A}_{\mathbf{D},\mathbf{g}}\mathcal{B}_{\mathbf{D},\mathbf{g}}$, with $\mathcal{A}_{\mathbf{D},\mathbf{g}}$ and $\mathcal{B}_{\mathbf{D},\mathbf{g}}$ as in Proposition 13.3.4. If $\mathcal{B}_{\mathbf{D},\mathbf{g}}$ is a unit in $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ (and hence in $\Lambda_{\mathbf{D}}$) then R must contain $\mathcal{A}_{\mathbf{D},\mathbf{g}}$. As $\mathcal{A}_{\mathbf{D},\mathbf{g}}\in\mathbb{I}[\![\Gamma_{\mathcal{K}}^+]\!]$, it follows that $R=P\Lambda_{\mathbf{D}}$ for some height one prime $P\subseteq\mathbb{I}[\![\Gamma_{\mathcal{K}}^+]\!]$. Hence to complete the proof of the proposition it suffices to show the existence of $\Sigma\supset\Sigma_0$ and an $M_{\mathbf{D}}$ for which there exists a $\mathbf{g}$ as above with $\mathcal{B}_{\mathbf{D},\mathbf{g}}$ a unit.

Our first step in finding Σ, $M_{\mathbf{D}}$, and $\mathbf{g}$ is to choose an idele class character θ of $\mathbf{A}_{\mathcal{K}}^\times$ such that

- $\theta_\infty(z)=z^{-1}$;
- $\theta^c=|\cdot|_{\mathcal{K}}\theta^{-1}$;
- $\mathrm{Nm}(\mathfrak{f}_\theta)=M_\theta^2$ for some integer M_θ prime to p and such that $D_{\mathcal{K}}M|M_\theta$ and $\ell|M_\theta$ for all $\ell\in\Sigma_0$;
- for some $q|D_{\mathcal{K}}$, $\theta|_{\mathcal{O}_q^\times}$ has order divisible by q;
- $\Omega_\infty^{-1}L(1,\theta)$ is a p-adic unit, where Ω_∞ is the CM-period defined in [Fi06];
- $\theta_{p,2}(p)-1$ is a p-adic unit.

The existence of such a θ is a simple exercise in light of the main results of [Fi06]. Let θ_1 be any idele class character of $\mathbf{A}_{\mathcal{K}}^\times$ such that $\theta_{1,\infty}(z)=z^{-1}$, $\theta_1^c=|\cdot|_{\mathcal{K}}\theta_1^{-1}$, and θ_1 is unramified at places above p. Let Σ_1 be the set of primes at which θ_1 is ramified. For each $\ell\in\Sigma_0\cup\Sigma_1$, $\ell\neq p$, let $\psi^{(\ell)}$ be a finite order anticyclotomic idele class character of $\mathbf{A}_{\mathcal{K}}^\times$ of conductor and order a power of ℓ and such that $\mathrm{Nm}(\mathfrak{f}_{\psi^{(\ell)}})=\ell^{2a_\ell}$ with $a_\ell>\mathrm{ord}_\ell(MD_{\mathcal{K}}\mathrm{Nm}(\mathfrak{f}_{\theta_1}))$. Let $\theta_2:=\theta_1\prod_{\ell\in\Sigma_0\cup\Sigma_1,\ell\neq p}\psi^{(\ell)}$. Then θ_2 has the first four of the six sought-for properties. If the root number of θ_2 is $+1$, then we can appeal to the main result of [Fi06] to obtain a character with all six. Let $\ell_0\notin\Sigma_0\cup\Sigma_1$ be a prime that splits completely in $\mathcal{K}$. By Theorem 1.1 of [Fi06], for all but finitely many anticyclotomic characters ψ of conductor and order a power of ℓ_0, $\Omega_\infty^{-1}L(1,\theta_2\psi)$ is a p-adic unit (the result in *loc. cit.* applies to $\Omega_\infty^{-1}L(0,(\theta_2\psi)^{-1})$, but $L(0,(\theta_2\psi)^{-1})=L(1,(\theta_2\psi)^c)=L(1,\theta_2\psi)$). We may therefore choose ψ so that that $\theta:=\theta_2\psi$ also has the last two of the six sought-for properties (for the last property we note that $\psi_{p,2}(p)$ can be an ℓ-power root of unity of arbitrarily high order). We also take $\Sigma:=\Sigma_0\cup\Sigma_1\cup\{\ell_0\}$ and $M_{\mathbf{D}}:=M_\theta$.

If the root number of θ_2 is -1, then we can choose a quadratic extension $L/\mathbf{Q}$ with absolute discriminant D_L prime to $\mathrm{Nm}(\mathfrak{f}_{\theta_2})$ and associated quadratic character χ_L of

$\mathbf{A}^\times$ such that the root number of $\theta_2\cdot\chi_L\circ\mathrm{Nm}$ is $+1$ (this root number is just $\chi_L(-D_K)$ times the root number of θ_2). In this case we then choose ℓ_0 to also be prime to D_L and take θ to be a suitable twist of $\theta_2\cdot\chi_L\circ\mathrm{Nm}$ by a finite order anticyclotomic character of conductor a power of ℓ_0, as above, and we let $\Sigma:=\Sigma_0\cup\Sigma_1\cup\{\ell|\ell_0 D_L\}$ and $M_{\mathbf{D}}:=M_\theta$.

Let g_θ be the CM newform associated with θ; g_θ has weight 2, level $M_\theta^2 D_K$, and trivial character. The p-adic Galois representation ρ_{g_θ} is isomorphic to $\mathrm{Ind}_{G_\mathcal{K}}^{G_\mathbf{Q}}\sigma_\theta$. The fourth listed property of θ (regarding $\theta|_{\mathcal{O}_q^\times}$ at the prime $q|D_\mathcal{K}$) ensures that the reduction of $\mathrm{Ind}_{G_\mathcal{K}}^{G_\mathbf{Q}}\sigma_\theta$ is irreducible, hence (**irred**) holds for $\bar\rho_{g_\theta}$. As p splits in $\mathcal{K}$, the restriction of $\mathrm{Ind}_{G_\mathcal{K}}^{G_\mathbf{Q}}\sigma_\theta$ to G_p is just $\sigma_\theta|_{G_\mathfrak{p}}\oplus\sigma_{\theta^c}|_{G_\mathfrak{p}}$. As θ is unramified at all primes above p and $\theta_\infty(z)=z^{-1}$, $\theta_{p,2}(p)$ is a p-adic unit (so g_θ is ordinary at p) and $\sigma_{\theta^c}|_{G_\mathfrak{p}}$ is the unramified character sending frob_p to $\theta_{p,2}(p)$. As $\sigma_\theta|_{G_\mathfrak{p}}=\varepsilon\sigma_{\theta^c}^{-1}|_{G_\mathfrak{p}}$, it follows that (**dist**) holds for $\bar\rho_{g_\theta}$.

As noted above, we may assume that A is sufficiently large. In particular, we may assume that it contains the values of θ. We will show that we can find $\mathbf{g}$ as in Proposition 13.3.4 such that $\mathcal{B}_{\mathbf{D},\mathbf{g}}$ is a unit.

As in 13.3.3, let $R^+:=\mathbb{I}[\![\Gamma_\mathcal{K}^+]\!]$ with a new $\Lambda_{W,A}$-algebra structure via $1+W\mapsto(1+W)^{-1}\gamma_+$. Let $\mathbf{g}\in\mathcal{M}^{\mathrm{ord}}(M_\theta^2 D_\mathcal{K},\mathbf{1};R^+)$ be the ordinary CM newform associated with θ. This has the property that if $\phi\in\mathcal{X}_{R^+,A}$ is such that $\phi(\gamma_+)=1=\phi(1+W)$, then $\mathbf{g}_\phi$ is the ordinary p-stabilization g of g_θ: $\mathbf{g}_\phi(w)=g(w):=g_\theta(w)-\theta_{p,1}(p)g_\theta(pw)$. We have

- $\mathbf{g}\otimes\chi_\mathcal{K}=\mathbf{g}$;
- $(\mathbf{irred})_\mathbf{g}$ and $(\mathbf{dist})_\mathbf{g}$ hold.

The first property follows from $\mathbf{g}$ being a family of CM forms, and the second follows from $\rho_{\mathfrak{m}_\mathbf{g}}\cong\bar\rho_{g_\theta}$ and the previously observed fact that (**irred**) and (**dist**) hold for $\bar\rho_{g_\theta}$.

Let $\mathcal{B}_{\mathbf{D},\mathbf{g}}\in\mathbb{I}[\![\Gamma_\mathcal{K}]\!]$ be as in Proposition 13.3.4. Let $\phi\in\mathcal{X}_\mathbf{D}^a$ be such that $\phi|\Gamma_\mathcal{K}=1$ and $\phi(1+W)=1$. Then $\mathbf{g}_\phi=g$ (g being the ordinary p-stabilization of g_θ, as above) and

(13.4.1.a)
$$\phi(\mathcal{B}_{\mathbf{D},\mathbf{g}})=\eta_g\frac{< E'(-)E'(M_\mathbf{D}^2(-)),g^c|_2\begin{pmatrix} & -1\\ pM_\mathbf{D}^2D_\mathcal{K} & \end{pmatrix}>_\Gamma}{< g,g^c|_2\begin{pmatrix} & -1\\ pM_\mathbf{D}^2D_\mathcal{K} & \end{pmatrix}>_\Gamma},$$

where $E'(w)=E(\chi_\mathcal{K},w)-E(\chi_\mathcal{K};pw)$ as in (13.3.4.e) and $\Gamma:=\Gamma_0(pM_\mathbf{D}^2D_\mathcal{K})$; this follows from Proposition 13.3.4 and (13.3.4.e). We will show that the right-hand side of (13.4.1.a) is a p-adic unit and thus that $\mathcal{B}_{\mathbf{D},\mathbf{g}}$ is a unit.

Let $h(w):=E'(w)E'(M_\mathbf{D}^2w)$. Our next step is to identify $< h,g^c|_2\begin{pmatrix} & -1\\ pM_\mathbf{D}^2D_\mathcal{K} & \end{pmatrix}>_\Gamma$ with a Rankin-Selberg convolution integral. We begin by noting that

$$E(\chi_\mathcal{K};w)|_1\begin{pmatrix} & -1\\ D_\mathcal{K} & \end{pmatrix}=\pm E(\chi_\mathcal{K};w).$$

Here and in what follows '$\pm$' will signify a quantity up to undetermined sign; the exact value of the sign will not matter but may vary from one usage to the next. It follows that

$$E'(w)|_1 \begin{pmatrix} & -1 \\ pD_{\mathcal{K}} & \end{pmatrix} = \pm p^{-1/2}(E(\chi_{\mathcal{K}};w) - pE(\chi_{\mathcal{K}};pw)$$

and so that

$$h|_2 \begin{pmatrix} & -1 \\ pM_{\mathbf{D}}^2 D_{\mathcal{K}} & \end{pmatrix} = p^{-1}E''(w)E''(M_{\mathbf{D}}^2 w), \quad E''(w) = E(\chi_{\mathcal{K}};w) - pE(\chi_{\mathcal{K}};pw).$$

Therefore

(13.4.1.b) $$< h, g^c|_2 \begin{pmatrix} & -1 \\ pD_{\mathcal{K}} & \end{pmatrix} >_\Gamma = p^{-1}M_{\mathbf{D}}^{-2} < E''(-)E''(M_{\mathbf{D}}^2(-)), g^c >_\Gamma .$$

For $s \in \mathbf{C}$ let $f_s : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ be the function defined by

$$f_s(g) = \begin{cases} \chi_{\mathcal{K}}(d)|a/d|^{s+1/2} j(k,i)^{-1} & g = \begin{pmatrix} a & b \\ & d \end{pmatrix} uk, u \in U_1(D_{\mathcal{K}}), k \in K'_\infty \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$c(s) := \frac{L(2s+1,\chi_{\mathcal{K}})D_{\mathcal{K}}}{-2\pi i G(\chi_{\mathcal{K}})}$$

and $F(s,g) := c(s)\sum_{\gamma \in B'(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{Q})} f_s(\gamma g)$. If $g \in \mathrm{GL}_2^+(\mathbf{R})$ and $w = g(i)$ we put $F(s,w) := \det g^{-1/2} j(g,i)F(s,g)$ (this depends only on w). Then

$$F(s,w) = c(s)\mathrm{Im}(w)^s \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(D_{\mathcal{K}})} \chi_{\mathcal{K}}(d_\gamma) j(\gamma,w)^{-1}|j(\gamma,w)|^{-2s},$$

where $\Gamma_\infty := \{\gamma \in \Gamma_0(D_{\mathcal{K}}) \ ; \ c_\gamma = 0\}$. It is a classical result that $F(s,w)$ converges for $\mathrm{Re}(s) > 1/2$ and is holomorphic in such s and has a meromorphic continuation to all $s \in \mathbf{C}$ that is holomorphic at $s = 0$ and even that $F(0,w) = E(\chi_{\mathcal{K}},w)$ (cf. [Mi89]; for the last equality compare Lemma 7.2.19(3) and (7.2.62) and (7.2.60) of *loc. cit.*, with $\chi = \chi_{\mathcal{K}}$ in the latter).

Let $h_s : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ be defined just as f_s was but with $U_1(D_{\mathcal{K}})$ replaced with $U_1(pM_{\mathbf{D}}^2 D_{\mathcal{K}})$. Define $H(s,g)$ and $H(s,w)$ as $F(s,g)$ and $F(s,w)$ but with h_s replacing f_s, so

$$H(s,w) = c(s)\mathrm{Im}(w)^s \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \chi_{\mathcal{K}}(d_\gamma) j(\gamma,w)^{-1}|j(\gamma,w)|^{-2s}.$$

We can relate f_s to h_s as follows. Let $S := \{\ell | pM_{\mathbf{D}} : \ell \nmid D_{\mathcal{K}}\}$. Let $N_0 := \prod_{\ell \in S} \ell$ and $N_1 := pM_{\mathbf{D}}^2/N_0$. Let

$$h'_s(g) := \prod_{\ell \in S}(1 - \chi_{\mathcal{K},\ell}(\ell)\ell^{2s+1})^{-1} \times \sum_{T \subseteq S}(-1)^{\#T}(\prod_{\ell \in T}\chi_{\mathcal{K},\ell}(\ell)\ell^{s+1/2}) f_s(g\prod_{\ell \in T}\begin{pmatrix} \ell^{-1} & \\ & 1 \end{pmatrix}_\ell).$$

It is easily checked that h'_s is supported on $B'(\mathbf{A})U_1(N_0)K'_\infty$ and that $h'_s(\left(\begin{smallmatrix} a & b \\ & d\end{smallmatrix}\right) uk) = \chi_{\mathcal{K}}(d)|a/d|^{s+1/2}j(k,i)^{-1}$. Therefore,

$$\begin{aligned} h_s(g) &= N_1^{-s-1/2} h'_s(g)\left(\begin{smallmatrix} N_1^{-1} & \\ & 1\end{smallmatrix}\right)_f) \\ &= \prod_{\ell\in S}(1-\epsilon_{\mathcal{K},\ell}\ell^{2s+1})^{-1} \times \sum_{T\subseteq S}(-1)^{\#T}(\prod_{\ell\in T}\chi_{\mathcal{K},\ell}(\ell)\ell^{s+1/2})N_1^{-s-1/2} \\ &\qquad \times f_s(g\prod_{\ell\in T}\left(\begin{smallmatrix} \ell^{-1} & \\ & 1\end{smallmatrix}\right)_\ell\left(\begin{smallmatrix} N_1^{-1} & \\ & 1\end{smallmatrix}\right)_f). \end{aligned}$$

It then follows that

$$H(0,w) = \prod_{\ell\in S}(1-\epsilon_{\mathcal{K},\ell}\ell^{2s+1})^{-1} \times \sum_{T\subseteq S}(-1)^{\#T}(\prod_{\ell\in T}\chi_{\mathcal{K},\ell}(\ell)\ell)E(\chi_{\mathcal{K}};N_1 d_T w), \quad d_T = \prod_{\ell\in T}\ell.$$

Therefore

$$E''(M_{\mathbf{D}}^2 w) = \pm p N_0^{-1}\left(\prod_{\ell\in S}(1-\chi_{\mathcal{K},\ell}(\ell))\right)H(0,w) + H'(w)$$

with $H'(w)$ a sum of forms that are old at some prime dividing $M_{\mathbf{D}}$. Hence (13.4.1.c)

$$< E''(-)E''(M_{\mathbf{D}}^2(-)), g^c >_\Gamma = \pm p N_0^{-1}\left(\prod_{\ell\in S}(1-\chi_{\mathcal{K},\ell}(\ell))\right) < E'' H(0,-), g >_\Gamma .$$

Let $I(s) :=< E''H(s,-), g >_\Gamma$; this is meromorphic as a function of s. By the usual unfolding argument, if $\mathrm{Re}(s)$ is sufficiently large then $I(s) = c(s)(2\pi)^{-(s+1)}\Gamma(s+1)D(E'',g;s+1)$, where $D(E'',g;s+1) := \sum_{n=1}^\infty a(n)b(n)n^{-s-1}$ if $E''(w) = \sum_{n=1}^\infty a(n)q^n$ and $g(w) = \sum_{n=1}^\infty b(n)q^n$. Recalling that $E''(w) = E(\chi_{\mathcal{K}};w) - pE(\chi_{\mathcal{K}};w)$ and $g(w) = g_\theta(w) - \theta_{p,1}(p)g_\theta(pw)$, we find that

$$D(E'',g;s+1) = \frac{L^{\{p\}}(g_\theta,s+1)L^{\{p\}}(g_\theta,\chi_{\mathcal{K}},s+1)(1-\theta_{p,2}(p)p^{-s})}{L^\Sigma(2s+1;\chi_{\mathcal{K}})(1-\theta_{p,2}(p)p^{-s-1})^2}.$$

Since $L(g_\theta,\chi,s) = L(g_\theta,s) = L(\theta,s)$, it follows from uniqueness of meromorphic continuation that

$$I(0) = \frac{L(\theta,1)^2(1-\theta_{p,2}(p))(1-\theta_{p,1}(p)p^{-1})^2}{(2\pi)c(0)L^\Sigma(1,\chi_{\mathcal{K}})}.$$

Combining this with (13.4.1.c) and (13.4.1.b) we find that

$$\begin{aligned}< h, g^c|_2 \left({}_{pM_{\mathbf{D}}^2 D_{\mathcal{K}}}{}^{-1} \right) >_\Gamma &= \pm N_0^{-1} I(0) \prod_{\ell \in S} (1 - \epsilon_{\mathcal{K},\ell}(\ell)\ell) \\ &= \pm I(0) \prod_{\ell \in S} (1 - \epsilon_{\mathcal{K},\ell}(\ell)\ell^{-1}) \\ &= \pm c(0) \frac{L(1,\chi_{\mathcal{K}})L(1,\theta)^2(1-\theta_{p,2}(p))(1-\theta_{p,1}(p)p^{-1})^2}{(2\pi)\theta_{p,2}(p)} \\ &= \pm \frac{D_{\mathcal{K}} L(1,\theta)^2(1-\theta_{p,2}(p))(1-\theta_{p,1}(p)p^{-1})^2}{i(-2\pi i)^2 \mathfrak{g}(\chi_{\mathcal{K}})\theta_{p,2}(p)^2}.\end{aligned}$$

By Lemma 12.2.2

$$\eta_g = u \frac{< g, g^c|_2 \left({}_{pM_{\mathbf{D}}^2 D_{\mathcal{K}}}{}^{-1} \right) >_\Gamma}{\Omega_g^+ \Omega_g^-}$$

with u a p-adic unit. Therefore, if $\phi|_{\Gamma_{\mathcal{K}}} = 1$ and $\phi(1+W) = 1$,

$$\phi(\mathcal{B}_{\mathbf{D},\mathbf{g}}) = \pm u \frac{\pm D_{\mathcal{K}} L(1,\theta)^2(1-\theta_{p,2}(p))(1-\theta_{p,1}(p)p^{-1})^2}{G(\chi_{\mathcal{K}}) i \theta_{p,2}(p)^2 (2\pi)^2 \Omega_g^+ \Omega_g^-}.$$

By the choice of θ, both $\theta_{p,2}(p)$ and $1-\theta_{p,2}(p) = \theta_{p,2}(p)(1-\theta_{p,1}(p)p^{-1})$ are p-adic units, hence $\phi(\mathcal{B}_{\mathbf{D},\mathbf{g}})$ is a p-adic unit if $L(1,\theta)/2\pi i\Omega_g^{\pm}$ is. We claim that $2\pi i\Omega_g^{\pm}$ is a p-adic unit multiple of Ω_∞. From the choice of θ it then follows that $L(1,\theta)/2\pi i\Omega_g^{\pm}$ is a p-adic unit, completing the proof of the proposition.

To prove the claim we first note that $\Omega_g^{\pm}$ is a p-adic unit multiple of $\Omega_{g_\theta}^{\pm}$. Also, $\Omega_{g_\theta}^{-}$ is a p-adic unit multiple of $\Omega_{g_\theta}^{+}$ since $g_\theta \otimes \chi_{\mathcal{K}} = g_\theta$ and $p \nmid D_{\mathcal{K}}$. So it suffices to show that $2\pi i\Omega_{g_\theta}^{+}$ is a p-adic unit multiple of Ω_∞. We certainly have that $2\pi i\Omega_{g_\theta}^{+} = a\Omega_\infty$ for some $a \in \overline{\mathbf{Q}}_p$. We also have that for any even Dirichlet character λ of conductor prime to $pM_{\mathbf{D}}$, $L(g_\theta,\lambda,1)/2\pi i\Omega_{g_\theta}^{+}$ and $L(1,\theta\lambda)/\Omega_\infty$ are p-adic integers. It follows that $a^{-1}L(1,\theta\lambda)/\Omega_\infty$ is also a p-adic integer. By our choice of θ, $L(1,\theta)/\Omega_\infty$ is a p-adic unit, and so it follows that a^{-1} is a p-adic integer. On the other hand, by [St82, Thm. 2.1] there exists λ such that $L(g,\lambda,1)/2\pi i\Omega_g^{+}$ is a p-adic unit, so a must also be a p-adic integer. Therefore a is a p-adic unit. ∎

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